

Evasion from a Group of Pursuers with Double Integrator Kinematics

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Abstract—We consider the problem of characterizing an evading strategy for an agent traversing a convex polygon populated by a group of pursuers. We address the problem by associating it with a generalized Voronoi partitioning problem, which encodes information about the proximity relations between the evader and the pursuers based on the value function of a pursuit-evasion game involving the evader and each pursuer from the group individually. The generalized Voronoi partition furnishes a collection of continuous paths which have the following property: When the evader travels along any of these paths, none of the pursuers will have a unilateral incentive to initiate the pursuit against it. With the proposed approach, the problem of evasion from the group of pursuers admits an elegant geometric solution, which can be computed by means of known computational techniques. Numerical simulations that illustrate the theoretical developments are presented.

I. INTRODUCTION

We address the problem of evasion of an agent from a group of pursuers that are distributed inside a convex polygon. Instead of directly addressing the problem using the framework of multi-player differential games, we propose a geometric solution technique which allows the evader to account for the presence of a pursuer only if the two players are sufficiently “close,” with respect to an appropriate pseudo-metric, to each other.

Previous work: The first attempt to study pursuit evasion games (PEGs, for short) within a self-contained mathematical framework is attributed to Isaacs [1], who generalized the theory of zero-sum games from the classical game theory [2] to problems with dynamic constraints. The approach of Isaacs focuses on problems involving two strictly competitive players, whose solution requires the computation of the so-called *value function* of the differential game, a concept similar to the cost-to-go function from dynamic programming, which satisfies a nonlinear partial differential equation known as the Hamilton-Jacobi-Isaacs equation. Bergovitz introduced an alternative framework to address PEGs based on variational techniques [3]. PEGs for players with linear dynamics and quadratic reward/loss functions were first studied in [4]. The results of Isaacs and Bergovitz on differential games with only two (strictly competitive) players were extended to problems involving multiple players in [5]–[7]. In particular, [7] discusses linear quadratic differential games that, in contradistinction with [4], involve players that do not necessarily have strictly competitive objectives and thus admit different solution concepts similar to, for example, Nash equilibria

and Pareto optimal solutions from the theory of non-zero-sum games. Some excellent expositions of differential games, with extensive discussions on PEGs, can be found in [4], [8]–[10].

A characteristic class of differential games involving multiple players are the so-called *group* PEGs, which involve, in general, a group of pursuers and a group of evaders [11]–[19]. A special subclass of group PEGs, known as the *group pursuit* problems, deals with the case when a group of pursuers aims at capturing a unique evader. In our previous work, we have addressed a special class of group pursuit problems [20]–[22], by making use of a particular class of generalized Voronoi diagrams, whose proximity metric is the minimum time that would be required for each pursuer to individually capture the evader provided that all the other pursuers did not participate in the pursuit of the latter. In these references, it was assumed that all the pursuers had a priori partial or complete knowledge of the feedback strategy of the evader; an assumption that essentially reduced the PEG to a problem of pursuit with anticipation [8], which is an optimal control problem. The idea of using Voronoi diagrams in PEGs has also been employed recently in [23].

Main Contributions: In this work, we consider the problem of determining a collection of paths for an evader that aims at traversing a given convex polygon, which is populated by a group of pursuers, in such a way that will not excite any pursuer to go after it. The problem is addressed by associating it with a particular class of generalized Voronoi diagram / partitioning problems, whose solution can be computed by means of available techniques from computational geometry. In particular, the space to be partitioned is the convex polygon that the evader has to traverse, and the point-set that generates this partition (the set of generators) consists of the initial positions of the pursuers. Furthermore, the proximity metric of the generalized Voronoi diagram, that is, the generalized distance function that determines the proximity relations between the generators (that is, the initial locations of the pursuers) and an arbitrary point in the polygon (which corresponds to any possible location of the evader) is the value function of a well-known PEG, namely the *isotropic rocket pursuit* (evasion) problem. In this way, the pursuers are not required to have a priori knowledge of the strategy of the evader in order to determine which one of them is the “closest,” in terms of the capture time, to the latter. Another important fact that distinguishes this work from our previous one has to do with the fact that the feedback strategy of the evader depends explicitly on the computed generalized Voronoi partition. In particular, the

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boundaries of the cells that comprise the generalized Voronoi partition will furnish a collection of paths, which we refer to as the *evading roadmap*, a subset of which solves the evasion problem.

Structure of the paper: The rest of the paper is organized as follows. Section II presents the formulation of the evasion problem, which is subsequently addressed in Section III. Section IV presents numerical simulations, and finally, Section V concludes the paper with a summary of remarks.

II. FORMULATION OF THE DIFFERENTIAL GAME PROBLEM

Let us consider a group of n pursuers, where each pursuer is located, at time $t = 0$, at $\bar{x}_i \in \mathbb{R}^2$ with a prescribed initial velocity $\bar{v}_i \in \mathbb{R}^2$, where $i \in \mathcal{I}_n := \{1, \dots, n\}$. We denote by $\bar{\mathcal{X}} := \{\bar{x}_i \in \mathbb{R}^2 : i \in \mathcal{I}_n\}$ and $\bar{\mathcal{V}} := \{\bar{v}_i \in \mathbb{R}^2 : i \in \mathcal{I}_n\}$, respectively, the sets of the initial positions and velocities of the pursuers. The motion of the i -th pursuer, where $i \in \mathcal{I}_n$, is described by the following set of equations

$$\dot{x}_i = v_i, \quad x_i(0) = \bar{x}_i, \quad (1a)$$

$$\dot{v}_i = u_i, \quad v_i(0) = \bar{v}_i, \quad (1b)$$

where $x_i := [x_i, y_i]^\top \in \mathbb{R}^2$ and $v_i := [v_i, w_i]^\top \in \mathbb{R}^2$ are, respectively, the position and velocity vectors of the i -th vehicle at time t . In addition, u_i denotes the control input of the i -th pursuer, which attains values in the set $U_p := \{v \in \mathbb{R}^2 : |v| \leq \bar{u}_p\}$, where \bar{u}_p corresponds to the maximum acceleration of each pursuer.

In addition, we consider an evader whose motion is described by the following single integrator kinematic model

$$\dot{x}_e = u_e, \quad x_e(0) = \bar{x}_e, \quad (2)$$

where $x_e := [x_e, y_e]^\top \in \mathbb{R}^2$ and $\bar{x}_e := [\bar{x}_e, \bar{y}_e]^\top \in \mathbb{R}^2$ are, respectively, the position vectors of the evader at time t and $t = 0$. Finally, u_e is the control input of the evader (velocity vector), which attains values in the set $U_e = \{v \in \mathbb{R}^2 : |v| \leq \bar{u}_e\}$, where \bar{u}_e is the maximum attainable speed of the evader.

Up to this point, we have not mentioned anything about the set of admissible control inputs, that is, the type of control inputs that the pursuers and the evader can employ. For now, we will assume that both u_i and u_e depend on the time t and the composite state vector $[x_1^\top, \dots, x_n^\top, x_e^\top]^\top$ and they satisfy, in addition, regularity conditions that guarantee the existence and uniqueness of solutions to the initial value problems described by Eqs. (1a)-(1b) and (2), respectively. We will denote by \mathcal{U}_p and \mathcal{U}_e the set of admissible control inputs for the i -th pursuer and the evader, respectively. Furthermore, we henceforth write $t \mapsto x_i(t; \bar{x}_i, u_i)$, and $t \mapsto x_e(t; \bar{x}_e, u_e)$ to denote the solutions of the initial value problems given in Eqs. (1a) and (2), respectively, for a given n -tuple of inputs $(u_1, \dots, u_n) \in \mathcal{U}_p^n$ and an input $u_e \in \mathcal{U}_e$.

Next, we formulate the problem of evasion from a group of pursuers.

Problem 1: Let $\mathcal{S} \subset \mathbb{R}^2$ be a convex polygon whose boundary $\partial\mathcal{S}$ consists of $m_{\mathcal{S}}$ edges, denoted by E_μ , where $\mu \in \mathcal{M}_{\mathcal{S}} := \{1, \dots, m_{\mathcal{S}}\}$, and let $\bar{\mathcal{X}} \subset \text{int}(\mathcal{S})$. In addition, let us consider a group of pursuers, where the i -th pursuer from the group commences at a point $\bar{x}_i \in \bar{\mathcal{X}}$ with an initial velocity $\bar{v}_i \in \bar{\mathcal{V}}$ and its motion is described by Eqs. (1a)-(1b), and an evader commencing at $\bar{x}_e \in E_\mu \subset \partial\mathcal{S}$, for some $\mu \in \mathcal{M}_{\mathcal{S}}$, whose motion is described by Eq. (2). Then, find an input $u_e \in \mathcal{U}_e$ with the application of which the evader will traverse the set \mathcal{S} such that

$$\max_{i \in \mathcal{I}_n} |x_e(t; \bar{x}_e, u_e) - x_i(t; \bar{x}_i, u_i)| > \epsilon_c,$$

for all $t \in [0, T_{\mathcal{S}}]$ and for all $u_i \in \mathcal{U}_p$, where $\epsilon_c > 0$ is the *capturability radius* of the evasion problem and $T_{\mathcal{S}}$ is the first time at which $x_e(T_{\mathcal{S}}) \in \partial\mathcal{S} \setminus \{E_\mu\}$.

Remark 1 Problem 1 is a differential game that involves $n + 1$ players. Typically, problems of this kind require the solution of a system of coupled nonlinear PDEs [6], [24], which is a computationally intractable task. In this work, we devise a different geometric approach, which will allow us to avoid dealing directly with the multi-player differential game.

Before we address the differential game that involves $n + 1$ players, we shall first examine the PEG that involves only two players, namely, the i -th pursuer and the evader. In the latter case, no other pursuer except from the i -th pursuer is allowed to participate in the pursuit of the evader. We shall refer to this problem as the i -th pursuit-evasion game (i -th PEG, for short). We assume that the i -th PEG terminates, if there exists a control pair $(u_i, u_e) \in \mathcal{U}_p \times \mathcal{U}_e$ and a time $\tau \geq 0$ such that

$$|x_i(\tau; \bar{x}_i, u_i) - x_e(\tau; \bar{x}_e, u_e)| \leq \epsilon_c.$$

In addition, for a given pair of inputs $(u_i, u_e) \in \mathcal{U}_p \times \mathcal{U}_e$, we define the capture time, denoted by $T(u_i, u_e)$, as follows

$$T(u_i, u_e) := \inf\{t \geq 0 : |x_i(t; \bar{x}_i, u_i) - x_e(t; \bar{x}_e, u_e)| \leq \epsilon_c\}, \quad (3)$$

provided that the set on the right hand side of (3) is non-empty; otherwise, $T(u_i, u_e) := \infty$. Note that $T(u_i, u_e)$ denotes the time at which the i -th pursuer driven by the control input u_i will capture the evader, which is driven by the input u_e , for the first time.

Definition 1: The i -th PEG is feasible, if there exists a pair of control inputs $(u_i, u_e) \in \mathcal{U}_p \times \mathcal{U}_e$ such that $T(u_i, u_e) < \infty$.

Next, we formulate the i -th PEG, which amounts to the characterization of a pair of control inputs (u_i^*, u_e^*) that furnish a saddle point of the capture time $T(u_i, u_e)$.

Problem 2 (i -th PEG): Let us consider the i -th pursuer and the evader, whose motion is described, respectively, by Equations (1a)-(1b) and (2). In addition, let $\mathcal{U}_p^K \subset \mathcal{U}_p$ and $\mathcal{U}_e^K \subset \mathcal{U}_e$ denote the sets that consists of all the time-invariant feedback laws $u_i = u_i(x_e, x_i, v_i)$ and $u_e =$

$u_e(x_e, x_i, v_i)$, respectively, that are K -strategies¹. Then, find a pair of K -strategies $(u_i^*, u_e^*) \in \mathcal{U}_p^K \times \mathcal{U}_e^K$, such that $T(u_i^*, u_e^*) < \infty$ and $T(u_i^*, u_e^*) = T_{\text{sp}}$, where

$$T_{\text{sp}} := \max_{u_e \in \mathcal{U}_e} \min_{u_i \in \mathcal{U}_p} T(u_i, u_e) = \min_{u_i \in \mathcal{U}_p} \max_{u_e \in \mathcal{U}_e} T(u_i, u_e). \quad (4)$$

Remark 2 Note that the min and max operators that appear in Eq. (4) commute; something that, although true for the value function of the i -th PEG, does not hold true in general [10].

The i -th PEG is known in the literature as the *isotropic rocket* pursuit-evasion game [1]. The time T_{sp} corresponds to the *value function* of this zero-sum differential game. We wish to highlight at this point that in the formulation of Problem 2, the inputs u_i^* and u_e^* are constrained to be time-invariant feedback laws, and in particular, K -strategies, in accordance with the treatment of PEGs by Isaacs [1]. We shall henceforth denote by $T_{\text{sp}}(x_e, x_i, v_i)$ the value function of the i -th PEG when at time t the evader is located at x_e whereas, at the same time, the i -th pursuer is commencing at the point x_i with velocity v_i .

Proposition 1: Let $\bar{y}_i = \bar{x}_i - \bar{x}_e$ and $\bar{v}_i \in \mathbb{R}^2$ be given. If there exists a pair $(u_i^*, u_e^*) \in \mathcal{U}_p^K \times \mathcal{U}_e^K$ that solves the i -th PEG, then the value function T_{sp} of the i -th PEG satisfies the following quartic equation

$$\begin{aligned} \bar{y}_i^T \bar{y}_i + 2\bar{y}_i^T \bar{v}_i T_{\text{sp}} + \bar{v}_i^T \bar{v}_i T_{\text{sp}}^2 &= Q^2(T_{\text{sp}}), \\ Q(T_{\text{sp}}) &= \epsilon_c - \bar{u}_e T_{\text{sp}} + \bar{u}_p T_{\text{sp}}^2/2, \end{aligned} \quad (5)$$

if $|\bar{y}_i| > \epsilon_c$, and $T_{\text{sp}} = 0$, otherwise (trivial capture).

Proof: The value function T_{sp} and the corresponding pair of inputs (u_i^*, u_e^*) are determined by the solution of the following PDE (known as the Isaacs-Hamilton-Jacobi or the *main equation*)

$$\begin{aligned} \nabla_{x_i} T_{\text{sp}}(x_e, x_i, v_i) v_i + \min_{u_i \in \mathcal{U}_p} \nabla_{v_i} T_{\text{sp}}(x_e, x_i, v_i) u_i \\ + \max_{u_e \in \mathcal{U}_e} \nabla_{x_e} T_{\text{sp}}(x_e, x_i, v_i) u_e &= -1, \end{aligned} \quad (6)$$

where $\nabla_{x_\ell} := [\partial_{x_\ell}, \partial_{y_\ell}]$, for $\ell \in \{i, e\}$, and $\nabla_{v_i} := [\partial_{v_i}, \partial_{w_i}]$. It turns out that Equation (6) can be solved semi-analytically to furnish Eq. (5). The reader may refer to [1, pp. 106–112] for more details. ■

Remark 3 Proposition 1 implies that both the strategies and the value function of the i -th PEG depend only on the relative position vector y_i between the two players, where $y_i := x_i - x_e$, and the i -th pursuer's velocity v_i . Now let $t \mapsto y_i(t; \bar{y}_i, u_i, u_e)$ denote the solution of the following initial value problem

$$\dot{y}_i = \dot{x}_i - \dot{x}_e = v_i - u_e, \quad y_i(0) = \bar{y}_i, \quad (7)$$

where v_i satisfies Eq. (1b) and $\bar{y}_i := \bar{x}_i - \bar{x}_e$. Then, the i -th PEG admits a solution, if there exists a control pair $(u_i, u_e) \in \mathcal{U}_p \times \mathcal{U}_e$ such that $y_i(\tau) \in \mathcal{C} := \{y \in \mathbb{R}^2 : |y| \leq \epsilon_c\}$, where

\mathcal{C} constitutes the *target set* of the i -th PEG. Note that the target set, when expressed in terms of the relative position y , becomes a fixed subset of \mathbb{R}^2 .

Figure 1 illustrates the level sets of the value function T_{sp} of the i -th PEG as a function of the initial position of the evader, when the i -th pursuer is initially located at the origin ($\bar{x}_i = 0$) and its velocity vector \bar{v}_i is parallel to the negative y -axis. We observe that the i -th pursuer can capture the evader relatively fast provided that the latter is sufficiently close to the former at $t = 0$ and, in addition, its velocity points towards the evader. The collection of the initial positions of the evader that are more favorable to the i -th pursuer belong to the region that is confined between the curve segments \mathcal{B} and \mathcal{D} and the circular arc \mathcal{Y} (thick dashed black line), as is illustrated in Fig. 1. In particular, \mathcal{Y} corresponds to the so-called *usable part* of the boundary $\partial\mathcal{C}$ of the target set of the i -th PEG, whereas the curve segments \mathcal{B} and \mathcal{D} , which intersect the target set $\partial\mathcal{C}$ tangentially, constitute the *barrier* of the i -th PEG. The barrier determines a manifold that cannot be crossed during optimal play by both players and along which the value function T_{sp} undergoes discontinuous jumps (more precisely, the manifold where T_{sp} undergoes discontinuous jumps is determined by the barrier and the set $\mathcal{C} \setminus \mathcal{Y}$). As shown in [1], if \mathcal{B} and \mathcal{D} intersect with each other, then the set that will be enclosed by them and the usable part \mathcal{Y} of \mathcal{C} will correspond to the so-called *capture zone*. The last term describes the set of initial positions of the evader from which the latter can always be captured by the i -th pursuer regardless of the future actions of the evader. It turns out that the curve segments \mathcal{B} and \mathcal{D} do not intersect with each other, provided that

$$2\bar{u}_p \epsilon_c > \bar{u}_e^2, \quad (8)$$

in which case the pursuer can always enforce the capture of the evader in finite time. The inequality (8) is known as the *capture condition*. In this work, we will assume that the capture condition always holds; an assumption that actually renders our problem more challenging. It should also be mentioned that when the capture condition is satisfied but the evader commences at a point that does not belong to the region confined between the curve segments \mathcal{B} and \mathcal{D} and the circular arc \mathcal{Y} , then it can utilize swerve-like maneuvers in order to significantly delay its capture by the pursuer. For more details on the definition of the barrier and the usable part of the target set and their role in the solution of PEGs, the reader may refer to [1].

III. THE EVADING ROADMAP PROBLEM

We are interested in characterizing a collection of continuous paths that enjoy the following property: when they are traversed by the evader, the latter remains sufficiently “far away,” with respect to the value function T_{sp} , from each pursuer from a given group of pursuers that are distributed in \mathcal{S} . We shall refer to this family of paths as the *evading*

¹For the definition of K -strategies, the reader may refer to [1].

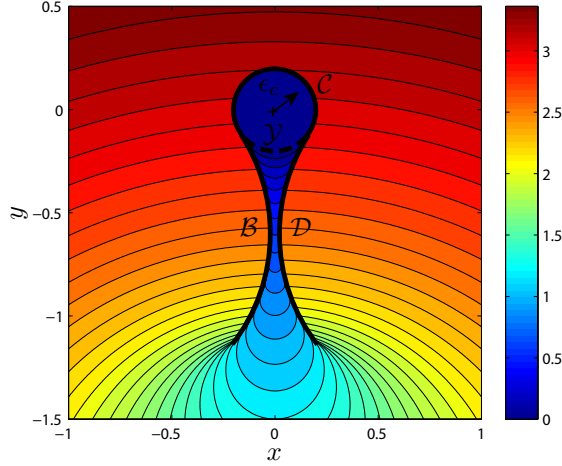


Fig. 1. Level sets of the value function T_{sp} of the i -th PE, when the i -th pursuer is initially located at the origin and its velocity is parallel to the negative y -axis. It is interesting to note the existence of a manifold (thick black line) where the value function undergoes discontinuous jumps.

roadmap. In order to make the meaning of “sufficiently far away” more precise, we introduce the following definition.

Definition 2: The i -th pursuer, which is located, at time $t = 0$, at a point \bar{x}_i with velocity \bar{v}_i , has a *unilateral incentive* to initiate a pursuit against the evader commencing at $\bar{x}_e \in \text{int}(\mathcal{S})$, at time $t \geq 0$, when

$$T_{sp}(\bar{x}_e, \bar{x}_i, \bar{v}_i) < T_{sp}(\bar{x}_e, \bar{x}_j, \bar{v}_j), \quad \text{for all } j \in \mathcal{I}_n \setminus \{i\}.$$

If none of the pursuers has a *unilateral incentive* to initiate a pursuit against the evader commencing at $\bar{x}_e \in \mathcal{S}$, then we will say that \bar{x}_e is an *equilibrium* position for Problem 1.

Proposition 2: A point $x_e \in \mathcal{S}$ is an equilibrium position for Problem 1 if, and only if, there exist $i, j \in \mathcal{I}_n$, where $i \neq j$, such that

$$T_{sp}(\bar{x}_e, \bar{x}_i, \bar{v}_i) = T_{sp}(\bar{x}_e, \bar{x}_j, \bar{v}_j) = \min_{\ell \in \mathcal{I}_n} T_{sp}(\bar{x}_e, \bar{x}_\ell, \bar{v}_\ell). \quad (9)$$

Problem 3 (The Evading Roadmap Problem): Let $\mathcal{S} \subset \mathbb{R}^2$ be a convex polygon, and let $\bar{\mathcal{X}} \subset \text{int}(\mathcal{S})$, where $\bar{\mathcal{X}} := \{\bar{x}_i \in \mathbb{R}^2, i \in \mathcal{I}_n\}$. Then, find a collection of continuous paths $\Gamma := \{\gamma_\ell, \ell \in \mathcal{L}\}$, where \mathcal{L} is an index set, and each path $\gamma_\ell : [0, 1] \mapsto \mathcal{S}$ satisfies the following properties: i) $\gamma_\ell(0), \gamma_\ell(1) \in \partial\mathcal{S}$, and $\gamma_\ell(t) \in \text{int}(\mathcal{S})$, for all $t \in]0, 1[$, ii) every point $x \in \text{trace}(\gamma_\ell)$, where $\text{trace}(\gamma_\ell) = \{x \in \mathcal{S} : x = \gamma_\ell(t), t \in [0, 1]\}$, is an equilibrium position for Problem 1.

Remark 4 We henceforth refer to the collection of continuous paths Γ that solve Problem 3 as the *evading roadmap*.

Next, we show that Problem 3 can admit a purely geometric solution, which is based, in turn, on a well known property enjoyed by the cells that comprise a generalized Voronoi partition. In particular, the common boundaries of two or more cells from a generalized Voronoi partition generated by a finite point-set (set of generators) are equidistant, with

respect to the proximity metric of the partition, from their corresponding generators. Let us now consider a generalized Voronoi partition, say \mathfrak{V} , generated by the point-set $\bar{\mathcal{X}}$ whose proximity metric is the value function T_{sp} of Problem 2. Then, the boundaries of the different cells of \mathfrak{V} would consist of points at which the evader is equidistant, with respect to T_{sp} , from at least two pursuers from the group of pursuers. Consequently, when the evader traverses the common boundaries of two or more cells of the generalized Voronoi partition \mathfrak{V} , then no pursuer will have a unilateral incentive to initiate the pursuit against the evader.

A. Formulation of the Partitioning Problem

Based on the previous discussion, we next formulate a generalized Voronoi partitioning problem with respect to the value function of Problem 2. The solution to this partitioning problem will allow us to address Problem 3 by making use of purely geometric arguments.

Problem 4: Let $\mathcal{S} \subset \mathbb{R}^2$ be a convex polygon. Given a collection of n distinct points $\bar{\mathcal{X}} := \{\bar{x}_i \in \mathbb{R}^2 : i \in \mathcal{I}_n\} \subset \text{int}(\mathcal{S})$, where $\bar{\mathcal{X}} := \{\bar{x}_i \in \mathbb{R}^2 : i \in \mathcal{I}_n\}$, and a corresponding collection of n velocity vectors $\bar{\mathcal{V}} := \{\bar{v}_i \in \mathbb{R}^2 : i \in \mathcal{I}_n\}$, then determine a partition $\mathfrak{V} = \{\mathfrak{V}^i : i \in \mathcal{I}_n\}$ of \mathcal{S} such that

- 1) $\mathcal{S} = \bigcup_{i \in \mathcal{I}_n} \mathfrak{V}^i$ and $\text{int}(\mathfrak{V}^i) \cap \text{int}(\mathfrak{V}^j) = \emptyset$.
- 2) For each $x \in \mathfrak{V}^i$, it holds that

$$T_{sp}(x, \bar{x}_i, \bar{v}_i) \leq T_{sp}(x, \bar{x}_j, \bar{v}_j), \quad \text{for all } j \in \mathcal{I}_n \setminus \{i\}.$$

Remark 5 Problem 4 is a generalized Voronoi partitioning problem, whose solution corresponds to a data structure that encodes information about the proximity relations between the pursuers and the evader. These relations are induced by the value function T_{sp} of the i -th PEG, when the evader is assumed to be initially located at an arbitrary point $x \in \mathcal{S}$. In this work, we will say that two or more generators from $\bar{\mathcal{X}}$, and their corresponding pursuers, are neighbors if the intersection of the boundaries of their associated cells is non-empty and non-trivial (not a singleton). We denote by \mathcal{N}_i the subset of \mathcal{I}_n that consists of the indices of all the pursuers that are neighbors of the i -th pursuer.

The following proposition underlines an important property enjoyed by the generalized Voronoi partition that solves Problem 4.

Proposition 3: Let $\mathfrak{V} = \{\mathfrak{V}^i, i \in \mathcal{I}_n\}$ correspond to the solution of Problem 4 for a given convex polygon \mathcal{S} , and let

$$\partial\mathfrak{V} := \bigcup_{i \in \mathcal{I}_n} \partial\mathfrak{V}^i \setminus \partial\mathcal{S}. \quad (10)$$

A point $x_e \in \text{int}(\mathcal{S})$ is an equilibrium position for Problem 1 if, and only if, $x_e \in \partial\mathfrak{V}$ for all $t \in]0, T_S[$, where T_S is the first time instant at which the evader, which commences at $\bar{x}_e \in E_\mu \subset \partial\mathcal{S}$, at time $t = 0$, reaches the set $\partial\mathcal{S} \setminus E_\mu$.

Proof: Note that the boundary of each cell $\mathfrak{V}^i \in \mathfrak{V}$ consists of points that belong to the intersection $\mathfrak{V}^i \cap \partial\mathcal{S}$,

when the latter set is nonempty, along with the points that belong to the intersection of \mathfrak{V}^i with its neighboring cells. Consequently, the set $\partial\mathfrak{V}$ consists of points that belong to the common boundaries of two or more cells exclusively. Therefore, if $x_e \in \partial\mathfrak{V}$, there exist $i \in \mathcal{I}_n$ and $j \in \mathcal{N}_i$ such that $x_e \in \partial\mathfrak{V}^i \cap \partial\mathfrak{V}^j$, which implies that $T_{\text{sp}}(x_e, \bar{x}_i, \bar{v}_i) = T_{\text{sp}}(x_e, \bar{x}_j, \bar{v}_j)$, and $T_{\text{sp}}(x_e, \bar{x}_k, \bar{v}_k) \geq T_{\text{sp}}(x_e, \bar{x}_i, \bar{v}_i)$, for all $k \in \mathcal{I}_n \setminus \{i, j\}$. Consequently, x_e is an equilibrium position for Problem 1. The converse can be shown similarly. ■

Remark 6 Proposition 3 can be interpreted as follows. In order to solve the Evading Roadmap Problem, we first need to compute the generalized Voronoi partition \mathfrak{V} that solves Problem 4, or more precisely, determine the boundaries of all the cells that comprise this partition. Subsequently, we have to remove from the latter sets the points that belong to $\partial\mathcal{S}$ in order to determine the set $\partial\mathfrak{V}$. The next step is to determine the paths whose trace belongs to the set $\partial\mathfrak{V}$. From these paths, the ones that belong to the roadmap Γ are the ones that traverse the polygon \mathcal{S} in accordance with the requirement ii) of the Evading Roadmap Problem.

B. Computation of the Solution to the Evading Roadmap Problem

Problem 4 cannot be directly associated with a class of generalized Voronoi partitioning problems for whose solution efficient computational techniques exist in the literature [25], [26]. A straightforward approach to address such partitioning problems is to use algorithms that compute approximations of the desired partitions by utilizing a discretization grid, say \mathcal{G} , over the space to be partitioned. For example, one such approach involves the use of computational techniques that are commonly employed for the numerical solution of partial differential equations, and in particular, *direct diffusion methods* [27]. The time complexity of the approach presented in [27] is $\mathcal{O}(\text{card}(\mathcal{G}))$, where $\text{card}(\mathcal{G})$ denotes the cardinality of the grid \mathcal{G} , which is, in turn, equal to the number of its nodes. An alternative approach is to compute the so-called lower envelope function that bounds from below the graphs of the proximity metric associated with each generator in $\mathcal{S} \times [0, \infty)$ (see, for example, [28]). In our problem, the lower envelope function, denoted by $T_{\text{sp}}^*(x)$, is defined as follows

$$T_{\text{sp}}^*(x) := \min_{i \in \mathcal{I}_n} T_{\text{sp}}(x, \bar{x}_i, \bar{v}_i).$$

The time complexity of the previous approach is $\mathcal{O}(n \text{card}(\mathcal{G}))$, where n is the number of the pursuers. Although, the algorithm that utilizes the direct diffusion methods runs faster than the algorithm that is based on the characterization of the lower envelope function, the latter one is, perhaps, easier to implement. Due to space limitations, we will not present the details of the computational techniques that we have just discussed very briefly. The reader is urged to refer to [27], [28] for more details.

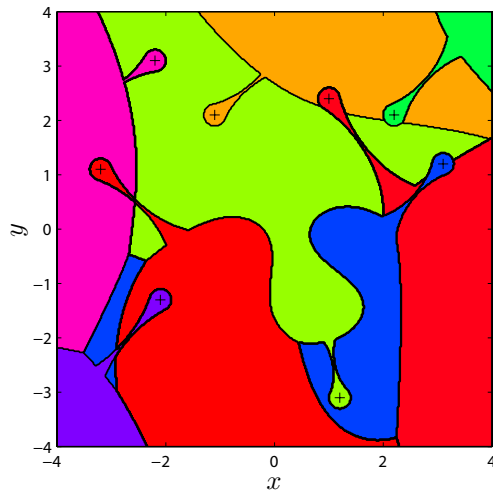
IV. NUMERICAL SIMULATIONS

Next, we present numerical simulations to illustrate the key points of the previous discussion. In particular, we consider a scenario with eight pursuers and use the following data: $\bar{u}_e = 0.6$, $\bar{u}_p = 1$, $\epsilon_c = 0.2$ and $\mathcal{S} = [-4, 4] \times [-4, 4]$. The generalized Voronoi partition that solves Problem 4 is illustrated in Fig. 2(a). The generalized Voronoi partition has been computed by utilizing the algorithm that is based on the characterization of the lower envelope function. Figure 2(b) illustrates the level sets of the lower envelope function $T_{\text{sp}}^*(x)$. Each arrow in Fig. 2(a) corresponds to the initial velocity vector $\bar{v}_i \in \bar{\mathcal{V}}$ of the i -th pursuer, which is initially located at $\bar{x}_i \in \bar{\mathcal{X}}$ (the locations of the generators are denoted by black crosses).

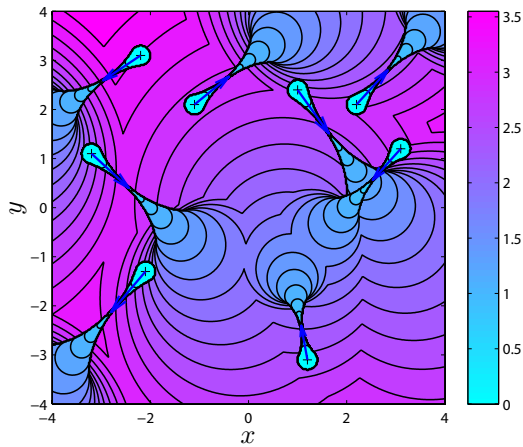
It is interesting to note that the solution to the partitioning problem suggests that, in some cases, the evader will have to traverse paths that circumvent one or more pursuers while staying relatively close to them. The evader may traverse such paths, which may appear counterintuitive, at a first glance, because in this way it avoids visiting points that are favorable to some pursuer from the group of pursuers. (Note that such points lie in the region between the two curve segments of the barrier and the usable part of the target set of the corresponding PEG). We also observe that the computed generalized Voronoi partition differs significantly from the corresponding standard Voronoi partition or other common types of generalized Voronoi partitions generated by the same point-set, which can be found in the relevant literature. This is mainly due to the following two reasons. The first one has to do with the non-uniform way that the level sets of the proximity metric (value function of the i -th PEG) expand along different directions through \mathcal{S} as a consequence of the fact that the proximity metric is an anisotropic, that is, a direction-dependent, pseudo-metric. In simple words, the value function T_{sp} of the i -th PEG expands faster along particular directions, and specifically, the directions that “differ less” from the direction of the initial velocity \bar{v}_i of the i -th pursuer located at \bar{x}_i . The second reason has to do with the existence of manifolds along which the value function T_{sp} of the i -th PEG undergoes discontinuous jumps. The existence of these discontinuous jumps results in situations where points that are close, in terms of the Euclidean distance, to a particular generator are at the same time “far away,” with respect to T_{sp} , from the same generator, and vice versa.

V. CONCLUSION

We have presented a solution technique to address a problem of evasion involving an evader and a group of pursuers, when the latter are distributed inside a convex polygon. The objective of the evader is to traverse the polygon while remaining sufficiently “far away” from the group of pursuers, when its proximity from them is characterized in terms of the value function of a corresponding pursuit-evasion game. We address the problem by associating it with a particular



(a) Generalized Voronoi partition that solves Problem 4.



(b) Level sets of the lower envelope function T_{sp}^* .

Fig. 2. The generalized Voronoi partition that solves Problem 4 and the level sets of the lower envelope function for a scenario with eight pursuers.

class of generalized Voronoi partitioning problems whose solution is computed via efficient, approximation algorithms. In particular, it turns out that the boundaries of the cells that comprise the proposed spatial partition furnish a collection of paths that enjoy the following property: When the evader traverses any of these paths, then no pursuer from the group will have a unilateral incentive to initiate a pursuit against it. The proposed geometric solution technique can be easily implemented and is significantly more computationally tractable than other more straightforward approaches aimed at directly addressing the corresponding multi-player differential game, which typically requires the solution of a system of coupled nonlinear PDEs.

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