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Time-Optimal Control of a Self-Propelled Particle in a Spatiotemporal Flow Field

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We address a minimum-time problem that constitutes an extension of the classical Zermelo navigation problem in higher dimensions. In particular, we address the problem of steering a self-propelled particle to a prescribed terminal position with free terminal velocity in the presence of a spatiotemporal flow field. Furthermore, we assume that the norm of the rate of change of the particle's velocity relative to the flow is upper bounded by an explicit upper bound. To address the problem, we first employ Pontryagin's Minimum Principle to parameterize the set of candidate time-optimal control laws in terms of a parameter vector that belongs to a compact set. Subsequently, we develop a simple numerical algorithm for the computation of the minimum time-to-come function that is tailored to the particular parametrization of the set of the candidate time-optimal control laws of our problem. The proposed approach bypasses the task of converting the optimal control problem to a parameter optimization problem, which can be computationally intense, especially when one is interested in characterizing the optimal synthesis of the minimum-time problem. Numerical simulations that illustrate the theoretical developments are presented.

Keywords: optimal control; minimum time; self-propelled particle; Pontryagin's Minimum Principle;

1. Introduction

We address the problem of driving a self-propelled or Newtonian particle to a prescribed terminal position with a free terminal velocity in the presence of a spatiotemporal flow field in minimum time. It is assumed that the control input of the particle is the rate of change of its velocity relative to the flow, whose 2-norm is bounded by an a priori known constant. Thus, the input value set of the particle is a “hyperball” and consequently, the control input attains its values in a set that remains symmetric with respect to the velocity of the particle at all times. This is in contradistinction with the steering problem in the presence of a constraint on the maximum norm of the particle's input, where the input value set is a “hypercube.” Moreover, the time-optimal control law in our problem turns out to be a continuous function of time, in contrast to the discontinuous, piece-wise constant control laws that solve the minimum-time problem when the input value set is a hypercube (Athans & Falb, 2007). The problem considered in this work belongs to a special class of minimum-time problems for linear or affine systems where the control input is constrained to attain values in a convex and compact set (Lee & Markus, 1986). For this class of problems, which were originally studied by Krasovskii, LaSalle and Hermes (Hermes & LaSalle, 1969; Krasovskii, 1957; LaSalle, 1960), there always exists a time-optimal control that attains its values in the boundary of the input value set exclusively; this result is widely known as the *bang-bang* principle for minimum-time problems (Hermes & LaSalle, 1969; LaSalle, 1960). For some interesting extensions of the bang-bang principle, the reader may refer to (Halkin, 1964; Kurzweil, 1965; Neustadt, 1965; Olech, 1966). It is well-known that in the case when the maximum norm of the input is constrained, one can unambiguously characterize the time-optimal control law in the time-invariant case provided that the system enjoys the so-called *normality* property, which practically means that the system is controllable with respect to each of its components individually. In our problem, the system does not enjoy the same normality property that plays a key role in the analysis of the minimum-time problem when the input value set is a hypercube. However, it turns out that when the input value

set is a hyperball, as in our problem, one can still unambiguously characterize a time-optimal control law without actually constraining the analysis to the time-invariant case.

The problem of steering a small boat in the presence of a spatiotemporal drift field in minimum time was first studied by Zermelo in one of his seminal papers (Zermelo, 1931). This minimum-time problem is known in the literature of calculus of variations and optimal control theory as the *Zermelo navigation problem* (Bryson & Ho, 1969; Carathéodory, 1999; Jurdjevic, 1997). Although, at a first glance, the ZNP appears as a rather simple problem, it turns out that the solution of the original formulation of this problem, as well as its subsequent extensions and variations, exhibits a number of interesting and intricate properties, which continue to attract the interest of researchers (Bakolas & Tsiotras, 2010, 2013; Bao et al., 2004; Rhoads et al., 2010; Serres, 2006, 2009). The main limitation of the ZNP is that the motion of the boat is described by a “single integrator” kinematic model, which allows one to directly control, and thus instantaneously change, the direction of the velocity of the boat relative to the flow. On the other hand, the problem of guiding a self-propelled (or Newtonian) particle of unit mass by means of a time-optimal controller under a constraint on the 2-norm of its acceleration has received considerable attention in the literature of applied mechanics (Akulenko & Shmatkov, 2002, 2007; Akulenko, 2008, 2011). The reader may also refer to (Aneesh & Bhat, 2006). None of the previous references deal, however, with the case when the self-propelled particle is traveling in the presence of a flow field. This problem, which constitutes a combination of two classical optimal control problems, namely the ZNP (Zermelo, 1931) and the minimum-time steering problem for a self-propelled particle (Akulenko & Shmatkov, 2002), was first proposed in our previous work (Bakolas, 2014), where it was addressed for the case when the velocity field of the flow is either a constant or time-varying, yet spatially invariant, function. The adopted approach in (Bakolas, 2014) is based on the transformation of the minimum-time problem to a parameter optimization problem, which essentially consists of a system of nonlinear algebraic equations in triangular form, whose numerical solution is thus straightforward. Unfortunately, in the case when the flow field has a spatiotemporal velocity field, the corresponding (finite-dimensional) parameter optimization obtained by transcribing the original (infinite-dimensional) optimal control problem (Betts, 2010), has a significantly more complex structure; consequently, its numerical solution is a computationally intense task, the success of which relies heavily on the identification of “good” initial guesses.

The main contribution of this paper is the presentation of the *time-optimal synthesis* of the problem of steering a self-propelled particle of unit mass to a prescribed position with a free terminal velocity in the presence of a spatiotemporal flow field in minimum time. The term *time-optimal synthesis* refers to the mapping that returns the time-optimal control and the corresponding minimum time as a function of the particle’s terminal position, assuming a fixed initial position and velocity. In this work, we assume that the velocity of the flow is approximated by a time-varying inhomogeneous linear (affine) field. We show that although our system does not enjoy the normality property even in the time-invariant case, the unambiguous characterization of the time-optimal control law is practically possible by solely utilizing Pontryagin’s Minimum Principle (Pontryagin et al., 1962). In particular, it turns out that the set of all the candidate time-optimal control laws can be parameterized in terms of a parameter vector that belongs to a compact set. Similar properties are typically enjoyed by the optimal synthesis of optimal control problems that do not admit abnormal minimum solutions and are often exploited in the development of numerical algorithms for such problems (see (Trélat, 2012) and references therein).

In our problem, the parameter vector corresponds to the initial value of a time-varying quantity that evolves according to a known ordinary differential equation, which in turn plays a role to the solution of our problem that is similar to that of the so-called *navigation formula* of the ZNP Bryson & Ho (1969); Carathéodory (1999). Essentially, the whole minimum-time steering problem reduces to the characterization of the parameter vector that will furnish the minimum arrival time. Subsequently, we develop an algorithm for the characterization of the level sets of the minimum time-to-come function, which exploits this particular parametrization of the candidate time-optimal control laws. Specifically, the main idea of the utilized algorithm is to expand the *ex-*

tremal fronts of the minimum-time problem, that is, the closed curves in the state space that consist of the endpoints of the candidate minimum-time trajectories for a certain arrival time. During the process of expanding these extremal fronts, we “filter out” the points of the extremal fronts that do not correspond to endpoints of minimum-time trajectories. The criterion for determining whether a point and its associated time of arrival determine an endpoint of a minimum-time trajectory is whether the extremal front that corresponds to this arrival time reaches this particular point for the first time or not. Our numerical simulations reveal that the minimum time is a discontinuous function of the terminal position of the particle.

It should be mentioned at this point that an alternative way to compute the optimal synthesis of the problem considered in this work is to utilize a dynamic programming approach. In particular, one can employ computational tools that are used for the numerical solution of the Hamilton-Jacobi-Bellman partial differential equation of the minimum-time problem (Osher & Sethian, 1988; Sethian, 1996; Tsitsiklis, 1995; Tsai et al., 2003). A few words of caution are in order here. First of all, the dimension of the minimum-time problem considered herein is not small, given that the state vector of our system consists of a position and a velocity vector component. Consequently, a high-dimensional spatial mesh that discretizes both the position and velocity spaces is required for the numerical solution of the partial differential equation. In addition, the utilized spatial mesh should be fine enough so that the numerical solution of the Hamilton-Jacobi-Bellman equation can successfully “capture” the singularities (e.g., discontinuities) of the minimum time function. Therefore, the complexity and the computational cost of the dynamic programming approach can be excessively high. The previous discussion justifies the need for the development of more specialized numerical algorithms that better exploit any information regarding the structure of the solution to the minimum-time problem, which can be obtained by means of a careful analysis based on Pontryagin’s Minimum Principle.

The structure of the remaining paper is laid out as follows. The optimal control problem is formulated in Section 2. The analysis of the minimum-time problem and the characterization of the structure of the time-optimal control law is presented in Section 3. A parametrization of the set of candidate time-optimal control laws in terms of a parameter vector that belongs to a compact set is presented in Section 4. Section 5 describes a numerical algorithm for the computation of the minimum time-to-come function. Numerical simulations are presented in Section 6. Finally, Section 7 concludes the paper with a summary of remarks.

2. Problem Formulation

2.1 Notation

We denote by \mathbb{R}^n the set of n -dimensional real vectors. Given $\alpha \in \mathbb{R}^{n_1}$, $\beta \in \mathbb{R}^{n_2}$, we denote by $\text{col}(\alpha, \beta)$, where $\text{col}(\alpha, \beta) \in \mathbb{R}^{n_1+n_2}$, the concatenation of the column vectors α, β . The 2-norm and the maximum norm of a vector $x \in \mathbb{R}^n$ are denoted by $\|x\|$ and $\|x\|_\infty$, respectively. Given a vector $\alpha \in \mathbb{R}^n$, where $\alpha = \text{col}(\alpha_1, \dots, \alpha_n)$, we denote by $\text{sgn}(\alpha)$ the vector $\text{col}(\text{sgn}(\alpha_1), \dots, \text{sgn}(\alpha_n))$, where $\text{sgn}(\alpha_i) = \alpha_i/|\alpha_i|$, if $\alpha_i \neq 0$, and $\text{sgn}(\alpha_i) = 0$, otherwise, for $i \in \{1, \dots, n\}$. In addition, we denote by e_i the unit vector in \mathbb{R}^n whose i -th element is equal to one. The unit sphere in \mathbb{R}^n , that is, the set $\{x \in \mathbb{R}^n : \|x\| = 1\}$, will be denoted by S^{n-1} , whereas the unit cube in \mathbb{R}^n , that is, the set $\{x \in \mathbb{R}^n : \|x\|_\infty = 1\}$ will be denoted by S_∞^{n-1} . Moreover, we write $\mathbb{R}_{\geq 0}$ to denote the set of non-negative real numbers. Finally, we denote the boundary of a set \mathcal{A} by $\text{bd}(\mathcal{A})$.

2.2 The Minimum-Time Steering Problem in a Flow-Field

We consider a self-propelled particle that travels in an n -dimensional Euclidean space in the presence of a flow whose velocity field varies both spatially and temporally. Specifically, it is assumed that the velocity field of the flow $w(\cdot) : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \mapsto \mathbb{R}^n$, can be approximated satisfactorily by an

inhomogeneous time-varying linear field, that is,

$$\mathbf{w}(t, \mathbf{x}) \approx \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t),$$

where $\mathbf{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}_{\geq 0}$ denote, respectively, the spatial and temporal variables. In addition, we assume that the matrix-valued mapping $\mathbf{A}(\cdot) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^{n \times n}$ and the vector-valued mapping $\mathbf{f}(\cdot) : \mathbb{R}_{\geq 0} \mapsto \mathbb{R}^n$ are piece-wise continuous functions of time t . Alternatively, we may assume instead that $\mathbf{A}(\cdot)$ and $\mathbf{f}(\cdot)$ are locally integrable on finite intervals of $\mathbb{R}_{\geq 0}$ (Hermes & LaSalle, 1969). Of course, the employed approximation of the velocity field of the flow becomes less accurate for large values of $\|\mathbf{x}\|$.

Now let $t_0 \geq 0$. Then, we assume that the motion of the self-propelled particle is described by the following set of equations

$$\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) + \mathbf{v}, \quad \mathbf{x}(t_0) = \mathbf{x}_0, \quad (1a)$$

$$\dot{\mathbf{v}} = \mathbf{u}(t) - \mu\mathbf{v}, \quad \mathbf{v}(t_0) = \mathbf{v}_0, \quad (1b)$$

where $\mathbf{x} \in \mathbb{R}^n$ ($\mathbf{x}_0 \in \mathbb{R}^n$) and $\mathbf{v} \in \mathbb{R}^n$ ($\mathbf{v}_0 \in \mathbb{R}^n$) are, respectively, the position vector and the velocity of the particle at time t (time $t = t_0$), $\mathbf{u}(t)$ is the control input at time t and μ is a non-negative constant. It is assumed that $\mathbf{u}(\cdot) \in \mathcal{U}$, where \mathcal{U} denotes the set of piecewise continuous functions $\mathbf{g}(\cdot) : [t_0, \infty) \mapsto \mathbb{R}^n$ that attain values in the set $U := \{\boldsymbol{\nu} \in \mathbb{R}^n : \|\boldsymbol{\nu}\| \leq \bar{u}\}$, where \bar{u} is a positive constant (maximum allowable 2-norm of the rate of change of the particle's velocity relative to the flow). We shall refer to all the control inputs that belong to \mathcal{U} as *admissible*. Moreover, we shall henceforth denote by \mathbf{z} (respectively, \mathbf{z}_0) the composite state vector at time t (resp., $t = t_0$) which is a concatenation of the position and the velocity vector, that is, $\mathbf{z} := \text{col}(\mathbf{x}, \mathbf{v})$ (resp., $\mathbf{z}_0 := \text{col}(\mathbf{x}_0, \mathbf{v}_0)$). The equations of motion of the particle in terms of the state vector \mathbf{z} are given by

$$\dot{\mathbf{z}} = \mathbf{F}(t)\mathbf{z} + \mathbf{G}\mathbf{u}(t) + \mathbf{\Gamma}\mathbf{f}(t), \quad \mathbf{z}(t_0) = \mathbf{z}_0, \quad (2)$$

where

$$\mathbf{F}(t) := \begin{bmatrix} \mathbf{A}(t) & \mathbf{I}_2 \\ \mathbf{0}_2 & -\mu\mathbf{I}_2 \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{I}_2 \end{bmatrix}, \quad \mathbf{\Gamma} := \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0}_2 \end{bmatrix}.$$

Note that to each admissible control input $\mathbf{u}(\cdot) \in \mathcal{U}$ corresponds an admissible trajectory $\mathbf{z}(\cdot; \mathbf{z}_0, t_0, \mathbf{u}(\cdot))$, where $\mathbf{z}(\cdot; \mathbf{z}_0, t_0, \mathbf{u}(\cdot)) = \text{col}(\mathbf{x}(\cdot; \mathbf{z}_0, t_0, \mathbf{u}(\cdot)), \mathbf{v}(\cdot; \mathbf{z}_0, t_0, \mathbf{u}(\cdot)))$, which is the corresponding solution of (2) in the sense of Carathéodory. It should also be highlighted that the role of the constant μ is to enforce indirectly a constraint on the maximum speed of the particle relative to the flow.

Proposition 1: *Let $\mu > 0$ and suppose that $\|\mathbf{v}_0\| \leq \bar{u}/\mu$. Then*

$$\|\mathbf{v}(t)\| \leq \bar{u}/\mu, \quad \text{for all } t \geq t_0. \quad (3)$$

Proof. By integrating (1b), it follows that

$$\mathbf{v}(t) = \exp(-\mu(t - t_0))\mathbf{v}_0 + \exp(-\mu t) \int_{t_0}^t \exp(\mu\sigma)\mathbf{u}(\sigma)d\sigma.$$

Therefore, by the triangle inequality together with the fact that $\|\mathbf{u}(t)\| \leq \bar{u}$ for all $t \geq t_0$, we have

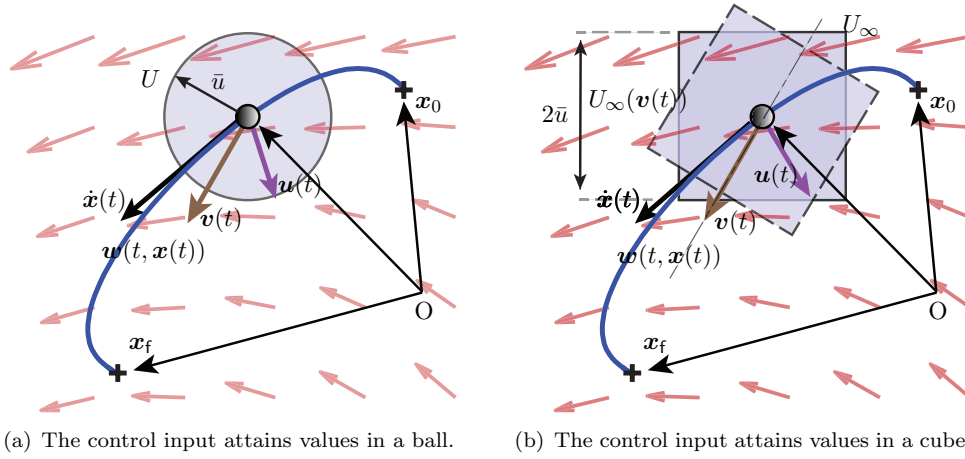


Figure 1. The minimum time steering problem in a spatiotemporal flow field in the presence of constraints on the norm of the input. If the 2-norm is employed, the input value set is always symmetric with respect to the velocity relative to the flow, v . This is not the case when one considers instead the maximum-norm (in which case the input value set is denoted by U_∞) unless the maximum norm constraint is taken to depend on the particle's velocity such that the corresponding input value set, which is denoted by $U_\infty(v)$, enjoys the desired symmetry property.

that

$$\begin{aligned} \|v(t)\| &\leq \|\exp(-\mu(t-t_0))v_0\| + \|\exp(-\mu t) \int_{t_0}^t \exp(\mu\sigma)u(\sigma)d\sigma\| \\ &\leq \exp(-\mu(t-t_0))\|v_0\| + \exp(-\mu t) \int_{t_0}^t \exp(\mu\sigma)\|u(\sigma)\|d\sigma \\ &\leq \bar{u}/\mu + \exp(-\mu(t-t_0))(\|v_0\| - \bar{u}/\mu). \end{aligned}$$

By hypothesis, the last term in the right hand side of the last inequality is non-positive. The result follows readily. \square

We wish to highlight at this point the rationale of our choice to consider a 2-norm input constraint rather than a maximum norm constraint. Specifically, our choice has to do with the fact that, in the case of the 2-norm input constraint, the admissible input value set U is always symmetric with respect to the particle's velocity, v , relative to the flow as one can observe in Fig. 1(a). This symmetry practically means that the constraints on the ability of the particle to control the tangential and the centripetal components of the rate of change of v are enforced consistently along its ensuing trajectory. This symmetry property is not enjoyed by the input value set in the presence of a maximum norm input constraint, which is denoted by U_∞ , unless we assume, in addition, that the input value set is time-varying, or more precisely, state-dependent. In particular, the maximum norm constraint should be enforced such that the input value set is symmetric with respect to v , in which case we denote it by $U_\infty(v)$. The situation is illustrated in Fig. 1(b). The corresponding optimal control problem with a state-dependent input value set is significantly more complicated, thus justifying the choice of a 2-norm input constraint.

Next, we state the minimum-time problem that we will study in this work.

Problem 1: Let $z_0 := \text{col}(x_0, v_0) \in \mathbb{R}^{2n}$ and let $x_f \in \mathbb{R}^n$ be given. Let us also consider the *terminal* manifold \mathcal{T}_f , where $\mathcal{T}_f := \{z_f = \text{col}(x_f, v_f) \in \mathbb{R}^{2n}, v_f \in \mathbb{R}^n\}$. Then, find a control input $u_\star(\cdot) \in \mathcal{U}$ and the minimum final time t_f^\star that will transfer the system described by Eq. (1a)-(1b) from the prescribed initial state z_0 to a state $z_f \in \mathcal{T}_f$ at time $t = t_f^\star$.

The existence of (optimal) solutions to Problem 1 follows readily from Theorem 13.1. (Hermes

& LaSalle, 1969).

Proposition 2: Let $\mathbf{z}_f := \text{col}(\mathbf{x}_f, \mathbf{v}_f) \in \mathbb{R}^{2n}$ be given and let $\mathbf{w}(t, \mathbf{x}) = \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t)$, where $\mathbf{A}(\cdot)$ and $\mathbf{f}(\cdot)$ are piecewise continuous functions of time t . If there exists an admissible trajectory that takes the particle, whose motion is described by (1a)-(1b), from \mathbf{z}_0 , at $t = t_0$, to \mathcal{T}_f , at some finite time $t = \tau$, then Problem 1 admits a solution, and vice versa.

Remark 1: Proposition 2 implies that the existence of an admissible trajectory that satisfies the boundary conditions of Problem 1 only, suffices for one to conclude that Problem 1 admits a (time-optimal) solution, and vice versa. The problem of existence of feasible solutions for arbitrary boundary conditions, also known as the problem of *complete controllability*, for time-varying systems with linear/affine dynamics in the presence of input constraints has received some attention in the literature. The reader may refer to (Conti, 1965).

3. Analysis of the Optimal Control Problem

Next, we employ a general formulation of Pontryagin's Minimum Principle (Pontryagin et al., 1962) in order to characterize the structure of the time-optimal control $\mathbf{u}_\star(\cdot)$ that solves the minimum-time Problem 1. In particular, let $\mathbf{z}_\star(\cdot) : [t_0, t_f^\star] \mapsto \mathbb{R}^{2n}$, where $\mathbf{z}_\star(\cdot) := \text{col}(\mathbf{x}_\star(\cdot), \mathbf{v}_\star(\cdot))$, denote the minimum-time trajectory generated with the application of the time-optimal control input $\mathbf{u}_\star(\cdot)$, for $t \in [t_0, t_f^\star]$. Then, there exists a scalar $p_0^\star \in \{0, 1\}$ and an absolutely continuous function $\mathbf{p}_\mathbf{z}^\star(\cdot) : [t_0, t_f^\star] \mapsto \mathbb{R}^{2n}$, known as the costate, where $\mathbf{p}_\mathbf{z}^\star(\cdot) := \text{col}(\mathbf{p}_\mathbf{x}^\star(\cdot), \mathbf{p}_\mathbf{v}^\star(\cdot))$, such that

- (i) $\|\mathbf{p}_\mathbf{x}^\star(t)\| + \|\mathbf{p}_\mathbf{v}^\star(t)\| + |p_0^\star| \neq 0$, for all $t \in [t_0, t_f^\star]$,
- (ii) For all $t \in [t_0, t_f^\star]$, the costate vectors $\mathbf{p}_\mathbf{x}^\star$ and $\mathbf{p}_\mathbf{v}^\star$ satisfy the following (canonical) differential equations

$$\dot{\mathbf{p}}_\mathbf{x}^\star = -\frac{\partial \mathcal{H}(t, \mathbf{z}_\star, \mathbf{p}_\mathbf{z}^\star, \mathbf{u}_\star, p_0^\star)}{\partial \mathbf{x}} = -\mathbf{A}^T(t)\mathbf{p}_\mathbf{x}^\star, \quad \mathbf{p}_\mathbf{x}^\star(t_f^\star) \in \mathbb{R}^n, \quad (4a)$$

$$\dot{\mathbf{p}}_\mathbf{v}^\star = -\frac{\partial \mathcal{H}(t, \mathbf{z}_\star, \mathbf{p}_\mathbf{z}^\star, \mathbf{u}_\star, p_0^\star)}{\partial \mathbf{v}} = -\mathbf{p}_\mathbf{x}^\star + \mu \mathbf{p}_\mathbf{v}^\star, \quad \mathbf{p}_\mathbf{v}^\star(t_f^\star) = 0, \quad (4b)$$

where $\mathcal{H}(\cdot) : [t_0, \infty) \times \mathbb{R}^{2n} \times \mathbb{R}^{2n} \times U \times \mathbb{R} \mapsto \mathbb{R}$ with

$$\mathcal{H}(t, \mathbf{z}, \mathbf{p}_\mathbf{z}, \mathbf{u}, p_0) := \langle \mathbf{p}_\mathbf{x}, \mathbf{A}(t)\mathbf{x} + \mathbf{f}(t) + \mathbf{v} \rangle + \langle \mathbf{p}_\mathbf{v}, \mathbf{u} - \mu \mathbf{v} \rangle + p_0$$

is the variational Hamiltonian of the optimal control problem.

- (iii) The Hamiltonian \mathcal{H} satisfies the following transversality condition:

$$\min_{\boldsymbol{\nu} \in U} \mathcal{H}(t_f^\star, \mathbf{z}_\star(t_f^\star), \mathbf{p}_\mathbf{z}^\star(t_f^\star), \boldsymbol{\nu}, p_0^\star) = 0. \quad (5)$$

- (iv) Furthermore, the time-optimal control $\mathbf{u}_\star(\cdot)$ necessarily minimizes the Hamiltonian evaluated along the optimal state and costate trajectories $\mathbf{z}_\star(\cdot)$ and $\mathbf{p}_\mathbf{z}^\star(\cdot)$, respectively, that is,

$$\mathbf{u}_\star(t) = \arg \min_{\boldsymbol{\nu} \in U} \mathcal{H}(t, \mathbf{z}_\star(t), \mathbf{p}_\mathbf{z}^\star(t), \boldsymbol{\nu}, p_0^\star), \quad \text{for all } t \in [t_0, t_f^\star], \quad (6)$$

which implies

$$\mathbf{u}_\star(t) = \arg \min_{\boldsymbol{\nu} \in U} \langle \mathbf{p}_\mathbf{v}^\star, \boldsymbol{\nu} \rangle, \quad \text{for all } t \in [t_0, t_f^\star]. \quad (7)$$

It is easy to show that Eq. (7) implies that the candidate time-optimal control satisfies the following equation:

$$\mathbf{u}_\star(t) = \begin{cases} -\bar{u} \frac{\mathbf{p}_v^\star(t)}{\|\mathbf{p}_v^\star(t)\|}, & \text{if } \mathbf{p}_v^\star(t) \neq 0, \\ \boldsymbol{\nu} \in U, & \text{otherwise.} \end{cases}$$

To refine the characterization of the candidate time-optimal control, we need to obtain more information regarding the structure of the costate $\mathbf{p}_v^\star(\cdot)$. To this aim, we will need the following lemma, whose proof can be found, for example in (Brockett, 1970, p. 44).

Lemma 1: *Let $t \geq t_0$ and let $\Phi(t, t_0)$ denote the state transition matrix of the homogeneous linear system $\dot{\mathbf{x}} = \mathbf{A}(t)\mathbf{x}$, that is, the vector $\mathbf{x}(t) = \Phi(t, t_0)\mathbf{x}_0$ solves the initial value problem described by (1a)-(1b). Let also $\Phi_{\mathcal{A}}(t, t_0)$ denote the state transition matrix of the adjoint system, which is described, in turn, by the following equation: $\dot{\boldsymbol{\chi}} = -\mathbf{A}^T(t)\boldsymbol{\chi}$, that is, $\boldsymbol{\chi}(t) = \Phi_{\mathcal{A}}(t, t_0)\boldsymbol{\chi}(t_0)$. Then,*

$$\Phi_{\mathcal{A}}(t, t_0) = \Phi^{-T}(t, t_0) = \Phi^T(t_0, t), \quad (8)$$

for all $t \geq t_0$.

Next, we integrate Equations (4a)-(4b) from time t_0 to time t to obtain

$$\mathbf{p}_x^\star(t) = \Phi_{\mathcal{A}}(t, t_0)\mathbf{p}_x^\star(t_0), \quad (9a)$$

$$\mathbf{p}_v^\star(t) = \exp(\mu(t - t_0))\mathbf{p}_v^\star(t_0) + \Psi(t, t_0)\mathbf{p}_x^\star(t_0), \quad (9b)$$

where $\Psi(t, t_0) := -\int_{t_0}^t \exp(\mu(t - \sigma))\Phi_{\mathcal{A}}(\sigma, t_0)d\sigma = -\int_{t_0}^t \exp(\mu(t - \sigma))\Phi^{-T}(\sigma, t_0)d\sigma$, in light of Lemma 1. In addition, the boundary condition $\mathbf{p}_v^\star(t_f^\star) = 0$ implies that

$$\mathbf{p}_v^\star(t_0) = -\exp(\mu(t_0 - t_f^\star))\Psi(t_f^\star, t_0)\mathbf{p}_x^\star(t_0). \quad (10)$$

Therefore,

$$\mathbf{p}_v^\star(t) = (\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0). \quad (11)$$

We can then express the candidate time-optimal control input as follows:

$$\mathbf{u}_\star(t) = \begin{cases} -\bar{u} \frac{(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)}{\|(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)\|}, & \text{if } \mathbf{p}_v^\star(t) \neq 0, \\ \boldsymbol{\nu} \in U, & \text{otherwise.} \end{cases}$$

An important observation here is that in the light of the so-called *bang-bang* principle (Theorem 13.2 (Hermes & LaSalle, 1969)), there is always a time-optimal control that solves Problem 1 and takes its values on the boundary of the set U , $\text{bd}(U)$, exclusively. Therefore, a time-optimal control of Problem 1 satisfies the equation

$$\mathbf{u}_\star(t) = \begin{cases} -\bar{u} \frac{(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)}{\|(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)\|}, & \text{if } \mathbf{p}_v^\star(t) \neq 0, \\ \boldsymbol{\nu} \in \text{bd}(U), & \text{otherwise,} \end{cases}$$

where $\text{bd}(U)$ denotes the boundary of the set U , that is, $\text{bd}(U) := \{\boldsymbol{\nu} \in \mathbb{R}^n : \|\boldsymbol{\nu}\| = \bar{u}\}$.

It is interesting to highlight here the structure of the time-optimal control in the case when the set of admissible control laws consisted of piecewise continuous functions that attain values in the set $U_\infty := \bar{u} \mathbb{S}_\infty^{n-1}$ instead. Specifically, in this case, a time-optimal control of Problem 1 would be given by $\mathbf{u}_\star(\cdot) = \text{col}(u_\star^1(\cdot), \dots, u_\star^n(\cdot))$, where

$$u_\star^i(t) = \begin{cases} -\bar{u} \text{sgn}[\langle \mathbf{e}_i, (\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0) \rangle], & \text{if } \langle \mathbf{e}_i, \mathbf{p}_v^\star(t) \rangle \neq 0, \\ \nu \in [-\bar{u}, \bar{u}], & \text{otherwise,} \end{cases} \quad (12)$$

for $i \in \{1, \dots, n\}$. In the time invariant case, one can actually go one step further and check whether the system enjoys the so-called *normality* property, that is, whether the system is controllable with respect to each of its input components independently (see Definition 4.15 (Athans & Falb, 2007)). If the system was normal, then the first equation in (12) would unambiguously determine the time-optimal control law (except perhaps on a set of time instants of measure zero), which is actually the unique time-optimal control. It is easy to show that even when $\mathbf{A}(\cdot) \equiv 0$ and $\mathbf{f}(\cdot) \equiv 0$, the system described by Eqs. (1a)-(1b) does not enjoy this property (obviously, for example, the first component of \mathbf{v} cannot be affected by the second component of the input \mathbf{u}). Interestingly, as we show next, a time-optimal control law in our problem, where the input value set is a hyperball, is practically determined by the following equation:

$$\mathbf{u}_\star(t) = -\bar{u} \frac{\mathbf{p}_v^\star(t)}{\|\mathbf{p}_v^\star(t)\|} = -\bar{u} \frac{(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)}{\|(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)\|},$$

for all times $t \in [t_0, t_f^\star]$ except from the time instants that belong to the set $\mathfrak{T}_s := \{\tau \geq t_0 : \mathbf{p}_v^\star(\tau) = 0\}$. As is shown next, however, \mathfrak{T}_s turns out to be either an empty set or a set of measure zero and consequently will not play any practical role in our analysis.

Proposition 3: *For any τ_0 and τ_1 in $]t_0, t_f^\star[$ with $\tau_0 < \tau_1$, we have that $\mathbf{p}_v^\star(t) \neq 0$, for all $t \in]\tau_0, \tau_1[$.*

Proof. Suppose on the contrary that there exist τ_0 and τ_1 in $]t_0, t_f^\star[$ with $\tau_0 < \tau_1$ such that $\mathbf{p}_v^\star(t) = 0$, for all $t \in]\tau_0, \tau_1[$. Then, we also have that $\mathbf{p}_v^\star(t) = 0$, for all $t \in]\tau_0, \tau_1[$, which implies, in view of Eq. (4b) and Eq. (9a) that $\mathbf{p}_x^\star(t) = \Phi_{\mathcal{A}}(t, t_0)\mathbf{p}_x^\star(t_0) = 0$ for all $t \in]\tau_0, \tau_1[$. Therefore, $\mathbf{p}_x^\star(t_0) = 0$, which implies, in light of Eq. (9a), that $\mathbf{p}_x^\star(\cdot) \equiv 0$ in $[t_0, t_f^\star]$. In addition, in view of Eq. (9b) for $t \in]\tau_0, \tau_1[$ together with the facts that $\mathbf{p}_x^\star(t_0) = 0$ and $\mathbf{p}_v^\star(t) = 0$ for $t \in]\tau_0, \tau_1[$, we take $\mathbf{p}_v^\star(t_0) = 0$. Consequently, Eq. (9b) gives $\mathbf{p}_v^\star(\cdot) \equiv 0$ in $[t_0, t_f^\star]$. Then, the transversality condition (5) yields $p_0^\star = 0$. Therefore,

$$\|\mathbf{p}_x^\star(t)\| + \|\mathbf{p}_v^\star(t)\| + |p_0^\star| = 0, \quad \text{for any } t \in [t_0, t_f^\star],$$

which contradicts Pontryagin's Minimum Principle (condition (i)). \square

Remark 2: Proposition 3 implies that no *singular arcs* appear in the solution to Problem 1. Consequently, the PMP practically allows one to unambiguously determine a time-optimal control law that solves the minimum-time problem considered herein.

Proposition 4: *Let $\mathbf{z}_0 \in \mathbb{R}^{2n}$ and let $\mathbf{x}_f \in \mathbb{R}^n$ be given and suppose that Problem 1 admits a solution for this particular set of boundary conditions. Then, the time-optimal control law satisfies the following equation:*

$$\mathbf{u}_\star(t) = -\bar{u} \frac{\mathbf{p}_v^\star(t)}{\|\mathbf{p}_v^\star(t)\|} = -\bar{u} \frac{(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)}{\|(\Psi(t, t_0) - \exp(\mu(t - t_f^\star))\Psi(t_f^\star, t_0))\mathbf{p}_x^\star(t_0)\|}, \quad (13)$$

for all $t \in [t_0, t_f^\star] \setminus \mathfrak{T}_s$, where $\mathfrak{T}_s := \{\tau \in [t_0, t_f^\star] : \mathbf{p}_v^\star(\tau) = 0\}$. In addition, the set \mathfrak{T}_s is either empty or has measure zero and consequently, in the latter case, the time-optimal control $\mathbf{u}_\star(t)$ can attain

arbitrary values in U for all $t \in \mathfrak{T}_s$.

Proof. It follows directly from Proposition 3. \square

Remark 3: The set \mathfrak{T}_s will play no practical role in the solution to our problem because it is either an empty set or a set of measure zero. Thus, neither the evolution of the dynamical system nor the performance of the optimal control problem can be affected by this set. Henceforth, we will write “for all $t \in [t_0, t_f^*]$ ” in lieu of “for all $t \in [t_0, t_f^*] \setminus \mathfrak{T}_s$ ” with a slight abuse of notation.

Remark 4: It is interesting to note that the time-optimal control $\mathbf{u}_*(\cdot)$ is a continuous function of time, a desirable feature for practical applications. By contrast, the time-optimal control law given in Eq. (12), which solves the minimum-time problem when the input value set is the set U_∞ is, in general, a piecewise constant (and hence discontinuous) function of time (Athans & Falb, 2007).

Remark 5: Because the characterization of the time-optimal control $\mathbf{u}_*(\cdot)$ depends essentially on the knowledge of the quantity $\mathbf{p}_x^*(t_0)$, we will often write $\mathbf{u}_*(\cdot; \mathbf{p}_x^*(t_0))$ to explicitly denote this dependence. Thus, the vector $\mathbf{p}_x^*(t_0)$ parameterizes the set of the candidate time-optimal control laws; however, this parametrization has little practical value given that $\mathbf{p}_x^*(t_0)$ can be, in principle, any n -dimensional vector. Later on, we will show that it is possible to parameterize the set of the candidate time-optimal control laws of our problem by means of a vector that belongs to a compact set.

4. The Navigation Formula of the Minimum-Time Problem

In this section, we present a particular parametrization of the set of candidate time-optimal control laws of Problem 1 that will allow us to characterize the time-optimal synthesis of Problem 1. To this aim, let us consider the following vector quantities:

$$\boldsymbol{\pi}(t) := \mathbf{p}_v^*(t) / \|\mathbf{p}_x^*(t)\|, \quad \mathbf{a}(t) := \mathbf{p}_x^*(t) / \|\mathbf{p}_x^*(t)\|. \quad (14)$$

Note that $\mathbf{p}_x^*(t) \neq 0$, for all $t \in [t_0, t_f^*]$, as we have shown in the proof of Proposition 3 and thus all of the new quantities in (14) are well defined. The time-optimal control is given by

$$\mathbf{u}_*(t) = -\bar{u} \frac{\boldsymbol{\pi}(t)}{\|\boldsymbol{\pi}(t)\|}, \quad \text{for all } t \in [t_0, t_f^*]. \quad (15)$$

Now, it is easy to show that $\boldsymbol{\pi}(t)$ satisfies the following differential equation

$$\dot{\boldsymbol{\pi}}(t) = \frac{\dot{\mathbf{p}}_v^*(t)}{\|\mathbf{p}_x^*(t)\|} - \frac{\langle \mathbf{p}_x^*(t), \dot{\mathbf{p}}_x^*(t) \rangle \mathbf{p}_v^*(t)}{\|\mathbf{p}_x^*(t)\|^3}, \quad \boldsymbol{\pi}(t_0) = \frac{\mathbf{p}_v^*(t_0)}{\|\mathbf{p}_x^*(t_0)\|}, \quad (16a)$$

$$\dot{\mathbf{a}}(t) = \frac{\dot{\mathbf{p}}_x^*(t)}{\|\mathbf{p}_x^*(t)\|} - \frac{\langle \mathbf{p}_x^*(t), \dot{\mathbf{p}}_x^*(t) \rangle \mathbf{p}_x^*(t)}{\|\mathbf{p}_x^*(t)\|^3}, \quad \mathbf{a}(t_0) = \frac{\mathbf{p}_x^*(t_0)}{\|\mathbf{p}_x^*(t_0)\|}, \quad (16b)$$

where the identity $\frac{d}{dt} \|\boldsymbol{\gamma}(t)\| = \langle \boldsymbol{\gamma}, \dot{\boldsymbol{\gamma}} \rangle / \|\boldsymbol{\gamma}\|$, when $\boldsymbol{\gamma} \neq 0$, has been employed. Now, let $\mathbf{a}_0 := \mathbf{a}(t_0)$. Then, in light of (10), we have $\boldsymbol{\pi}(t_0) = \boldsymbol{\pi}_0(\mathbf{a}_0; t_0, t_f^*)$, where

$$\boldsymbol{\pi}_0(\mathbf{a}_0; t_0, t_f^*) := -\exp(\mu(t_0 - t_f^*)) \boldsymbol{\Psi}(t_f^*, t_0) \mathbf{a}_0. \quad (17)$$

Then, in view of Eqs. (4a)-(4b) and (14), Eq. (16a) can be written as follows:

$$\dot{\boldsymbol{\pi}}(t) = -\mathbf{a}(t) + (\mu + \langle \mathbf{a}(t), \mathbf{A}^T(t)\mathbf{a}(t) \rangle)\boldsymbol{\pi}(t), \quad \boldsymbol{\pi}(t_0) = \boldsymbol{\pi}_0(\mathbf{a}_0; t_0, t_f^*), \quad (18a)$$

$$\dot{\mathbf{a}}(t) = -\mathbf{A}^T(t)\mathbf{a}(t) + \langle \mathbf{a}(t), \mathbf{A}^T(t)\mathbf{a}(t) \rangle \mathbf{a}(t), \quad \mathbf{a}(t_0) = \mathbf{a}_0. \quad (18b)$$

Note that a particular vector $\mathbf{a}_0 \in \mathbb{S}^{n-1}$ determines unambiguously a candidate time-optimal control law that attains its values on the boundary of the set U , $\text{bd}(U)$. We will henceforth write $\mathbf{u}_\star(\cdot; \mathbf{a}_0)$ to denote the direct dependence of the (candidate) time-optimal control $\mathbf{u}_\star(\cdot)$ on \mathbf{a}_0 . We say that the vector \mathbf{a}_0 parameterizes the set of the candidate time-optimal control laws; we denote this set by \mathcal{U}^* , where $\mathcal{U}^* := \{\mathbf{u}_\star(\cdot; \mathbf{a}_0) \in \mathcal{U}, \mathbf{a}_0 \in \mathbb{S}^{n-1}\}$. In contradistinction with the parametrization of the time-optimal control laws in terms of the vector $\mathbf{p}_x^*(t_0)$, which can be any n -dimensional vector, the parametrization $\mathbf{u}_\star(\cdot; \mathbf{a}_0)$ is significantly more practical given that \mathbf{a}_0 belongs to a compact set, namely \mathbb{S}^{n-1} . For example, by discretizing \mathbb{S}^{n-1} into a finite mesh, a finite approximation of the set of candidate time-optimal control laws \mathcal{U}^* is readily obtained. This idea will be explored in the subsequent section, where a numerical scheme for the computation of the optimal synthesis of the minimum time problem (Problem 1) is presented.

The reader familiar with the solution of the ZNP should have probably noticed that Eqs. (18a)-(18b) resemble the so-called *navigation formula* of the ZNP. In the ZNP, the time-optimal control depends on an angle θ that satisfies a differential equation, which is known as the “navigation formula.” Thus, knowledge of the “correct” initial value of the angle θ suffices to completely characterize the time-optimal control input and the corresponding minimum-time trajectory for the ZNP. In our case, the parameter that determines the solution to the minimum-time problem is the initial value of the unit vector \mathbf{a}_0 .

It is important to also note that the parametrization of the set of candidate time-optimal control laws induces naturally a parametrization of the set of candidate minimum-time trajectories of Problem 1. In particular, to each vector \mathbf{a}_0 , we associate a candidate minimum-time trajectory $\mathbf{z}_\star(\cdot; \mathbf{z}_0, t_0, \mathbf{a}_0)$, which is simply the trajectory generated with the application of the candidate time-optimal control law $\mathbf{u}_\star(\cdot; \mathbf{a}_0)$; we also write $\mathbf{z}_\star(\cdot; \mathbf{z}_0, t_0, \mathbf{a}_0) = \text{col}(\mathbf{x}_\star(\cdot; \mathbf{z}_0, t_0, \mathbf{a}_0), \mathbf{v}_\star(\cdot; \mathbf{z}_0, t_0, \mathbf{a}_0))$. Moreover, given a terminal position vector \mathbf{x}_f , there exists a unit vector $\mathbf{a}_0^* \in \mathbb{S}^{n-1}$ such that the control law $\mathbf{u}_\star(\cdot; \mathbf{a}_0^*)$ is actually a time-optimal control law that solves Problem 1, $\mathbf{u}_\star(\cdot; \mathbf{a}_0^*) \equiv \mathbf{u}_\star(\cdot)$, and the corresponding trajectory $\mathbf{z}_\star(\cdot; \mathbf{z}_0, t_0, \mathbf{a}_0)$ is a minimum-time trajectory, $\mathbf{z}_\star(\cdot; \mathbf{z}_0, t_0, \mathbf{a}_0^*) \equiv \mathbf{z}_\star(\cdot; \mathbf{z}_0, t_0)$ that brings the particle to \mathbf{x}_f in minimum time t_f^* . In this way, we obtain a family of candidate minimum-time trajectories parameterized by the unit vector \mathbf{a}_0 . To this parametrization, we associate a set-valued map $\mathcal{F}_x(\cdot; \mathbf{z}_0, t_0, \mathbb{S}^{n-1}) : [t_0, \infty[\rightrightarrows \mathbb{R}^n$, where

$$\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1}) := \{\mathbf{x}_f \in \mathbb{R}^n : \mathbf{x}_f = \mathbf{x}_\star(t; \mathbf{z}_0, t_0, \mathbf{a}_0), \mathbf{a}_0 \in \mathbb{S}^{n-1}\}.$$

Note that the set $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$ consists of the endpoints of candidate minimum-time trajectories of the system (1a)-(1b) generated with the application of the candidate time-optimal control law $\mathbf{u}(\cdot) = \mathbf{u}_\star(\cdot; \mathbf{a}_0)$ in the time interval $[t_0, t]$. We shall refer to $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$ as the *t-extremal front* of the minimum-time problem. It should be stressed here that the *t-extremal front* $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$ does not coincide with the *t-level set* of the minimum time-to-come function, which is denoted by $\mathcal{L}_x^*(t; \mathbf{z}_0, t_0)$. The *t-level set* is defined as the image of the set-valued map $\mathcal{L}_x^*(\cdot; \mathbf{z}_0, t_0) : [t_0, \infty[\rightrightarrows \mathbb{R}^n$, where

$$\mathcal{L}_x^*(t; \mathbf{z}_0, t_0) := \{\mathbf{x}_f \in \mathbb{R}^n : t_f^*(\mathbf{x}_f; \mathbf{z}_0, t_0) = t\}.$$

Therefore, each *t-level set* consists of endpoints of minimum-time trajectories exclusively.

Proposition 5: *The t-level set of the minimum time-to-come function $t_f^*(\cdot; \mathbf{z}_0, t_0)$ is a subset of the t-extremal front, that is, $\mathcal{L}_x^*(t; \mathbf{z}_0, t_0) \subseteq \mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$, for all $t \geq t_0$.*

Proof. Suppose to the contrary that there exists a point $\mathbf{y} \in \mathbb{R}^n$ such that $\mathbf{y} \in \mathcal{L}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0)$ but $\mathbf{y} \notin \mathcal{F}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$. Then, there exists a vector $\mathbf{a}_0^* \in \mathbb{S}^{n-1}$, where $\mathbf{a}_0^* = \mathbf{a}_0^*(\mathbf{y})$ such that $\mathbf{y} = \mathbf{x}_*(t; \mathbf{z}_0, t_0, \mathbf{u}_*(\cdot))$, where $\mathbf{u}_*(\cdot) = \mathbf{u}_*(\cdot; \mathbf{a}_0^*)$. Because, by definition, $\mathbf{u}_*(\cdot) = \mathbf{u}_*(\cdot; \mathbf{a}_0^*) \in \mathcal{U}^*$, it follows that $\mathbf{y} \in \mathcal{F}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$, which is a contradiction. Therefore, every point in $\mathcal{L}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0)$ will also belong to $\mathcal{F}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$. \square

Proposition 5 implies that for a given $t \geq t_0$, there may exist a point \mathbf{y} that belong to the t -extremal front $\mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$ that can be reached before time t , that is, $t_f^*(\mathbf{y}; \mathbf{z}_0, t_0) < t$. Equivalently, it is not necessarily true that all points in $\mathcal{F}_{\mathbf{x}}(t_s; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$ are endpoints of minimum-time trajectories.

Now let $\mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0)$ denote the so-called reachable set, which is the set comprised of all the position vectors \mathbf{x}_f that can be reached by the self-propelled particle at some time $t \in [t_0, \tau]$ with the application of an admissible control input $\mathbf{u}(\cdot) \in \mathcal{U}$ in the time interval $[t_0, t]$,

$$\mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0) := \bigcup_{t \in [t_0, \tau]} \{\mathbf{x}(t; \mathbf{x}_0, t_0, \mathbf{u}(\cdot)), \mathbf{u}(\cdot) \in \mathcal{U}\}.$$

It turns out that the union of all the t -extremal fronts $\mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$, for all $t \in [t_0, \tau]$, can “fill out” $\mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0)$.

Proposition 6: *Let $\tau > t_0$. Then*

$$\mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0) = \bigcup_{t \in [t_0, \tau]} \mathcal{L}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0) = \bigcup_{t \in [t_0, \tau]} \mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1}). \quad (19)$$

Proof. The fact that $\mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0) = \bigcup_{t \in [t_0, \tau]} \mathcal{L}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0)$ follows immediately from Proposition 2 (if a point can be reached by means of an admissible input, then it can also be reached with the application of a time-optimal control input, and vice versa). On the other hand, because $\mathcal{U}^* \subseteq \mathcal{U}$, we have that $\bigcup_{t \in [t_0, \tau]} \mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1}) \subseteq \mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0)$ and in light of Proposition 5, it follows that $\bigcup_{t \in [t_0, \tau]} \mathcal{L}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0) \subseteq \bigcup_{t \in [t_0, \tau]} \mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$. Therefore, we have that

$$\bigcup_{t \in [t_0, \tau]} \mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1}) \subseteq \mathcal{R}_{\mathbf{x}}(\tau; \mathbf{z}_0, t_0) = \bigcup_{t \in [t_0, \tau]} \mathcal{L}_{\mathbf{x}}^*(t; \mathbf{z}_0, t_0) \subseteq \bigcup_{t \in [t_0, \tau]} \mathcal{F}_{\mathbf{x}}(t; \mathbf{z}_0, t_0, \mathbb{S}^{n-1})$$

and the result follows readily. \square

5. A Numerical Algorithm for the Computation of the Minimum Cost-to-Come Function

In this section, we present a simple algorithm for the computation of the optimal synthesis of the minimum-time problem (Problem 1) in the two-dimensional case (planar motion case). The three-dimensional case or the more general case when $n \geq 3$ can be handled similarly after the necessary modifications have been carried out, which we will briefly highlight at the end of this section.

Specifically, our objective is to characterize the minimum time t_f^* required for the transfer of the self-propelled particle from a prescribed initial state $\mathbf{z}_0 = \text{col}(\mathbf{x}_0, \mathbf{v}_0) \in \mathbb{R}^4$ to an arbitrary position vector $\mathbf{x}_f \in \mathbb{R}^2$ with free terminal velocity (minimum cost-to-come function). Because the function $t_f^*(\cdot; \mathbf{z}_0, t_0) : \mathcal{D} \mapsto \mathbb{R}_{\geq 0}$, where $\mathcal{D} \subset \mathbb{R}^2$ is a certain domain of interest, cannot be characterized in a closed form, in general, we propose an algorithm that computes an approximation of the graph of this function $\mathcal{G} := \{(\mathbf{x}_f, t_f^*(\mathbf{x}_f; \mathbf{z}_0, t_0)), \mathbf{x}_f \in \mathcal{D}\}$.

The proposed algorithm exploits the structure of the solution to the minimum-time problem and in particular, the parametrization of the set of candidate time-optimal control laws in terms

of the vector $\mathbf{a}_0 \in \mathbb{S}^1$, which was described previously. In particular, the algorithm “expands” the extremal fronts of the minimum-time problem for an increasing sequence of time instants while filtering out the points that belong to these sets that are not endpoints of minimum-time trajectories. Because at each time $t > t_0$, the set $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^1)$ will most likely consist of an infinite number of states, we will have to approximate it with an appropriate finite set. To this aim, we consider a finite discretization of the unit circle \mathbb{S}^1 induced by a finite partition of the interval $[0, 2\pi - \varepsilon]$, $\mathcal{P} := \{\theta^1, \dots, \theta^N\}$, where $\theta^1 = 0$ and $\theta^N = 2\pi - \varepsilon$, and $0 < \varepsilon \ll 1$. In this way, the unit circle \mathbb{S}^1 is approximated by the finite set $\mathbf{S} := \{\mathbf{a}_0(\theta) \in \mathbb{S}^1 : \mathbf{a}_0(\theta) = \text{col}(\cos \theta, \sin \theta), \theta \in \mathcal{P}\}$. Then, the set $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbf{S})$, where $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbf{S}) := \{\mathbf{x}_f \in \mathbb{R}^2 : \mathbf{x}_f = \mathbf{x}(t; \mathbf{z}_0, t_0, \mathbf{a}_0), \mathbf{a}_0 \in \mathbf{S}\}$, will constitute a finite, discrete approximation of $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbb{S}^1)$, for any $t \geq t_0$.

Let us now take the set \mathcal{D} to be the τ -reachable set $\mathcal{R}_x(\tau; \mathbf{z}_0, t_0)$ and let $\Sigma := \{\sigma_1, \dots, \sigma_m\}$, where $\sigma_1 = t_0$ and $\sigma_m = \tau$, be a finite partition of the interval $[t_0, \tau]$. Our first objective is to approximately “fill out” the set $\mathcal{R}_x(\tau; \mathbf{z}_0, t_0)$ with a finite collection of t -extremal fronts, where $t \in \Sigma$. When we say “fill out,” we mean that for an arbitrary point $\mathbf{x}_f \in \mathcal{R}_x(\tau; \mathbf{z}_0, t_0)$, there is $\sigma_k \in \Sigma$ such that \mathbf{x}_f is sufficiently close to the σ_k -extremal front $\mathcal{F}_x(\sigma_k; \mathbf{z}_0, t_0, \mathbf{S})$, in terms of the Hausdorff metric in \mathbb{R}^2 . In this case, the time t can be assigned to the point \mathbf{x}_f and it will be the candidate value for the minimum time of arrival from \mathbf{z}_0 to \mathbf{x}_f . However, as we have already mentioned, the fact that $\mathbf{x}_f \in \mathcal{F}_x(\sigma_k; \mathbf{z}_0, t_0, \mathbf{S})$ does not necessarily imply that $t_f^*(\mathbf{x}_f; \mathbf{z}_0, t_0) = \sigma_k$ (see Proposition 5). This is because it is possible that there may exist another time instant $\sigma_\ell < \sigma_k$, where $\ell < k$, such that $\mathbf{x}_f \in \mathcal{F}_x(\sigma_\ell; \mathbf{z}_0, t_0, \mathbf{S})$; obviously, in this case $t_f^*(\mathbf{x}_f; \mathbf{z}_0, t_0) \leq \sigma_\ell < \sigma_k$. However, if a point \mathbf{x}_f belongs to $\mathcal{F}_x(\sigma_k; \mathbf{z}_0, t_0, \mathbf{S})$ but does not belong to the front $\mathcal{F}_x(\sigma_\ell; \mathbf{z}_0, t_0, \mathbf{S})$, for any $\ell < k$, then one can conclude that $\sigma_k \approx t_f^*(\mathbf{x}_f; \mathbf{z}_0, t_0)$, or more precisely, $\sigma_{k-1} \leq t_f^*(\mathbf{x}_f; \mathbf{z}_0, t_0) \leq \sigma_k$.

Based on the previous discussion, we propose a numerical algorithm that computes an approximation of \mathcal{G} by “expanding” the extremal fronts $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbf{S})$ in m steps, by taking $t = \sigma_k$, $\sigma_k \in \Sigma$, at the k -th step. Moreover, at the k -th step during the expansion of the extremal fronts, we filter out points that do not correspond to endpoints of minimum-time trajectories from the set $\mathcal{F}_x(\sigma_k; \mathbf{z}_0, t_0, \mathbf{S})$. In this way, only the position vectors \mathbf{x}_f that are reached during the time interval $[\sigma_{k-1}, \sigma_k]$ for the first time are marked as *visited*. For the implementation of the previous idea, we utilize a spatial mesh, call it \mathcal{M} , that discretizes a compact set that contains the set $\mathcal{R}_x(\tau; \mathbf{z}_0, t_0)$, such that at each step we record the nodes of the mesh which are the closest to the points visited for the first time by the corresponding extremal front. Of course, the points of a t -extremal front $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbf{S})$ will not exactly coincide with the nodes of the mesh; at these points, one may compute an approximation of the minimum-time function by using, for example, a simple bilinear interpolation scheme.

Next, we provide the main steps of the baseline numerical algorithm:

- (1) Set computation parameters, namely, the number of steps m , the upper bound on the final time τ , the partition \mathcal{P} , and the mesh \mathcal{M} . In addition, set $k = 1$.
- (2) Compute the t -extremal front $\mathcal{F}_x(t; \mathbf{z}_0, t_0, \mathbf{S})$ for $t = \sigma_k$.
- (3) Assign the time value σ_k to all the nodes of the mesh \mathcal{M} that belong to the region enclosed by $\mathcal{F}_x(\sigma_k; \mathbf{z}_0, t_0, \mathbf{S})$ and have not been visited at a previous step, if $k > 1$. Subsequently, mark these nodes as *visited*.
- (4) Set $k \leftarrow k + 1$. Repeat steps (2)–(3) until either $k > m$ or all the nodes of the mesh \mathcal{M} have been marked as *visited*.

Remark 6: As we have already mentioned, the output of the algorithm is an approximation of the graph \mathcal{G} of the minimum time-to-come function.

Remark 7: The extension of the previous numerical approach to the case $n \geq 3$, requires a finite discretization of the unit sphere \mathbb{S}^{n-1} . This is easily accomplished by employing spherical coordinates, for $n = 3$, and the so-called hyperspherical coordinates, for $n > 3$, to parameterize a unit vector $\mathbf{a}_0 \in \mathbb{S}^{n-1}$. In this way, we express the coordinates of \mathbf{a}_0 as the product of appropriate trigonometric functions whose arguments belong to the compact interval $[0, 2\pi - \varepsilon]$, for some

$$0 < \varepsilon \ll 1.$$

6. Numerical Simulations

In this section, we present numerical simulations to illustrate the previous theoretical developments. In particular, we consider the motion of a self-propelled particle in the two-dimensional Euclidean plane ($n = 2$), for three different flow fields. For each field, we consider the cases when $\mu = 0$ and $\mu = 1$ for $\bar{u} = 1$. Figures 2 and 3 illustrate the level sets $\mathcal{L}_x^*(\sigma_k; \mathbf{z}_0, t_0)$ of the minimum time function $t_f^*(\cdot; \mathbf{z}_0, t_0)$, for $k \in \{1, \dots, 40\}$, $\sigma_1 = 0$ and $\sigma_{40} = 6$. For our simulations, we have used

the following data: *i*) $\mathbf{A}(t) = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}$ (constant matrix), $\mathbf{v}_0 = \text{col}(0, 1)$, $\mathbf{f}(\cdot) \equiv 0$ (Figs. 2(a), 3(a)),

ii) $\mathbf{A}(t) = \begin{bmatrix} \cos t & \sin t \\ -\sin t & \cos t \end{bmatrix}$ (periodic matrix), $\mathbf{v}_0 = \text{col}(-1, 0)$, $\mathbf{f}(\cdot) \equiv 0$ (Figs. 2(b), 3(b)), and *iii*)

$\mathbf{A}(t) = \begin{bmatrix} 0 & 0.4t \\ -0.4t & 0 \end{bmatrix}$ (non-periodic matrix), $\mathbf{v}_0 = \text{col}(1, -1)$, $\mathbf{f}(\cdot) \equiv 0$ (Figs. 2(c), 3(c)).

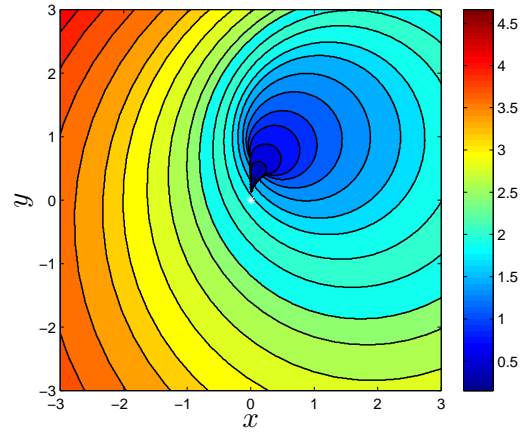
One important observation is that the minimum time-to-come function undergoes discontinuous jumps in all the three scenarios we consider. We also observe that the vector $\mathbf{x}_f = 0$ is not an interior point of the set $\mathcal{R}_x(t; \mathbf{z}_0, t_0)$, for all $t > t_0$.

7. Conclusion

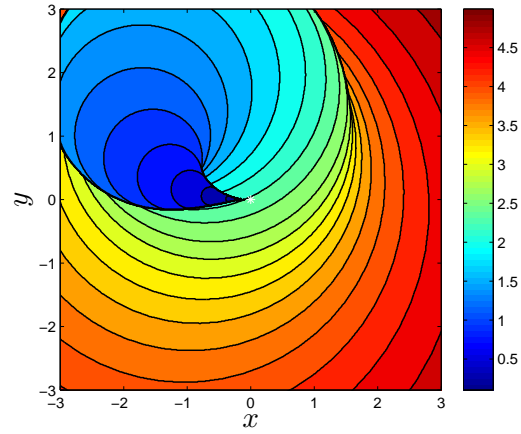
In this paper, we have addressed a minimum-time problem, which is essentially the combination of two classical optimal control problems, namely the Zermelo navigation problem and the problem of steering a self-propelled particle with a constraint on the 2-norm of its acceleration. In particular, we have addressed the problem of characterizing the time-optimal control law that will steer a self-propelled particle to a prescribed terminal position with a free terminal velocity in the presence of a spatiotemporal flow field. We have characterized the optimal synthesis of the problem by associating it with a family of initial value problems based on the parametrization of the set of the candidate time-optimal control laws. Our proposed numerical algorithm is simple and easy to implement. More importantly, our algorithm does not suffer from the complexities and limitations of other approaches, which either are based on converting or transcribing the optimal control problem to a parameter optimization problem or attempt to directly solve the corresponding Hamilton-Jacobi-Bellman partial differential equation (dynamic programming approach). Specifically, it is well known that the first approach requires a good initial guess for each terminal position so that the algorithm that solves the parameter optimization problem can converge to a solution. On the other hand, the second approach requires a high-dimensional spatial mesh discretizing both the position and velocity space that has to be sufficiently fine so that the numerical solution of the partial differential equation can accurately catch the discontinuities of the minimum-time function. In our future work, we intend to examine the problem when the flow field is nonlinear and perhaps uncertain. We also plan to consider the case when both the terminal position and the terminal velocity of the particle are prescribed.

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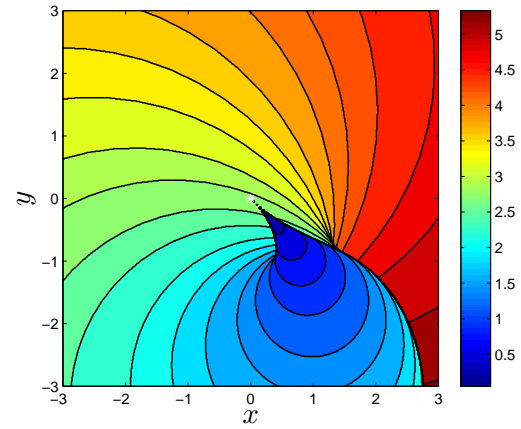
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(a) Level sets of the minimum time when $\mathbf{A}(t)$ is a constant matrix.



(b) Level sets of the minimum time when $\mathbf{A}(t)$ is periodic.

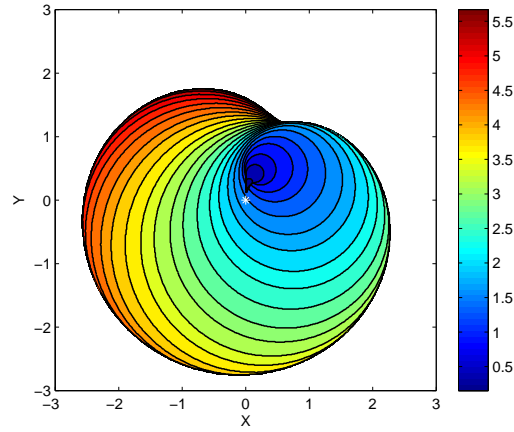


(c) Level sets of the minimum time when $\mathbf{A}(t)$ is not periodic.

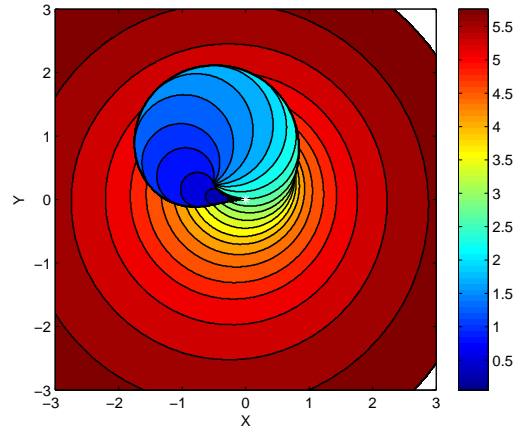
Figure 2. Level sets of the minimum time-to-come function $t_{\text{f}}^*(\cdot; 0, 0)$ for $\mu = 0$.

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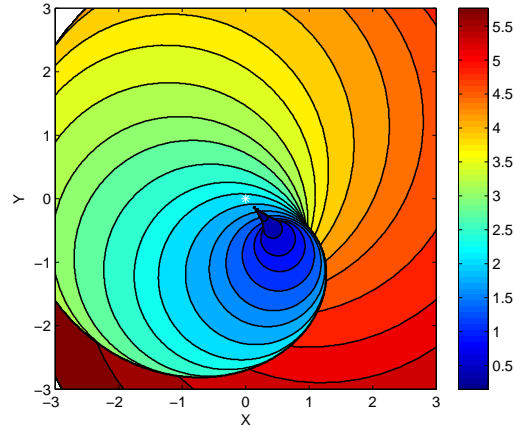
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(a) Level sets of the minimum time when $\mathbf{A}(t)$ is a constant matrix.



(b) Level sets of the minimum time when $\mathbf{A}(t)$ is periodic.



(c) Level sets of the minimum time function $\mathbf{A}(t)$ is not periodic.

Figure 3. Level sets of the minimum time-to-come function $t_f^*(\cdot; 0, 0)$ for $\mu = 1$.

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