Optimal Covariance Control for Stochastic Linear Systems Subject to Integral Quadratic State Constraints

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***Abstract*— This work is concerned with an optimal covariance control problem for stochastic linear systems subject to quadratic state integral constraints. In particular, our objective is to design a feedback control law that will steer the covariance matrix of the (random) terminal state vector of a stochastic linear system to a designated positive semi-definite matrix while minimizing the expected value of the control effort required for this "covariance transition" or "Schrödinger bridge" subject to integral quadratic state inequality constraints. We address this problem by imbedding it into a one-parameter family of unconstrained covariance control problems that are more tractable, both analytically and computationally, than the original, constrained covariance control problem. In this way, the original problem is essentially reduced to a finite-dimensional optimal parameter selection problem, which can be addressed by means of gradient descent-type algorithms.**

I. Introduction

This work deals with the problem of steering the (random) state of a stochastic linear system, which is drawn from a known Gaussian distribution, to a terminal state which is also drawn from a Gaussian distribution at a given terminal time subject to mean integral state constraints. We refer to this problem as the finite time horizon covariance control problem. The motivation for studying this type of problems stems from broad classes of control applications involving stochastic control systems in which the control objectives are more appropriately described in terms of the root-mean-square values of the terminal state vector [1], [2].

Literature Review: The importance of developing a "covariance control theory" was first highlighted in the literature by Hotz and Skelton [1], [2]. Ideas and techniques presented in these references were subsequently extended in a series of papers (see, for example, [3]–[6]). In all these references, the emphasis is placed on covariance control problems with an infinite time horizon and the objective is to steer the covariance of the state of a stochastic linear system to a "reachable" *steady state* covariance matrix. In general, the solvability of the infinite time horizon covariance control problem requires the satisfaction of rather stringent conditions and typically, one has to first characterize a parametrization of the subset of the convex cone of positive definite matrices that consists of all the "reachable" *steady state* covariance matrices for a given linear stochastic system.

The interest in this class of problems has been recently revived in light of the significant results reported in [7], [8], where the covariance control problem was placed in a stochastic optimal control framework and connections with the theory of *Schrödinger bridges* were also established. In particular, the authors of [7], [8] identified two different subclasses of stochastic optimal control problems whose performance index is the expected value of the control effort or energy. In the first subclass [7], the time horizon is finite and it is also assumed that the uncertainty is injected to the system via the same channels as the control inputs. It turns out that in this case, the problem can be reduced, under some mild technical assumptions, to a system of two Riccati equations that are coupled only via their boundary conditions and practically admits a closed-form solution. The case when the input channels and process noise channels are not necessarily the same is treated in [8]. In this case, the existence of solutions to the minimum effort covariance control problem is equivalent to the solvability of a Schrödinger nonlinear system, which involves a system of coupled Riccati equations with coupled boundary conditions, which is an open problem. For this class of problems, one has to resort to numerical techniques of convex optimization in order to characterize suboptimal solutions [9].

Main Contribution: In this paper, we consider an optimal covariance control problem using the framework of finite-horizon stochastic optimal control similar in spirit to the Refs. [7], [8], which have been the main source of inspiration for this work. What distinguishes this work from the previous two references is the fact that in our problem formulation, we explicitly account for the presence of mean integral quadratic state inequality constraints, which are intended to confine the sample trajectories of the stochastic system within a "narrow" tube as is illustrated in Fig. 1. Herein, we will refer to the extension of the classic finite-horizon LQG (Linear Quadratic Gaussian) problem [10], [11] with full-state observation but with boundary conditions in terms of the covariance of the state vector as the unconstrained LQG covariance control (unconstrained LQGCC) problem, whereas the same problem subject to the (mean) integral quadratic state inequality constraints will be referred as the *constrained* LQG covariance control (constrained LQGCC) problem. It should be mentioned here that the solution to the unconstrained LQG covariance control problem (in which both the state and the input appear in the running cost) can easily be recovered from the solution to the constrained LQGCC problem.

To address the constrained optimal covariance control problem, we imbed it into a one-parameter family of unconstrained LQGCC problems. Each parametric unconstrained LQGCC problem is subsequently associated with the minimum effort covariance control problem addressed in [7], [8] via a time-varying input transformation. The final step is to solve the (finite-dimensional) optimal parameter selection problem, which seeks for the optimal parameter that will

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determine the optimal control law that solves the constrained stochastic optimal control problem. The proposed approach is based on standard results from the theory of Lagrangian duality as in [12], [13].

Structure of the paper: The rest of the paper is organized as follows. The LQGCC problem is formulated in Section II and is subsequently imbedded in a family of unconstrained LQGCC problems in Section III. In Section IV, we associate the constrained LQGCC problem with a finite-dimensional optimal parameter selection problem. Illustrative numerical simulations are included in Section V, and finally, Section VI concludes the paper with a summary of remarks.

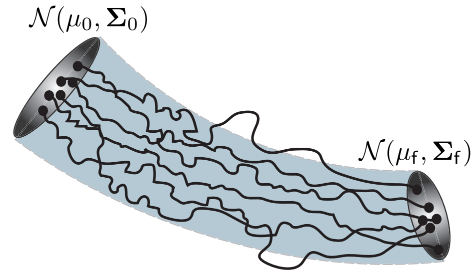


Fig. 1. The problem of steering the uncertain state of a stochastic control system, which is drawn from a known Gaussian distribution $\mathcal{N}(\mu_0, \mathbf{\Sigma}_0)$ to a terminal state that is also drawn from a known distribution $\mathcal{N}(\mu_{\mathsf{f}}, \mathbf{\Sigma}_{\mathsf{f}})$, at a given final time (finite time horizon covariance control problem). By considering state integral constraints, the sample paths of the Schrödinger bridge are expected to lie within a "narrow" tube.

II. Problem Formulation

A. Notation

We denote by $\mathbb{R}^n$ the set of n-dimensional real vectors. The space of continuous functions over an interval $[\alpha, \beta]$ that are taking values in $\mathbb{R}^{m\times n}$ is denoted by $\mathcal{C}([\alpha,\beta]; \mathbb{R}^{m\times n})$. Furthermore, given a probability space $(\Omega, \mathfrak{F}, P)$, we denote by $\mathcal{L}_2^n([\alpha,\beta]; \Omega, \mathfrak{F}, P)$ the Hilbert space of mean square integrable stochastic processes $\{x(t) \ : \ t \in [\alpha,\beta]\}$ in $(\Omega, \mathfrak{F}, P)$, where $x(t)$ is a n-dimensional (random) vector at each $t \in [\alpha, \beta]$. Finally, we will denote the cone of $n\times n$ (symmetric) positive definite and positive semi-definite matrices by $\mathbb{P}_n$ and $\mathbb{PS}_n$, respectively.

B. Formulation of the Optimal Covariance Control Problem

We consider a stochastic linear system that satisfies the following stochastic differential equation:

$$\mathrm{d}x(t) = \mathbf{A}(t)x(t)\mathrm{d}t + \mathbf{B}(t)u(t)\mathrm{d}t + \mathbf{C}(t)\mathrm{d}w(t), \qquad (1)$$

with $x(t_0) = x_0$, for $t \in [t_0, t_{\mathsf{f}}]$, where $\{x(t),\ t \in [t_0, t_{\mathsf{f}}]\}$ and $\{u(t) \ : \ t \in [t_0, t_{\mathsf{f}}]\}$ denote the state and the input stochastic processes, respectively, in a probability space $(\Omega, \mathfrak{F}, P)$. At each time $t \in [t_0, t_{\mathsf{f}}]$, $x(t)$ and $u(t)$ are n-dimensional and m-dimensional vectors, respectively. The input process $\{u(t) \ : \ t \in [t_0, t_{\mathsf{f}}]\}$ is assumed to belong to $\mathcal{L}_2^m([t_0, t_{\mathsf{f}}]; \Omega, \mathfrak{F}, P)$ and to have finite k-moments for all $k > 0$. We will henceforth refer to an input process that satisfies these properties as *admissible*. In addition, $\{w(t) \ : \ t \in [t_0, t_{\mathsf{f}}]\}$ is a standard Brownian motion, which is adapted to an increasing family of σ-algebras $\{\mathfrak{F}(t_0, t),\ t \in [t_0, t_{\mathsf{f}}]\}$, that is, $\mathfrak{F}(t_0, t_1) \subseteq \mathfrak{F}(t_0, t_2)$ for all $t_0 \le t_1 \le t_2 \le t_{\mathsf{f}}$, where $\mathfrak{F}(t_0, t) \subseteq \mathfrak{F}$, for all $t \in [t_0, t_{\mathsf{f}}]$. At each time t the (random) vector $w(t)$ is p-dimensional. We will be considering input and state processes that are non-anticipative with respect to $\{\mathfrak{F}(t_0, t),\ t \in [t_0, t_{\mathsf{f}}]\}$. Finally, $\mathbf{A}(\cdot) \in \mathcal{C}([t_0, t_{\mathsf{f}}]; \mathbb{R}^{n\times n})$, $\mathbf{B}(\cdot) \in \mathcal{C}([t_0, t_{\mathsf{f}}]; \mathbb{R}^{n\times m})$ and $\mathbf{C}(\cdot) \in \mathcal{C}([t_0, t_{\mathsf{f}}]; \mathbb{R}^{n\times p})$.

Now, let us assume that the initial state x_0 is a random vector drawn from the multi-variate normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_0)$, where $\mathbf{\Sigma}_0 \in \mathbb{P}_n$ is the initial covariance of system's state, that is, $\mathbb{E}\{x_0 x_0^{\mathrm{T}}\} = \mathbf{\Sigma}_0$. Recently, the problem of steering the state of the stochastic linear system (1) from the initial (random) vector $x_0 \sim \mathcal{N}(0, \mathbf{\Sigma}_0)$ to a terminal (random) vector x_{f} at a given time $t = t_{\mathsf{f}}$, where $x_{\mathsf{f}} \sim \mathcal{N}(0, \mathbf{\Sigma}_{\mathsf{f}})$, and where $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{P}_n$ is the terminal covariance, that is, $\mathbb{E}\{x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\} = \mathbf{\Sigma}_{\mathsf{f}}$, via a control input that minimizes the expected value of the control effort was addressed in [7], [8]. In this work, we examine a similar optimal covariance control problem when, in addition, a mean integral quadratic state (inequality) constraint is enforced. The motivation stems from classical linear quadratic control problems, in which, the objective is to steer the state of the system to the origin in a way that strikes a balance between using "reasonable" control effort while also keeping the deviations of the system's state from the origin small on average (see Fig. 1). In the framework proposed in [7], [8], the aspect of penalizing state deviations explicitly and in particular, via mean integral quadratic state constraints, is missing. This work is intended to fill this gap.

Problem 1: Constrained LQGCC Problem: Given $0 \le t_0 < t_{\mathsf{f}} < \infty$, $c > 0$, and positive definite (symmetric) matrices $\mathbf{Q}_0$, $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_{\mathsf{f}}$ find an admissible control input $u^{\circ}(\cdot)$ that minimizes the performance index

$$J(u(\cdot); t_0, t_{\mathsf{f}}) := \mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} u(t)^{\mathrm{T}} u(t)\mathrm{d}t\}, \qquad (2)$$

subject to the (stochastic) dynamic constraints (1), the mean integral quadratic state constraint $h(x(\cdot)) \le c$, with

$$h(x(\cdot)) := \mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} x(t)^{\mathrm{T}} \mathbf{Q}(t) x(t)\mathrm{d}t\}, \qquad (3)$$

where $\mathbf{Q}(t) - \mathbf{Q}_0 \in \mathbb{SP}_n$ for all $t \in [t_0, t_{\mathsf{f}}]$, and the boundary conditions in terms of the covariance of the (random) state vector $x(t)$ at time $t = t_0$ and $t = t_{\mathsf{f}}$:

$$\mathbb{E}\{x_0 x_0^{\mathrm{T}}\} = \mathbf{\Sigma}_0, \qquad \mathbb{E}\{x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\} = \mathbf{\Sigma}_{\mathsf{f}}, \qquad (4)$$

where $x_0 = x(t_0)$ and $x(t_{\mathsf{f}}) = x_{\mathsf{f}}$.

Problem 1 can be imbedded in a one-parameter family of unconstrained covariance control problems, which are significantly more tractable, both analytically and computationally. The objective in this class of problems is to address the covariance control problem for the stochastic linear system (1) with boundary conditions (4) while minimizing the cost function J_λ, where

$$J_\lambda(u(\cdot;\lambda); t_0, t_{\mathsf{f}}) := \mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} u(t;\lambda)^{\mathrm{T}} u(t;\lambda)\mathrm{d}t\} + \lambda h(x;\lambda), \qquad (5)$$

for a given $\lambda \geq 0$. Note that the hard integral state inequality constraint $h(x) \leq c$, is replaced by a "soft constraint" that is reflected in the new running cost. The exact formulation of the previously described problem is given next.

Problem 2: Parametric, Unconstrained LQGCC Problem: Let $\lambda \geq 0$ and $0 \leq t_0 < t_\mathsf{f} < \infty$ be given. Then find an admissible control input $u^\circ(\cdot;\lambda)$ that minimizes the augmented performance index given in Eq. (5) subject to (1) and the boundary conditions in terms of the covariance of the (random) state vector $x^\circ(t;\lambda)$, which are given in Eq. (4).

After we address Problem 2, we will be able to reduce Problem 1 to a finite-dimensional optimal parameter selection problem. In particular, the latter problem, will furnish an optimal parameter λ° such that the optimal control input that solves Problem 1, provided that the latter problem admits a solution, is given by

$$u^\circ(t) = u^\circ(t;\lambda^\circ), \qquad t \in [t_0, t_\mathsf{f}]. \tag{6}$$

It is important to highlight at this point that, for a given non-negative scalar λ, Problem 2 is essentially an unconstrained LQG problem [11], [14] but with boundary conditions in terms of the covariance of the state vector. Next we characterize the solution to Problem 2 provided that the set of minimizers of J_λ is non-empty (the existence of solutions to Problem 2 will be revisited later on).

Proposition 1: Let $\lambda \geq 0$ and $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_\mathsf{f} \in \mathbb{P}_n$ be given. Then, if Problem 2 admits a solution, an optimal control law that solves this problem will be given by

$$u^\circ(t;\lambda) = -\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda)x^\circ(t;\lambda), \tag{7}$$

where $\{x^\circ(t;\lambda),\ t \in [t_0, t_\mathsf{f}]\}$ denotes the stochastic state process generated by the optimal stochastic input process $\{u^\circ(t;\lambda),\ t \in [t_0, t_\mathsf{f}]\}$ and $\mathbf{S}(\cdot;\lambda)$ satisfies the following Riccati (matrix) equation:

$$\begin{aligned}-\dot{\mathbf{S}}(t;\lambda) = \lambda\mathbf{Q}(t) + \mathbf{S}(t;\lambda)\mathbf{A}(t) + \mathbf{A}(t)^\mathrm{T}\mathbf{S}(t;\lambda)\\ - \mathbf{S}(t;\lambda)\mathbf{B}(t)\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda),\end{aligned} \tag{8}$$

with boundary condition $\mathbf{S}(t_\mathsf{f};\lambda) = \mathbf{S}_\mathsf{f}(\lambda)$, where $\mathbf{S}_\mathsf{f}(\lambda) \in \mathbb{SP}_n$ is such that the covariance of the state of the system driven by the control input $u^\circ(\cdot;\lambda)$, which is denoted by $\mathbf{\Sigma}(t;\lambda) := \mathbb{E}\{x^\circ(t;\lambda)(x^\circ(t;\lambda))^\mathrm{T}\}$ and evolves according to the following first-order linear differential matrix equation:

$$\begin{aligned}\dot{\mathbf{\Sigma}}(t;\lambda) = (\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda))\mathbf{\Sigma}(t;\lambda)\\ + \mathbf{\Sigma}(t;\lambda)(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda))^\mathrm{T}\\ + \mathbf{B}(t)\mathbf{B}(t)^\mathrm{T},\end{aligned} \tag{9}$$

satisfies at time $t = 0$ and $t = t_\mathsf{f}$ the following boundary conditions:

$$\mathbf{\Sigma}(t_0;\lambda) = \mathbf{\Sigma}_0, \qquad \mathbf{\Sigma}(t_\mathsf{f};\lambda) = \mathbf{\Sigma}_\mathsf{f}. \tag{10}$$

Proof: Let $\mathbf{S}(t;\lambda)$ be the solution to the Riccati Eq. (9). Then, in view of Lemma 7.1 and its proof [14, pg. 287–289], we have that

$$\begin{aligned}\int_{t_0}^{t_\mathsf{f}} \left[u(t)^\mathrm{T}u(t) + \lambda x(t)^\mathrm{T}\mathbf{Q}(t)x(t)\right]\mathrm{d}t =\\ + \int_{t_0}^{t_\mathsf{f}} |u(t) + \mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda)x(t)|^2\mathrm{d}t\\ + \int_{t_0}^{t_\mathsf{f}} \mathrm{tr}(\mathbf{S}(t;\lambda)\mathbf{C}(t)\mathbf{C}(t)^\mathrm{T})\mathrm{d}t\\ + \int_{t_0}^{t_\mathsf{f}} \mathrm{d}w(t)^\mathrm{T}\mathbf{C}(t)^\mathrm{T}\mathbf{S}(t;\lambda)x(t)\\ + \int_{t_0}^{t_\mathsf{f}} x(t)^\mathrm{T}\mathbf{S}(t;\lambda)\mathbf{C}(t)\mathrm{d}w(t)\\ - x(t_\mathsf{f})^\mathrm{T}\mathbf{S}(t_\mathsf{f};\lambda)x(t_\mathsf{f}) + x(t_0)^\mathrm{T}\mathbf{S}(t_0;\lambda)x(t_0),\end{aligned}$$

which in turn implies

$$\begin{aligned}\mathbb{E}\{\int_{t_0}^{t_\mathsf{f}} \left[u(t)^\mathrm{T}u(t) + \lambda x(t)^\mathrm{T}\mathbf{Q}(t)x(t)\right]\mathrm{d}t\} =\\ \mathbb{E}\{\int_{t_0}^{t_\mathsf{f}} |u(t) + \mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda)x(t)|^2\mathrm{d}t\}\\ + \int_{t_0}^{t_\mathsf{f}} \mathrm{tr}(\mathbf{S}(t;\lambda)\mathbf{C}(t)\mathbf{C}(t)^\mathrm{T})\mathrm{d}t\\ - \mathrm{tr}(\mathbf{S}(t_\mathsf{f};\lambda)\mathbf{\Sigma}_\mathsf{f}) + \mathrm{tr}(\mathbf{S}(t_0;\lambda)\mathbf{\Sigma}_0),\end{aligned} \tag{11}$$

where we have used the fact that

$$\begin{aligned}\mathbb{E}\{x(t)^\mathrm{T}\mathbf{S}(t;\lambda)x(t)\} = \mathbb{E}\{x(t)^\mathrm{T}\}\mathbf{S}(t;\lambda)\mathbb{E}\{x(t)\}\\ + \mathrm{tr}(\mathbf{S}(t;\lambda)\mathbf{\Sigma}(t;\lambda))\end{aligned} \tag{12}$$

together with $\mathbb{E}\{x(t_0)\} = \mathbb{E}\{x(t_\mathsf{f})\} = 0$, and the fact that, formally, $\mathbb{E}\{\mathrm{d}w(t)\} = 0$ (given that the increment $\mathrm{d}w(t)$ of a Brownian motion process follows a Gaussian distribution with zero mean and covariance $\mathrm{d}t\,\mathbf{I}$, that is, $\mathrm{d}w(t) \sim \mathcal{N}(0, \mathrm{d}t\,\mathbf{I})$ [15]), for $t \in [t_0, t_\mathsf{f}]$. Eq. (7) follows readily from (11). The fact that the optimal control satisfies Eq. (7) in turn allows to show by direct computation that Eq. (9) also holds true. (Note that we have skipped a number of steps in this proof, which are based on standard arguments and techniques that can be found in the literature of stochastic linear-quadratic optimal control [11], [14]). ■

In light of Proposition 1 and its proof, we have that for a given $\lambda \geq 0$, the optimal cost of Problem 2, $J^\circ(\lambda) := J_{\lambda^\circ}(u(\cdot;\lambda^\circ))$ satisfies the following equation:

$$\begin{aligned}J^\circ(\lambda) = \int_{t_0}^{t_\mathsf{f}} \mathrm{tr}(\mathbf{C}(t)^\mathrm{T}\mathbf{S}(t;\lambda)\mathbf{C}(t))\mathrm{d}t\\ + \mathrm{tr}\left(\mathbf{S}(t_0;\lambda)\mathbf{\Sigma}_0 - \mathbf{S}(t_\mathsf{f};\lambda)\mathbf{\Sigma}_\mathsf{f}\right).\end{aligned} \tag{13}$$

In light of Proposition (1), the main challenge for solving Problem 2 is for the state covariance $\mathbf{\Sigma}(\cdot;\lambda)$ to satisfy the prescribed boundary conditions at time $t = t_0$ and $t = t_\mathsf{f}$, which are given in (10). As is highlighted in [7], [8], this is a challenging problem given that the Riccati equations (8) and (9) are coupled. More precisely, (9) depends on both $\mathbf{S}(t;\lambda)$ and $\mathbf{\Sigma}(t;\lambda)$ whereas (8) depends, at a first glance, only on $\mathbf{S}(t;\lambda)$. Unfortunately, it turns out that the system of the two Riccati equations is not in triangular form because the boundary condition for Eq. (8) is not prescribed; instead, it has to be chosen so that the state covariance $\mathbf{\Sigma}(\cdot;\lambda)$ satisfy the prescribed boundary conditions at time $t = t_0$ and $t = t_{\mathsf{f}}$. As is shown in [7], [8], under the assumption that $\mathbf{\Sigma}_0 \in \mathbb{P}_n$, which implies that the state covariance matrix $\mathbf{\Sigma}(t;\lambda)$ belongs to $\mathbb{P}_n$ for all $[t_0, t_{\mathsf{f}}]$ as implied immediately from (9) whose initial condition is a matrix in $\mathbb{P}_n$), the time-varying matrix $\mathbf{H}(t;\lambda) := \mathbf{\Sigma}(t;\lambda)^{-1} - \mathbf{S}(t;\lambda)$ is well defined for all $t \in [t_0, t_{\mathsf{f}}]$. Then, in light of the identity $\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(t;\lambda)^{-1} = -\mathbf{H}(t;\lambda)^{-1}\frac{\mathrm{d}}{\mathrm{d}t}\mathbf{H}(t;\lambda)\mathbf{H}(t;\lambda)^{-1}$, it is easy to show that

$$
\begin{aligned}
-\dot{\mathbf{H}}(t;\lambda) = \mathbf{A}(t)^{\mathrm{T}}\mathbf{H}(t;\lambda) + \mathbf{H}(t;\lambda)\mathbf{A}(t) \\
+ \mathbf{H}(t;\lambda)\mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}}\mathbf{H}(t;\lambda) - \lambda\mathbf{Q}(t) \\
- (\mathbf{\Sigma}(t;\lambda) + \mathbf{H}(t;\lambda))(\mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}} - \mathbf{C}(t)\mathbf{C}(t)^{\mathrm{T}}) \\
\times (\mathbf{\Sigma}(t;\lambda) + \mathbf{H}(t;\lambda)).
\end{aligned} \tag{14}
$$

As explained in [8], the system of the two coupled Riccati equations (9) and (14), corresponds to a nonlinear Schrödinger system. The solvability of such systems cannot be determined easily in the more general case in which the matrices $\mathbf{B}(t)$ and $\mathbf{C}(t)$ do not coincide for all times t. In the latter case, one may have to confine the search to suboptimal solutions of (9) and (14), which can be characterized numerically via known convex optimization techniques.

In the special case when $\mathbf{B}(t) = \mathbf{C}(t)$, for $t \in [t_0, t_{\mathsf{f}}]$ (identical input and noise channels), that is, when the noise affects the control system through the input channels, Eq. (14) reduces to

$$
\begin{aligned}
-\dot{\mathbf{H}}(t;\lambda) = \mathbf{A}(t)^{\mathrm{T}}\mathbf{H}(t;\lambda) + \mathbf{H}(t;\lambda)\mathbf{A}(t) \\
+ \mathbf{H}(t;\lambda)\mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}}\mathbf{H}(t;\lambda) - \lambda\mathbf{Q}(t),
\end{aligned} \tag{15}
$$

which is now decoupled from $\mathbf{S}(t;\lambda)$. However, even though, in the latter case, (9) and (15) may appear to form a system of decoupled Riccati matrix equations, this is actually incorrect given that these two equations are still coupled via their boundary conditions. In particular, the boundary conditions (4) imply that

$$\mathbf{\Sigma}_0^{-1} = \mathbf{S}(t_0;\lambda) + \mathbf{H}(t_0;\lambda) \tag{16a}$$

$$\mathbf{\Sigma}_{\mathsf{f}}^{-1} = \mathbf{S}(t_{\mathsf{f}};\lambda) + \mathbf{H}(t_{\mathsf{f}};\lambda). \tag{16b}$$

In the unconstrained case, that is, when $\lambda = 0$, and when $\mathbf{B}(t) = \mathbf{C}(t)$, for all $t \in [t_0, t_{\mathsf{f}}]$, one can actually obtain solvability conditions as well as closed form expressions for the boundary conditions of $\mathbf{S}(t_0;\lambda)$ and $\mathbf{H}(t_0;\lambda)$, when the problem admits a solution, as was recently shown in [7]. In the constrained case, however, the existence of the term $\lambda\mathbf{Q}(t)$ destroys the homogeneity of the Riccati matrix equations (8) and (15) in terms of $\mathbf{S}(t;\lambda)$ and $\mathbf{H}(t;\lambda)$, respectively. Consequently, one can no more reduce the system of Riccati equations (8) and (15) into a decoupled system of first order linear matrix equations in terms of $\mathbf{S}(t;\lambda)^{-1}$ and $\mathbf{H}(t;\lambda)^{-1}$, in contradistinction with [7].

III. Reduction of the Parametric, Unconstrained LQGCC Problem to a Minimum Control Effort Covariance Problem

In this section, we will show how to associate Problem 2 with the finite-horizon minimum effort covariance control problem addressed in [7], [8]. In a nutshell, the minimum effort covariance control problem addressed in these references corresponds to Problem 1 in the special case when $\lambda = 0$. The cost function in this case is the expected value of the minimum control effort and is denoted by J_0, that is,

$$J_0(u_0(\cdot); t_0, t_{\mathsf{f}}) = \mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} u(t)^{\mathrm{T}} u(t)\mathrm{d}t\}, \tag{17}$$

and the corresponding minimum effort control input at time t, which is denoted by $u^{\circ}(t;0)$, is given by

$$u^{\circ}(t;0) := -\mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t;0)x^{\circ}(t;0), \tag{18}$$

where $\mathbf{S}(t;0)$, $t \in [t_0, t_{\mathsf{f}}]$, denotes the solution to the Riccati equation (8), when $\lambda = 0$, with boundary condition $\mathbf{S}(t_{\mathsf{f}};0) = \mathbf{S}_{\mathsf{f}}(0)$. Note that the matrix $\mathbf{S}_{\mathsf{f}}(0) \in \mathbb{PS}_n$ is chosen such that the covariance $\mathbf{\Sigma}(t;0) := \mathbb{E}\{x^{\circ}(t;0)(x^{\circ}(t;0))^{\mathrm{T}}\}$ satisfies the boundary conditions given in Eq. (10) at time $t = t_0$ and $t = t_{\mathsf{f}}$. We proceed by presenting a time-varying input transformation that will allow us to associate the parametric unconstrained LQGCC problem with a minimum control effort covariance control problem with the same boundary conditions on the state covariance but subject to stochastic (linear) dynamic constraints that will be different from those of the unconstrained LQGCC problem.

Theorem 1: Let $\lambda \geq 0$ and let $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{P}_n$ be given. In addition, suppose that there exists $\mathbf{S}_{\mathsf{f}} \in \mathbb{PS}_n$ such that the Riccati Equation (8) admits a symmetric solution $\mathbf{S}(t;\lambda)$ that is well-defined for all $t \in [t_0, t_{\mathsf{f}}]$ and satisfies the boundary condition $\mathbf{S}(t_{\mathsf{f}};\lambda) = \mathbf{S}_{\mathsf{f}}$. Then, the control input $u^{\circ}(\cdot;\lambda)$ given in (18) solves Problem 2 with boundary conditions (4) if, and only if, the problem of minimizing the performance index

$$J_0(v(\cdot); t_0, t_{\mathsf{f}}) := \mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} v(t)^{\mathrm{T}} v(t)\mathrm{d}t\} \tag{19}$$

subject to the dynamic constraints:

$$\mathrm{d}x(t) = \tilde{\mathbf{A}}(t)x(t)\mathrm{d}t + \mathbf{B}(t)v(t)\mathrm{d}t + \mathbf{C}(t)\mathrm{d}w(t), \tag{20}$$

where $\tilde{\mathbf{A}}(t) := \mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t;\lambda)$, for $t \in [t_0, t_{\mathsf{f}}]$, and the boundary conditions given in Eq. (4), admits an optimal solution $v^{\circ}(\cdot)$.

Proof: After we apply the following input transformation:

$$v(t) = u(t) + \mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t;\lambda)x(t), \quad t \in [t_0, t_{\mathsf{f}}], \tag{21}$$

to Equation (11), we take

$$
\begin{aligned}
\mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} \left[u(t)^{\mathrm{T}}u(t) + \lambda x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t)\right]\mathrm{d}t\} = \\
\mathbb{E}\{\int_{t_0}^{t_{\mathsf{f}}} v(t)^{\mathrm{T}}v(t)\mathrm{d}t\} + \int_{t_0}^{t_{\mathsf{f}}} \mathrm{tr}(\mathbf{S}(t;\lambda)\mathbf{C}(t)\mathbf{C}(t)^{\mathrm{T}})\mathrm{d}t \\
- \mathrm{tr}(\mathbf{S}(t_{\mathsf{f}};\lambda)\mathbf{\Sigma}_{\mathsf{f}}) + \mathrm{tr}(\mathbf{S}(t_0;\lambda)\mathbf{\Sigma}_0),
\end{aligned} \tag{22}
$$

where $\mathbf{S}(t;\lambda)$ is the solution of (8), which is well defined for all $t \in [t_0, t_{\mathsf{f}}]$, by hypothesis. The substitution $u(t) = v(t) - \mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t;\lambda)x(t)$ in Eq. (1) yields the stochastic linear system given by (20). Now, the last two terms in (22) cannot be affected directly by the control input and the result follows. ■

Theorem 1, which associates the unconstrained LQGCC problem (for a given $\lambda \geq 0$) to the minimum effort covariance control problem, has significant practical value, in the light of the recent results presented in [7], [8]. Note that the proposed input transformation (21) requires only the solution to the Riccati equation (8) that is well defined in $[t_0, t_\mathsf{f}]$ for an appropriately chosen boundary condition $\mathbf{S}_\mathsf{f} \in \mathbb{PS}_n$. The matrix $\mathbf{S}_\mathsf{f}$ is irrelevant to enforcing the boundary conditions (10) in terms of the covariance matrix $\mathbf{\Sigma}(\cdot;\lambda)$ of Problem 2. The boundary conditions (10) in terms of the covariance matrices will be accounted only in the solution to the minimum effort covariance control problem subject to the new stochastic linear dynamic constraints (20), which are derived after the application of the time-varying input transformation (21).

To see the practical benefit of associating Problem 2 with the minimum effort LQGCC problem, let us consider the case when $\mathbf{B}(t) = \mathbf{C}(t)$ for all $t \in [t_0, t_\mathsf{f}]$, in which the two Riccati equations, namely Eq. (9) and Eq. (15) are only coupled via their boundary conditions, as we have already explained. However, as we have previously underlined, in this case both Riccati equations are non-homogeneous due to the presence of the term $\lambda\mathbf{Q}(t)$. This fact does not allow us to reduce Eq. (9) and Eq. (15) to a system of first-order linear (differential) matrix equations as in [7]. By following the approach proposed in this section, Problem 2 will be reduced to a minimum effort covariance control problem subject to the new stochastic linear dynamic constraints given in (20), which are determined by the time-varying input transformation defined in Theorem 2. The important nuance here is that this transformation does not affect the $\mathbf{B}(t)$ matrix, in other words, the $\mathbf{B}(t)$ matrix in (1) and (20) is exactly the same. Therefore, if it is true that $\mathbf{B}(t) = \mathbf{C}(t)$, for all $t \in [t_0, t_\mathsf{f}]$, for the optimal covariance Problem 2, in which the stochastic dynamic constraints are given in Eq. (1), then the same holds true for the minimum effort covariance control problem, which is equivalent to Problem 2 and whose stochastic dynamic constraints are given by (20). Therefore, in principle, we will be able to find the solution to Problem 2 in closed form as in [7]. Another advantage of associating Problem 2 with a minimum effort covariance control problem via the time-varying input transformation is that the question of existence of solutions to Problem 2 can be answered in a more straightforward way.

Corollary 1: Suppose that $\mathbf{A}(t) = \mathbf{B}(t)$, for all $t \in [t_0, t_\mathsf{f}]$ and let $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_\mathsf{f} \in \mathbb{P}_n$ be given. Let $\tilde{\mathbf{A}}(t) := \mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t_\mathsf{f};\lambda)$, for $t \in [t_0, t_\mathsf{f}]$, where $\mathbf{S}(t_\mathsf{f};\lambda)$ is a symmetric solution to the Riccati Equation (8) for some terminal condition $\mathbf{S}_\mathsf{f}(\lambda) \in \mathbb{PS}_n$ that is well defined for all $t \in [t_0, t_\mathsf{f}]$. If the deterministic time-varying linear system

$$\dot{x}(t) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) \tag{23}$$

is uniformly controllable over $[t_0, t_\mathsf{f}]$, then the LQGCC problem subject to the stochastic dynamic constraints given in (1) always admits an optimal solution.

Proof: The proof of this corollary is based on the classic result (see, for example, Theorem 5.5.2 in [16]), which states that the uniform controllability over $[t_0, t_\mathsf{f}]$ of the (deterministic) time-varying linear system (23) implies the uniform controllability over $[t_0, t_\mathsf{f}]$ of the deterministic time-varying linear system

$$\dot{x}(t) = \tilde{\mathbf{A}}(t)x(t) + \mathbf{B}(t)v(t),$$

where $v(t) := u(t) - \mathbf{K}(t)x(t)$, $\tilde{\mathbf{A}}(t) := \mathbf{A}(t) + \mathbf{B}(t)\mathbf{K}(t)$, for $t \in [t_0, t_\mathsf{f}]$ and for any $\mathbf{K}(\cdot) \in \mathcal{C}([t_0, t_\mathsf{f}]; \mathbb{R}^{m\times n})$. Now for $\mathbf{K}(t) = -\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t;\lambda)$ and in light of Theorem 8 from [7], we have that the problem of minimizing J_0 subject to (20) and the boundary conditions given in Eq. (4) admits a solution. Equivalently, by virtue of Theorem 1, Problem 2 admits a solution and the proof is complete. ■

IV. Optimal Parameter Selection Problem

In this section, we address the optimal parameter selection problem that will allow us to find the optimal solution to Problem 1, which we formulate next.

Problem 3: Find the parameter λ° that maximizes the objective function $\phi : [0, \infty) \mapsto \mathbb{R}$, where $\phi(\lambda) := J^\circ(\lambda) - \lambda c$, provided that the set of local maximizers of $\phi(\cdot)$ is non-empty.

Based on standard results from the theory of Lagrangian duality [17], it is easy to show using similar arguments as in [13] that the parameter λ° that solves Problem 3, which is the maximizer of $\phi(\lambda)$ over $[0, \infty)$, will furnish the optimal control u° that solves the constrained LQGCC problem (Problem 1). In particular, the optimal control input $u^\circ(\cdot)$ that solves Problem 1 will satisfy the following equation: $u^\circ(t) = u^\circ(t;\lambda^\circ)$, for $t \in [t_0, t_\mathsf{f}]$, where $u^\circ(\cdot;\lambda^\circ)$ solves Problem 2 for $\lambda = \lambda^\circ$.

Theorem 2: Suppose that for any $\lambda \geq 0$, there is an admissible pair $(x(\cdot;\lambda), u(\cdot;\lambda))$ such that $h(x(\cdot;\lambda)) \leq c$. Then, Problem 3 admits a solution $\lambda^\circ \geq 0$ such that $u^\circ(t) = u^\circ(t;\lambda^\circ)$, for $t \in [t_0, t_\mathsf{f}]$.

Proof: The proof of this theorem is similar to that in Theorem 4.1 in [13]. We simply highlight the main ideas behind it. In particular, under the assumption of the existence of an admissible pair $(x(\cdot;\lambda), u(\cdot;\lambda))$ for some $\lambda \geq 0$, Problem 3 always admits a solution. Now, it follows from standard arguments based on the theory of Lagrangian duality [17] that the existence of a solution, λ°, to Problem 3 implies the existence of a solution to the constrained LQGCC problem and, in addition, $u^\circ(t) = u^\circ(t;\lambda^\circ)$, for $t \in [t_0, t_\mathsf{f}]$. ■

Ref. [13] also presents a numerical algorithm for the computation of the gradient $\frac{\partial}{\partial\lambda}J^\circ(\lambda)$ that can be used in, for instance, a gradient descent type algorithm. This class of algorithms will asymptotically converge to a critical point of $\phi(\lambda)$, which always exists in our case, in light of Theorem 2. It should be noted here that if $\lambda^\circ = 0$, then the solution to Problems is identical to the minimum effort covariance control problem; in other words, the integral inequality constraint remains inactive in this case.

For our problem, the process of computing the gradient of $J(\lambda)$ which will in turn allow us to characterize the critical points of $\phi(\lambda)$ is straightforward, given that we have considered a single integral state constraint. In particular, under the assumption that the gradient $\frac{\partial}{\partial\lambda}\mathbf{S}(t_0;\lambda)$ is well defined for all $\lambda > 0$ and for all $t \in [t_0, t_{\mathsf{f}}]$, one can directly compute the gradient $\frac{\partial}{\partial \lambda} J^{\circ}(\lambda)$ based on (13). In particular, we have

$$\begin{aligned}\tfrac{\partial}{\partial \lambda} J^{\circ}(\lambda) = &\int_{t_0}^{t_{\mathsf{f}}} \operatorname{tr}(\mathbf{C}(t)^{\mathrm{T}} \tfrac{\partial}{\partial \lambda} \mathbf{S}(t;\lambda) \mathbf{C}(t)) \mathrm{d}t \\ &+ \left(\tfrac{\partial}{\partial \lambda} \mathbf{S}(t_0;\lambda) \boldsymbol{\Sigma}_0 - \tfrac{\partial}{\partial \lambda} \mathbf{S}(t_{\mathsf{f}};\lambda) \boldsymbol{\Sigma}_{\mathsf{f}} \right), \qquad (24)\end{aligned}$$

where the gradient $\frac{\partial}{\partial \lambda} \mathbf{S}(t;\lambda)$ satisfies (formally) the following first-order linear (differential) matrix equation:

$$\begin{aligned}-\tfrac{\mathrm{d}}{\mathrm{d}t} \tfrac{\partial}{\partial \lambda} \mathbf{S}(t;\lambda) = &\tfrac{\partial}{\partial \lambda} \mathbf{S}(t;\lambda) \left(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}} \mathbf{S}(t;\lambda) \right) \\ &+ \left(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{B}(t)^{\mathrm{T}} \mathbf{S}(t;\lambda) \right)^{\mathrm{T}} \tfrac{\partial}{\partial \lambda} \mathbf{S}(t;\lambda) \\ &+ \mathbf{Q}(t) \qquad (25)\end{aligned}$$

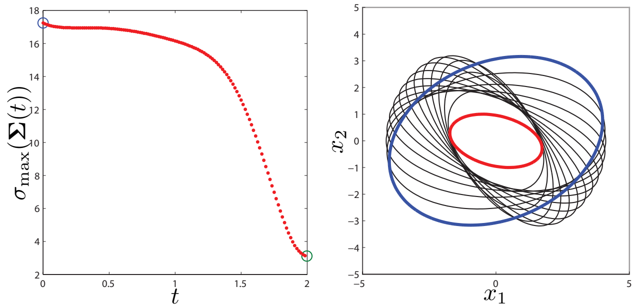
with boundary condition: $\frac{\partial}{\partial \lambda} \mathbf{S}(t_{\mathsf{f}};\lambda) = \frac{\partial}{\partial \lambda} \mathbf{S}_{\mathsf{f}}(\lambda)$. Therefore, the characterization of a critical point of $J^{\circ}(\lambda)$ can be achieved by means of a gradient descent-type algorithm, in which the gradient can be computed analytically via Eq. (24) after plugging the solution to (25) in the former equation.

V. Numerical Simulations

To illustrate the ideas of the previous sections, we present numerical simulations for a simple example. In particular, we consider the finite time horizon, constrained LQGCC problem subject to:

$$\mathrm{d}x_1(t) = x_2(t)\mathrm{d}t, \quad \mathrm{d}x_2(t) = -x_1(t)\mathrm{d}t + u(t)\mathrm{d}t + \mathrm{d}w(t).$$

Note that the above stochastic linear system corresponds to the controlled version of a "stochastic" harmonic oscillator. We assume that the initial state, x_0, at time $t = 0$ is drawn from the normal distribution $\mathcal{N}(0, \boldsymbol{\Sigma}_0)$ and the objective is to drive it to a terminal state, x_{f}, at time $t = 2$, which is drawn from the normal distribution $\mathcal{N}(0, \boldsymbol{\Sigma}_{\mathsf{f}})$, where $\boldsymbol{\Sigma}_0 = \left[\begin{smallmatrix} 16 & 5 \\ 5 & 16 \end{smallmatrix}\right]$, $\boldsymbol{\Sigma}_{\mathsf{f}} = \left[\begin{smallmatrix} 0.1 & 0 \\ 0 & 0.1 \end{smallmatrix}\right]$, while minimizing the cost: $\mathbb{E}\{\int_0^2 u(t)^{\mathrm{T}} u(t) \mathrm{d}t\}$, subject to the mean integral quadratic state constraint: $\mathbb{E}\{\int_0^2 x(t)^{\mathrm{T}} \left[\begin{smallmatrix} 1.5 & 0 \\ 0 & 3 \end{smallmatrix}\right] x(t) \mathrm{d}t\} \leq 70$. Figure 2 illustrates the time evolution of the covariance matrix $\boldsymbol{\Sigma}(t)$ of the closed-loop system, which is driven by the optimal stochastic control that solves the constrained LQGCC problem.



(a) Evolution of $\sigma_{\max}(\boldsymbol{\Sigma}(t))$. (b) Evolution of 1-level sets of $x^{\mathrm{T}} \boldsymbol{\Sigma}(t)^{-1} x$.

Fig. 2. The evolution of $\sigma_{\max}(\boldsymbol{\Sigma}(t))$ with time t allows us to observe the rate at which the covariance $\boldsymbol{\Sigma}(t)$ converges to the $\boldsymbol{\Sigma}_{\mathsf{f}}$ at $t = t_{\mathsf{f}}$. On the other hand, the evolution of the 1-level sets of the quadratic function $x^{\mathrm{T}} \boldsymbol{\Sigma}(t)^{-1} x$, at different time instants, reflects the "path" that the covariance matrix follows until it converges to $\boldsymbol{\Sigma}_{\mathsf{f}}$ at $t = t_{\mathsf{f}}$. In this figure, the blue and the red ellipsoids correspond to the 1-level sets of $x^{\mathrm{T}} \boldsymbol{\Sigma}_0^{-1} x$ and $x^{\mathrm{T}} \boldsymbol{\Sigma}_{\mathsf{f}}^{-1} x$, respectively.

VI. Conclusion

In this work, we have addressed the problem of steering the covariance of the uncertain state of a stochastic linear system subject to (hard) state integral constraints using a stochastic optimal control framework. By building upon some recent results on the finite time horizon covariance control problem, we have developed a more general framework that can account for the presence of mean integral quadratic state constraints. We have addressed this constrained LQGCC problem by imbedding it into a one-parameter family of unconstrained LQGCC problems, which can in turn be directly associated with the minimum effort covariance control problem, which has been addressed recently in the literature. In this way, we have achieved to reduce the original constrained covariance control problem to a finite-dimensional optimal parameter selection problem, which can be addressed via, for instance, gradient descent type algorithms. In the future, we plan to study LQGCC problems in which the covariance of the state of the system has to visit a sequence of prescribed positive definite matrices.

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