Covariance Control for Discrete-Time Stochastic Linear Systems with Incomplete State Information

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***Abstract*— This work deals with a finite-horizon covariance control problem for discrete-time stochastic linear systems with incomplete state information subject to constraints. We show that under the assumption that the class of admissible control policies for this stochastic optimal control problem is comprised of sequences of non-anticipative (causal) control laws that can be expressed as linear combinations of the past and present output measurements of the system, then the covariance control problem can be reduced to a finite-dimensional, deterministic nonlinear program with a convex performance index. In addition, we show that the nonlinear program can be associated with a convex program via a simple relaxation technique that allows us to express the non-convex matrix equality constraint induced by the boundary condition on the terminal state covariance as a positive semi-definite (convex) constraint.**

I. Introduction

We consider a finite-horizon covariance control problem for a discrete-time stochastic linear system with incomplete state information. In other words, we seek for a control policy, that is, a sequence of control laws which are measurable functions of all past and present output measurements, that will transfer the (uncertain) state of the stochastic linear system, which is initially drawn from a known Gaussian distribution, to a goal Gaussian distribution at a given (finite) final stage. In the formulation of the covariance control problem, we will also consider the presence of constraints on the expected value of finite sums of convex quadratic functions of the state and / or the input. The main idea of this work is to associate the stochastic optimal control problem with a tractable, deterministic, finite-dimensional optimization problem.

Literature Review: The covariance control problem was first introduced in the controls community by Hotz and Skelton [1], [2]. This class of problems has been studied in detail in the literature for both continuous-time and discrete-time stochastic linear systems (the reader may refer, for instance, to [3]–[6]). The previous references deal with the infinite-horizon covariance control problem and focus primarily on the controllability problem as well as the parametrization of the set of its solutions. It should be mentioned that in many cases, the solvability conditions on the infinite-horizon covariance control problem can be particularly restrictive.

It is interesting to point out at this point that for many years the finite-horizon covariance control problem has not been explored in the literature until very recently [7], [8].

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The authors of [7], [8] framed the finite-horizon covariance control problem as a stochastic optimal control problem assuming perfect state information. The authors of the same references have also explored a similar stochastic optimal control problem in the case of incomplete state information [9]. The stochastic optimal control problem considered in these references can be viewed as an extension / variation of the standard finite-horizon LQG (Linear Quadratic Gaussian) problem for stochastic linear systems [10]–[12]. The main difference between the standard finite-horizon LQG problem and the covariance control problem lies in their terminal state constraints. In particular, in the finite-horizon covariance control problem, the requirement on the terminal state is that its probability distribution will match a prescribed Gaussian distribution in contrast with the standard finite-horizon LQG problem in which terminal state constraints are enforced indirectly (soft constraints) via an appropriate terminal cost (typically, the expected value of a quadratic function of the terminal state).

Some extensions of the results presented in [7], [8] on the finite-horizon covariance control problem with perfect state information can be found in [13], [14]. In particular, [13] considers the finite-horizon covariance control problem for continuous-time stochastic linear systems subject to integral quadratic state constraints, whereas [14] addresses a finite-horizon covariance control problem with a soft constraint on the terminal state covariance based on the Wasserstein distance between the Gaussian distribution of the terminal state and the goal distribution. Finally, [15] presents a numerical framework for the solution of the finite-horizon covariance control problem for discrete-time stochastic linear systems with complete state information based on convex optimization techniques [16], [17].

Main Contribution: In this work, we address a finite-horizon covariance control problem for discrete-time stochastic linear systems subject to constraints on the expected value of finite sums of (convex) quadratic functions of the state and the input. It is also assumed that the available state information is incomplete (as we have already mentioned, the case of complete state information was considered in [15]). In order to simplify the computation of the solution to the covariance control problem with incomplete state information, we will only consider control policies that correspond to sequences of control laws that can be expressed as linear combinations of the past and present output measurements of the system. This particular assumption, which is very common in the synthesis of output feedback control laws for stochastic linear systems (see [18] and references therein), will allow us to make a connection between the covariance control problem and the rich literature on feedback control design for stochastic discrete-time linear systems [18]–[22]. In particular, we show that, by following a similar approach with the one proposed in [18], the stochastic optimal control problem can be reduced to a (finite-dimensional) deterministic nonlinear program. This nonlinear program can in turn be associated with a convex program via a simple relaxation technique that allows us to express the non-convex matrix equality constraint induced by the boundary condition on the terminal state covariance as a positive semi-definite (convex) constraint. We argue that the control policy associated with the relaxed convex program favors the generation of closed-loop trajectories whose endpoints are expected to be concentrated closer to the mean of the goal Gaussian distribution than a representative sample of points drawn from the latter distribution.

Structure of the paper: The rest of the paper is organized as follows. In Section II, we formulate the covariance control problem as a stochastic optimal control problem with incomplete state information. In Section III, we reduce the stochastic optimal control problem to a deterministic nonlinear program, which we subsequently associate with a convex program via a simple relaxation technique. Finally, Section IV concludes the paper with a summary of remarks.

II. Problem Formulation

A. Notation

We denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of n-dimensional real vectors and $m \times n$ real matrices, respectively. We write $\mathbb{Z}_{\geq 0}$ and $\mathbb{Z}_{>0}$ to denote the set of non-negative integers and strictly positive integers, respectively. Given a probability space $(\Omega, \mathfrak{F}, P)$ and $N \in \mathbb{Z}_{>0}$, we denote by $\ell_2^n(\{0,\dots,N\}; \Omega, \mathfrak{F}, P)$ the Hilbert space of mean square summable random sequences $\{x(t) : t \in \{0,\dots,N\} \subset \mathbb{Z}_{\geq 0}\}$ on $(\Omega, \mathfrak{F}, P)$, where $x(t)$ is a n-dimensional (random) vector at each $t \in \{0,\dots,N\}$. We write $\mathbb{E}[\cdot]$ to denote the expectation operator. Given a square matrix $\mathbf{A}$, we denote its trace by $\mathrm{trace}(\mathbf{A})$. We write $\mathbf{0}$ and $\mathbf{I}$ to denote the zero matrix and the identity matrix, respectively. Furthermore, we denote by $\mathrm{bdiag}(\mathbf{A}_1,\dots,\mathbf{A}_\ell)$ the block diagonal matrix whose diagonal elements are matrices $\mathbf{A}_i$, $i \in \{1,\dots,\ell\}$, of compatible dimensions. Finally, we will denote the convex cone of $n\times n$ symmetric positive definite and positive semi-definite matrices by $\mathbb{P}_n$ and $\overline{\mathbb{P}}_n$, respectively. The space of symmetric $n \times n$ matrices will be denoted by $\mathbb{S}_n$. Given a matrix $\mathbf{A} \in \mathbb{P}_n$ (resp. $\overline{\mathbb{P}}_n$), we will also write $\mathbf{A} \succ \mathbf{0}$ (resp., $\succeq \mathbf{0}$), where $\succ$ (resp. $\succeq$) denotes the Loewner partial order in $\mathbb{P}_n$ (resp. $\overline{\mathbb{P}}_n$).

B. Formulation of the Optimal Covariance Control Problem

For a given $N \in \mathbb{Z}_{>0}$, let $\{\mathbf{A}(t) \in \mathbb{R}^{n\times n} : t \in \{0,\dots,N-1\}\}$, $\{\mathbf{B}(t) \in \mathbb{R}^{n\times m} : t \in \{0,\dots,N-1\}\}$, $\{\mathbf{C}(t) \in \mathbb{R}^{n\times p} : t \in \{0,\dots,N-1\}\}$, $\{\mathbf{G}(t) \in \mathbb{R}^{n\times q} : t \in \{0,\dots,N-1\}\}$ and $\{\mathbf{N}(t) \in \mathbb{R}^{p\times r} : t \in \{0,\dots,N-1\}\}$ be known sequences of real matrices. Let us also consider a discrete-time stochastic linear system described in terms of the following stochastic difference equation and output equation, respectively:

$$x(t+1) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) + \mathbf{G}(t)w(t), \tag{1a}$$

$$y(t) = \mathbf{C}(t)x(t) + \mathbf{N}(t)\nu(t), \tag{1b}$$

for $t \in \{0,\dots,N-1\}$, where the initial state $x(0) = x_0$ is a random vector drawn from a known Gaussian distribution $\mathcal{N}(0, \mathbf{\Sigma}_0)$ with $\mathbf{\Sigma}_0 \in \mathbb{P}_n$. Furthermore, $\{x(t),\ t \in \{0,\dots,N\}\}$, $\{u(t) :\ t \in \{0,\dots,N-1\}\}$ and $\{y(t),\ t \in \{0,\dots,N-1\}\}$, or simply $\{x(t)\}_{t=0}^{N}$, $\{u(t)\}_{t=0}^{N-1}$, and $\{y(t)\}_{t=0}^{N-1}$ denote, respectively, the state, the control input, and the output (random) sequences (or processes) on a complete probability space $(\Omega, \mathfrak{F}, P)$. The control input sequence $\{u(t)\}_{t=0}^{N-1}$ is assumed to belong to $\ell_2^m(\{0,\dots,N-1\}; \Omega, \mathfrak{F}, P)$ and to have finite k-moments for all $k > 0$. We will henceforth refer to a control input sequence that satisfies these properties as *admissible*. In addition, $\{w(t) :\ t \in \{0,\dots,N-1\}\}$ and $\{\nu(t) :\ t \in \{0,\dots,N-1\}\}$, or simply $\{w(t)\}_{t=0}^{N-1}$ and $\{\nu(t)\}_{t=0}^{N-1}$, are sequences of independent normal random variables with zero mean and unit covariance, that is,

$$\mathbb{E}[w(t)] = \mathbf{0}, \qquad \mathbb{E}\left[w(t)w(\tau)^{\mathrm{T}}\right] = \delta(t,\tau)\mathbf{I}, \tag{2a}$$

$$\mathbb{E}[\nu(t)] = \mathbf{0}, \qquad \mathbb{E}\left[\nu(t)\nu(\tau)^{\mathrm{T}}\right] = \delta(t,\tau)\mathbf{I}, \tag{2b}$$

for all $t, \tau \in \{0,\dots,N-1\}$, where $\delta(t,\tau) = 1$ when $t = \tau$ and $\delta(t,\tau) = 0$, otherwise. It is assumed that x_0 and $\{w(t)\}_{t=0}^{N-1}$ as well as $\{w(t)\}_{t=0}^{N-1}$ and $\{\nu(t)\}_{t=0}^{N-1}$ are mutually independent, that is,

$$\mathbb{E}\left[w(t)\nu(\tau)^{\mathrm{T}}\right] = \mathbf{0}, \tag{3a}$$

$$\mathbb{E}\left[w(t)x_0^{\mathrm{T}}\right] = \mathbf{0}, \quad \mathbb{E}\left[\nu(t)x_0^{\mathrm{T}}\right] = \mathbf{0}, \tag{3b}$$

for $t, \tau \in \{0,\dots,N-1\}$.

Our objective is to steer the (uncertain) state $x(t)$ of the stochastic linear system (1a)-(1b), which is drawn at stage $t = 0$ from a given multivariate normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_0)$, with $\mathbf{\Sigma}_0 \in \mathbb{P}_n$, to a terminal state at stage $t = N$ that is drawn from a prescribed multivariate normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_{\mathsf{f}})$, with $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{P}_n$. Note that the mean of the two normal distributions at $t = 0$ and $t = N$ is assumed to be zero, without loss of generality (or perhaps, with minimal loss). In particular, all the solution techniques that we will present in this work can be easily modified to handle the non-zero mean case. Under the zero mean assumption, the boundary conditions in terms of the state covariance at $t = 0$ and $t = N$ can be written as follows:

$$\mathbb{E}\left[x_0 x_0^{\mathrm{T}}\right] = \mathbf{\Sigma}_0, \qquad \mathbb{E}\left[x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\right] = \mathbf{\Sigma}_{\mathsf{f}},$$

where $x_0 = x(0)$ and $x_{\mathsf{f}} = x(N)$. In addition, in order to be able to leverage some powerful techniques from the design of affine / linear controllers for discrete-time stochastic linear systems, we will only be considering control policies in which the control input at stage t depends on the history of the output measurements up to stage t, which is denoted by $\mathcal{Y}_t$ and defined as $\mathcal{Y}_t := \{y(\tau) :\ \tau \in \{0,\dots,N-1\}\}$. Furthermore, we will also consider constraints on the expected value of finite sums of (convex) quadratic functions of the state and the input. Next, we provide the exact formulation of the covariance control problem we just described.

Problem 1: Let $N \in \mathbb{Z}_{>0}$, $c > 0$, and $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_\mathsf{f} \in \mathbb{P}_n$ be given. In addition, assume that for all $t \in \{0, \ldots, N-1\}$ the matrices $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ belong to $\overline{\mathbb{P}}_n$ and $\mathbb{P}_m$, respectively. Furthermore, the matrices $\mathbf{Q_c}(t)$ and $\mathbf{R_c}(t)$ belong to $\overline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}_m$, respectively. Let $\mathcal{Y}_t$ denote the information set of the system (1a)-(1b) at time t, which consists of all the output measurements up to stage t, that is, $\mathcal{Y}_t := \{y(\tau) : \tau \in \{0, \ldots, t\}\}$. In addition, let Π denote the class of admissible control policies $\pi := \{\mu(\mathcal{Y}_t; t) : t \in \{0, \ldots N-1\}\}$, where $\mu(\cdot; t)$ is a causal (non-anticipative) feedback law which maps the random set $\mathcal{Y}_t$ to a random m-dimensional vector; in particular,

$$\mu(\mathcal{Y}_t; t) := \sum_{\tau=0}^{t} \mathbf{K}(t, \tau), y(\tau),$$

for $t \in \{0, \ldots, N-1\}$, where $\mathbf{K}(t,\tau) \in \mathbb{R}^{m \times p}$ for all $(t, \tau) \in \{0, \ldots, N-1\} \times \{0, \ldots, N-1\}$ with $t \geq \tau$. Then, find an optimal control policy $\pi^\circ := \{\mu^\circ(\mathcal{Y}_t; t) : t \in \{0, \ldots N-1\}\} \in \Pi$ that minimizes the performance index:

$$J(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^\mathrm{T}\mathbf{Q}(t)x(t) + u(t)^\mathrm{T}\mathbf{R}(t)u(t)\Big], \tag{4}$$

subject to (1a)-(1b), the following inequality constraint:

$$h(\pi) \leq c,$$

where

$$h(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^\mathrm{T}\mathbf{Q_c}(t)x(t) + u(t)^\mathrm{T}\mathbf{R_c}(t)u(t)\Big], \tag{5}$$

and the boundary conditions in terms of the state covariance of the (random) state vector $x(t)$ of the closed-loop system at the stages $t = 0$ and $t = N$:

$$\mathbb{E}\left[x_0 x_0^\mathrm{T}\right] = \mathbf{\Sigma}_0, \quad \mathbb{E}\left[x_\mathsf{f} x_\mathsf{f}^\mathrm{T}\right] = \mathbf{\Sigma}_\mathsf{f}, \tag{6}$$

where $x_0 = x(0)$ and $x_\mathsf{f} = x(N)$.

Remark 1 Note that in the formulation of Problem 1, we have explicitly required that the optimal control policy, π°, is comprised of control laws that can be written as linear combinations of the past and present output measurements of the system. This requirement reflects an implicit assumption on the validity of the so-called *separation principle* [23] in our problem. The main practical benefit of this approach is that it can allow us to forge direct connections with the literature on the control design of affine / linear controllers for discrete-time stochastic linear systems, which offers many powerful computational tools based on convex optimization techniques [16], [17].

In the absence of the inequality constraint given in (5), that is, when $\mathbf{Q_c}(t) \equiv \mathbf{0}$ and $\mathbf{R_c}(t) \equiv \mathbf{0}$, one can conjecture, based on the results presented in [9] for the same problem but for the continuous-time case, that the optimal control policy $\pi^\circ = \{\mu^\circ(\mathcal{Y}_t; t) : t \in \{0, \ldots, N-1\}\}$ is a sequence of output feedback control laws $\mu^\circ(\mathcal{Y}_t; t)$ with

$$\mu^\circ(\mathcal{Y}_t; t) = \mathbf{K}^\circ(t)\hat{x}(t), \quad t \in \{0, \ldots, N-1\}, \tag{7}$$

where $\hat{x}(t)$ corresponds to the conditional mean of $x(t)$ given $\mathcal{Y}_t$, that is, $\hat{x}(t) := \mathbb{E}\left[x(t)|\mathcal{Y}_t\right]$. The evolution of $\hat{x}$ is determined by the following recursive scheme (Kalman filtering algorithm [24, pp. 174-175]):

$$\begin{aligned} &\hat{x}(t|t-1) = \\ &\qquad (\mathbf{A}(t-1) + \mathbf{B}(t-1)\mathbf{K}^\circ(t-1))\hat{x}(t-1), && \text{(8a)} \\ &\hat{x}(t) = \hat{x}(t|t-1) + \mathbf{\Lambda}^\circ(t)\big(y(t) - \mathbf{C}(t)\hat{x}(t|t-1)\big), && \text{(8b)} \end{aligned}$$

for $t \in \{1, \ldots, N-1\}$ and $\hat{x}(0) = \mathbb{E}(x_0) = \mathbf{0}$, where the optimal estimation gain matrix $\mathbf{\Lambda}^\circ(t)$ is determined by the following recursive scheme:

$$\begin{aligned} &\mathbf{P}(0|0) = \mathbb{E}[x_0 x_0^\mathrm{T}] = \mathbf{\Sigma}_0, && \text{(9a)} \\ &\mathbf{P}(t|t-1) = \mathbf{A}(t-1)\mathbf{P}(t-1|t-1)\mathbf{A}(t-1)^\mathrm{T} \\ &\quad + \mathbf{G}(t-1)\mathbf{G}(t-1)^\mathrm{T}, && \text{(9b)} \\ &\mathbf{\Lambda}^\circ(t) = \mathbf{P}(t|t-1)\mathbf{C}(t)^\mathrm{T} \\ &\quad \times \big[\mathbf{C}(t)\mathbf{P}(t|t-1)\mathbf{C}(t)^\mathrm{T} + \mathbf{N}(t)\mathbf{N}(t)^\mathrm{T}\big]^{-1}, && \text{(9c)} \\ &\mathbf{P}(t|t) = \big[\mathbf{I} - \mathbf{\Lambda}^\circ(t)\mathbf{C}(t)\big]\mathbf{P}(t|t-1), && \text{(9d)} \end{aligned}$$

for $t \in \{1, \ldots, N\}$.

Furthermore, the optimal control gain matrix $\mathbf{K}^\circ(t)$ that appears in (7) and (8a) satisfies the following equation:

$$\mathbf{K}^\circ(t) = -\mathbf{L}(t)\mathbf{A}(t), \tag{10}$$

where $\mathbf{L}(t)$ is given by the following equation:

$$\mathbf{L}(t) = \big(\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t+1)\mathbf{B}(t) + \mathbf{R}(t)\big)^{-1}\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t+1). \tag{11}$$

The matrix $\mathbf{S}(t)$ that appears in the previous equation satisfies the following recursive (Riccati-type) equation:

$$\begin{aligned} \mathbf{S}(t) = \mathbf{Q}(t) + \mathbf{A}(t)^\mathrm{T}\Big(&\mathbf{S}(t+1) - \mathbf{L}(t)^\mathrm{T}\big[\mathbf{B}(t)^\mathrm{T}\mathbf{S}(t+1)\mathbf{B}(t) \\ &+ \mathbf{R}(t)\big]\mathbf{L}(t)\Big)\mathbf{A}(t) \end{aligned} \tag{12}$$

with boundary condition $\mathbf{S}(N) = \mathbf{S}_\mathsf{f}$, where the matrix $\mathbf{S}_\mathsf{f}$ belongs to $\overline{\mathbb{P}}_n$. In addition, the matrix $\mathbf{S}_\mathsf{f}$ is such that the state covariance $\mathbf{\Sigma}(t) := \mathbb{E}\big[x(t)x(t)^\mathrm{T}\big]$ of the closed-loop system, which is driven by the control policy $\pi^\circ = \{\mathbf{K}^\circ(t)\hat{x}(t) : t \in \{0, \ldots, N-1\}\}$, satisfies the boundary conditions given in (6). Now, it is easy to show that

$$\mathbf{\Sigma}(t) = \hat{\mathbf{\Sigma}}(t) + \mathbf{P}(t|t), \tag{13}$$

for $t \in \{0, \ldots, N\}$, where $\mathbf{P}(t|t)$ is the covariance of the estimation error, that is, $\mathbf{P}(t;t) := \mathbb{E}\big[(x(t) - \hat{x}(t))(x(t) - \hat{x}(t))^\mathrm{T}\big]$, which satisfies Eq. (9a)–(9d), and $\hat{\mathbf{\Sigma}}(t)$ is the covariance of the state estimate $\hat{x}(t)$, that is, $\hat{\mathbf{\Sigma}}(t) := \mathbb{E}\big[\hat{x}(t)\hat{x}(t)^\mathrm{T}\big]$. (To establish (13), one simply has to note that the estimation error $x(t) - \hat{x}(t)$ is orthogonal to the estimate $\hat{x}(t)$). It is straightforward to show that the covariance $\hat{\mathbf{\Sigma}}(t)$ of the state estimate $\hat{x}(t)$ satisfies the following recursive Lyapunov (or Stein [25]) equation:

$$\begin{aligned}\hat{\mathbf{\Sigma}}(t+1) = (\mathbf{A}(t)+\mathbf{B}(t)\mathbf{K}^{\circ}(t))\hat{\mathbf{\Sigma}}(t)(\mathbf{A}(t)+\mathbf{B}(t)\mathbf{K}^{\circ}(t))^{\mathrm{T}} \\ + \mathbf{\Lambda}^{\circ}(t)\mathbf{N}(t)\mathbf{N}(t)^{\mathrm{T}}\mathbf{\Lambda}^{\circ}(t)^{\mathrm{T}},\end{aligned} \tag{14}$$

with $\hat{\mathbf{\Sigma}}(0) = \mathbf{0}$, for all $t \in \{0, \ldots, N-1\}$. Let us now denote as $\hat{\mathbf{\Sigma}}(t; \mathbf{S}_{\mathsf{f}})$, where $\hat{\mathbf{\Sigma}}(0; \mathbf{S}_{\mathsf{f}}) = \hat{\mathbf{\Sigma}}(0) = \mathbf{0}$, the solution to Eq. (14) when $\mathbf{K}^{\circ}(t) = \mathbf{K}^{\circ}(t; \mathbf{S}_{\mathsf{f}})$ for a given $\mathbf{S}_{\mathsf{f}} \in \overline{\mathbb{P}}_n$. Then, it follows readily from the previous discussion and (13) that Problem 1 reduces to the solution of the following (implicit) nonlinear algebraic matrix equation:

$$\mathbf{\Sigma}(N) = \hat{\mathbf{\Sigma}}(N; \mathbf{S}_{\mathsf{f}}) + \mathbf{P}(N|N) = \mathbf{\Sigma}_{\mathsf{f}}. \tag{15}$$

Unfortunately, the computation of $\mathbf{S}_{\mathsf{f}} \in \overline{\mathbb{P}}_n$ that solves Eq. (15) can be, in general, a very complex task.

III. Practical Numerical Solution Techniques

In this section, we will first reduce the stochastic optimal control problem, which was formulated in Problem 1, to a deterministic nonlinear program (NLP). The only non-convex element of this NLP will be a matrix equality constraint that results from the boundary condition on the terminal state covariance. Subsequently, we will associate the NLP with a convex program via a simple relaxation technique.

Next, we summarize the key steps for the transcription of Problem 1 (stochastic optimal control problem) into a deterministic, finite-dimensional nonlinear program. The first step is to express the solution to the difference equation (1a) and the corresponding output from equation (1b) in the following compact form:

$$\boldsymbol{x} = \boldsymbol{\mathcal{B}}\boldsymbol{u} + \boldsymbol{\mathcal{G}}\boldsymbol{w} + \boldsymbol{x}_0, \tag{16a}$$
$$\boldsymbol{y} = \boldsymbol{\mathcal{C}}\boldsymbol{x} + \boldsymbol{\mathcal{N}}\boldsymbol{\nu}, \tag{16b}$$

where the vector

$$\boldsymbol{x} := [x(0)^{\mathrm{T}}, \ldots, x(N)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n}$$

corresponds to the sequence of states for $t \in \{0, \ldots, N\}$ and the vectors

$$\begin{aligned}\boldsymbol{u} &:= [u(0)^{\mathrm{T}}, \ldots, u(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}, \\ \boldsymbol{y} &:= [y(0)^{\mathrm{T}}, \ldots, y(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Np}\end{aligned}$$

correspond to the sequence of inputs and outputs for $t \in \{0, \ldots, N-1\}$, respectively. Furthermore, the vectors

$$\begin{aligned}\boldsymbol{w} &:= [w(0)^{\mathrm{T}}, \ldots, w(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nq}, \\ \boldsymbol{\nu} &:= [\nu(0)^{\mathrm{T}}, \ldots, \nu(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nr}\end{aligned}$$

correspond to the sequence of process and measurement noise signals, respectively, for $t \in \{0, \ldots, N-1\}$. In addition, the matrices $\boldsymbol{\mathcal{B}} \in \mathbb{R}^{(N+1)n \times Nm}$, $\boldsymbol{\mathcal{G}} \in \mathbb{R}^{(N+1)n \times Nq}$ satisfy, respectively, the following equations:

$$\boldsymbol{\mathcal{B}} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{B}(0) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{\Phi}(2,1)\mathbf{B}(0) & \mathbf{B}(1) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Phi}(N,1)\mathbf{B}(0) & \mathbf{\Phi}(N,2)\mathbf{B}(1) & \ldots & \mathbf{B}(N-1) \end{bmatrix},$$

$$\boldsymbol{\mathcal{G}} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{G}(0) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{\Phi}(2,1)\mathbf{G}(0) & \mathbf{G}(1) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{\Phi}(N,1)\mathbf{G}(0) & \mathbf{\Phi}(N,2)\mathbf{G}(1) & \ldots & \mathbf{G}(N-1) \end{bmatrix},$$

where

$$\mathbf{\Phi}(t,\tau) := \mathbf{A}(t-1)\ldots\mathbf{A}(\tau), \qquad \mathbf{\Phi}(\tau,\tau) = \mathbf{I},$$

for $(t,\tau) \in \{0, \ldots, N\} \times \{0, \ldots, N\}$ with $t \geq \tau$. Furthermore,

$$\begin{aligned}\boldsymbol{\mathcal{C}} &:= \left[\mathrm{bdiag}(\mathbf{C}(0), \ldots, \mathbf{C}(N-1)), \mathbf{0}\right], \\ \boldsymbol{\mathcal{N}} &:= \mathrm{bdiag}(\mathbf{N}(0), \ldots, \mathbf{N}(N-1)).\end{aligned}$$

Finally, $\boldsymbol{x}_0 := \mathbf{\Gamma}x_0$, where

$$\mathbf{\Gamma} := \begin{bmatrix}\mathbf{I} & \mathbf{\Phi}(1,0)^{\mathrm{T}} & \ldots & \mathbf{\Phi}(N,0)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}}.$$

Since in the formulation of Problem 1, it is explicitly required that $\pi = \{\mu(\mathcal{Y}_t; t) : t \in \{0, \ldots, N-1\}\} \in \Pi$, which implies that the input $u(t) = \mu(\mathcal{Y}_t; t)$ can be written as a linear combination of the elements of $\mathcal{Y}_t$, that is,

$$u(t) = \mu(\mathcal{Y}_t; t) = \sum_{\tau=0}^{t} \mathbf{K}(t,\tau)y(\tau),$$

for $t \in \{0, \ldots, N-1\}$. The previous equation can be written more compactly as follows:

$$\boldsymbol{u} = \boldsymbol{\mathcal{K}}\boldsymbol{y}, \tag{17}$$

where

$$\boldsymbol{\mathcal{K}} := \begin{bmatrix} \mathbf{K}(0,0) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{K}(1,0) & \mathbf{K}(1,1) & \ldots & \mathbf{0} \\ \mathbf{K}(2,0) & \mathbf{K}(2,1) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ddots & \vdots \\ \mathbf{K}(N-1,0) & \mathbf{K}(N-1,1) & \ldots & \mathbf{K}(N-1,N-1) \end{bmatrix}.$$

It follows readily that

$$\boldsymbol{x} = \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\chi}, \tag{18a}$$
$$\boldsymbol{u} = \boldsymbol{\mathcal{U}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\xi}, \tag{18b}$$

where

$$\boldsymbol{\mathcal{X}}_{\boldsymbol{w}} := \boldsymbol{\mathcal{G}} + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}}, \tag{19a}$$
$$\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}} := \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{N}}, \tag{19b}$$
$$\boldsymbol{\mathcal{U}}_{\boldsymbol{w}} := \boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}}, \tag{19c}$$
$$\boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}} := \boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{N}}, \tag{19d}$$

and

$$\boldsymbol{\chi} := \boldsymbol{x}_0 + \boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{x}_0, \tag{20a}$$

$$\boldsymbol{\xi} := \boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}\boldsymbol{\mathcal{C}}\boldsymbol{x}_0. \tag{20b}$$

Note that the inverse of $\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}$ is always well defined given that $\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}}$ turns out to be a block lower triangular matrix whose block diagonal elements are zero matrices. In addition, the performance index can be written as follows:

$$\begin{aligned} J(\pi) &= \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}(t)u(t)\Big] \\ &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\boldsymbol{u}\Big], \end{aligned} \tag{21}$$

where $\boldsymbol{\mathcal{Q}} := \mathrm{diag}(\mathbf{Q}(0), \dots, \mathbf{Q}(N-1), \mathbf{0})$ and $\boldsymbol{\mathcal{R}} := \mathrm{diag}(\mathbf{R}(0), \dots, \mathbf{R}(N-1))$. In view of Eqs. (18a)–(20b), we can write the performance index as follows:

$$\begin{aligned} J(\pi) = \mathbb{E}\Big[&(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\chi})^{\mathrm{T}}\boldsymbol{\mathcal{Q}}(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\chi}) \\ &+ (\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\xi})^{\mathrm{T}}\boldsymbol{\mathcal{R}}(\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\xi})\Big], \end{aligned}$$

which, in view of Eqs. (2a)-(2b) and (3a)-(3b), can be written:

$$\begin{aligned} J(\pi) = \mathrm{trace}&(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\mathcal{Q}}\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}^{\mathrm{T}}) \\ &+ \mathrm{trace}(\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}^{\mathrm{T}} + \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}\boldsymbol{\mathcal{R}}\boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}^{\mathrm{T}}) \\ &+ \mathrm{trace}(\boldsymbol{\mathcal{Q}}\mathbb{E}[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}] + \boldsymbol{\mathcal{R}}\mathbb{E}[\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}]). \end{aligned} \tag{22}$$

As is highlighted in [18], Eq. (22) does not allow us by itself to conclude whether $J(\pi)$, with $\pi \in \Pi$, can be expressed as a convex function of the block lower triangular matrix $\boldsymbol{\mathcal{K}}$. To overcome this problem, we will make use of a bilinear transformation, which was suggested in [18]. This transformation will allow us to express the performance index as a convex function of a new decision variable, which is denoted as $\boldsymbol{\Psi}$ and is defined as follows:

$$\boldsymbol{\Psi} := \boldsymbol{\mathcal{K}}(\mathbf{I} - \boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}}\boldsymbol{\mathcal{K}})^{-1}. \tag{23}$$

Using similar arguments as those in the discussion following Eq. (20b), we conclude that $\boldsymbol{\Psi}$ is always well-defined and it is actually a block lower triangular matrix. In addition, from (23) we have that

$$\boldsymbol{\mathcal{K}} = (\mathbf{I} + \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{B}})^{-1}\boldsymbol{\Psi}, \tag{24}$$

where the right hand side of Eq. (24) is well defined based again on similar arguments as those in the discussion following Eq. (20b). In view of (24), Eqs. (19a)–(19d) and (20a)–(20b) become, respectively,

$$\boldsymbol{\mathcal{X}}_{\boldsymbol{w}} := (\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})\boldsymbol{\mathcal{G}}, \qquad \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}} := \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{N}}, \tag{25a}$$

$$\boldsymbol{\mathcal{U}}_{\boldsymbol{w}} := \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\mathcal{G}}, \qquad \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}} := \boldsymbol{\Psi}\boldsymbol{\mathcal{N}}, \tag{25b}$$

and

$$\boldsymbol{\chi} := (\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})\boldsymbol{x}_0, \quad \boldsymbol{\xi} := \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{x}_0. \tag{26}$$

Thus, we have that

$$\mathbb{E}[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}] = (\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}(\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})^{\mathrm{T}}, \tag{27a}$$

$$\mathbb{E}[\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}] = \boldsymbol{\Psi}\boldsymbol{\mathcal{C}}\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\mathcal{C}}^{\mathrm{T}}\boldsymbol{\Psi}^{\mathrm{T}}, \tag{27b}$$

where in the previous derivations, we have used the following identity

$$\mathbb{E}[\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}] = \boldsymbol{\Gamma}^{\mathrm{T}}\mathbb{E}[x_0x_0^{\mathrm{T}}]\boldsymbol{\Gamma} = \boldsymbol{\Gamma}^{\mathrm{T}}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}. \tag{28}$$

Therefore, in view of Eqs. (22), (25a)–(25b) and (27a)–(27b), and the fact that $\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}$, $\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}$, $\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}$ and $\boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}$ are linear or affine functions of the new decision variable $\boldsymbol{\Psi}$, it follows that the performance index $J(\pi)$, when $\pi \in \Pi$, can be expressed as a convex function of the elements of the new decision variable $\boldsymbol{\Psi}$.

Similarly, we can express $h(\pi)$, when $\pi \in \Pi$, as follows:

$$\begin{aligned} h(\pi) &:= \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}_{\mathbf{c}}(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}_{\mathbf{c}}(t)u(t)\Big] \\ &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_{\boldsymbol{c}}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}_{\boldsymbol{c}}\boldsymbol{u}\Big], \end{aligned} \tag{29}$$

where $\boldsymbol{\mathcal{Q}}_{\boldsymbol{c}} := \mathrm{diag}(\mathbf{Q}_{\mathbf{c}}(0), \dots, \mathbf{Q}_{\mathbf{c}}(N-1), \mathbf{0})$ and $\boldsymbol{\mathcal{R}}_{\boldsymbol{c}} := \mathrm{diag}(\mathbf{R}_{\mathbf{c}}(0), \dots, \mathbf{R}_{\mathbf{c}}(N-1))$, or equivalently,

$$\begin{aligned} h(\pi) &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_{\boldsymbol{c}}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}_{\boldsymbol{c}}\boldsymbol{u}\Big] \\ &= \mathrm{trace}(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{\mathcal{Q}}_{\boldsymbol{c}}\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\mathcal{Q}}_{\boldsymbol{c}}\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}^{\mathrm{T}}) \\ &\quad + \mathrm{trace}(\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}\boldsymbol{\mathcal{R}}_{\boldsymbol{c}}\boldsymbol{\mathcal{U}}_{\boldsymbol{w}}^{\mathrm{T}} + \boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}\boldsymbol{\mathcal{R}}_{\boldsymbol{c}}\boldsymbol{\mathcal{U}}_{\boldsymbol{\nu}}^{\mathrm{T}}) \\ &\quad + \mathrm{trace}(\boldsymbol{\mathcal{Q}}_{\boldsymbol{c}}\mathbb{E}[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}] + \boldsymbol{\mathcal{R}}_{\boldsymbol{c}}\mathbb{E}[\boldsymbol{\xi}\boldsymbol{\xi}^{\mathrm{T}}]), \end{aligned} \tag{30}$$

where in the last derivation, we have used Eqs. (2a)–(2b) and Eqs. (3a)–(3b). Based on very similar arguments with those used to demonstrate that the performance index $J(\pi)$ can be expressed as a convex quadratic function of $\boldsymbol{\Psi}$, we can show that $h(\pi)$, when $\pi \in \Pi$, can be expressed as a convex quadratic function of $\boldsymbol{\Psi}$ as well.

Next, we will express the matrix equality constraint on the terminal state covariance $\mathbb{E}[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}] - \boldsymbol{\Sigma}_{\mathsf{f}} = \mathbf{0}$ in terms of the new decision variable $\boldsymbol{\Psi}$. To this aim, we will first express $\mathbb{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}]$ as a function of $\boldsymbol{\Psi}$. Specifically, in view of Eqs. (2a)–(2b), (3a)–(3b) and (18b), it is straightforward to show that

$$\begin{aligned} \mathbb{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}] &= \mathbb{E}\big[(\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\chi}) \\ &\qquad \times (\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{w} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\nu} + \boldsymbol{\chi})^{\mathrm{T}}\big] \\ &= \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\mathbb{E}[\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}]\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\mathbb{E}[\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}]\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}^{\mathrm{T}} \\ &\qquad + \mathbb{E}[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}] \\ &= \boldsymbol{\mathcal{X}}_{\boldsymbol{w}}\boldsymbol{\mathcal{X}}_{\boldsymbol{w}}^{\mathrm{T}} + \boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}\boldsymbol{\mathcal{X}}_{\boldsymbol{\nu}}^{\mathrm{T}} + \mathbb{E}[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}]. \end{aligned} \tag{31}$$

In view of (25a)-(25b), (27a) and (28), Eq. (31) can be written as follows:

$$\begin{aligned} \mathbb{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}] &= (\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})(\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2} \\ &\qquad \times (\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2}(\mathbf{I} + \boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})^{\mathrm{T}}. \end{aligned} \tag{32}$$

Now, because $x_{\mathsf{f}} = x(N) = \boldsymbol{\mathcal{P}}_N\boldsymbol{x}$, where

$$\boldsymbol{\mathcal{P}}_N := [\mathbf{0} \dots \mathbf{I}] \in \mathbb{R}^{n \times (N+1)n},$$

we can write

$$\mathbb{E}[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}] = \boldsymbol{\mathcal{P}}_N\mathbb{E}[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}]\boldsymbol{\mathcal{P}}_N^{\mathrm{T}} = \mathbf{Z}\mathbf{Z}^{\mathrm{T}},$$

where $\mathbf{Z} := \boldsymbol{\mathcal{P}}_N(\mathbf{I}+\boldsymbol{\mathcal{B}}\boldsymbol{\Psi}\boldsymbol{\mathcal{C}})(\boldsymbol{\mathcal{G}}\boldsymbol{\mathcal{G}}^{\mathrm{T}}+\boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2}$. Note that $\mathbf{Z}$ is itself an affine function of $\boldsymbol{\Psi}$. In addition, the symmetric matrix-valued function $\boldsymbol{f}(\cdot) : \mathbb{R}^{n\times n} \to \mathbb{S}_n$, where $\boldsymbol{f}(\mathbf{Z}) := \mathbf{Z}\mathbf{Z}^{\mathrm{T}} - \boldsymbol{\Sigma}_{\mathrm{f}}$ is convex in the sense of Definition 6.6.44 given in [26]. Specifically, $\boldsymbol{f}$ is a convex quadratic function of $\boldsymbol{\Psi}$ as the composition of a convex quadratic function and an affine function of $\boldsymbol{\Psi}$. However, the $n(n+1)/2$ scalar equality constraints that are derived from the matrix equality constraint $\boldsymbol{f}(\boldsymbol{\Psi}) = \mathbf{0}$ are not necessarily convex. Therefore, in general, Problem 1 can only be associated with an NLP, whose numerical solution can be a complex task in general. To overcome this problem, we will employ a simple convex relaxation technique [16]. In particular, we will replace the equality constraint $\boldsymbol{f}(\mathbf{Z}) = \mathbf{0}$ with the following constraint $\boldsymbol{f}(\mathbf{Z}) \preceq \mathbf{0}$, which is convex. To see this, it suffices to note that the inequality $\boldsymbol{f}(\mathbf{Z}) \preceq \mathbf{0}$, which is equivalent to $\boldsymbol{\Sigma}_{\mathrm{f}} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}} \succeq \mathbf{0}$, can be written as a (convex) positive semi-definite constraint:

$$\boldsymbol{\mathcal{X}} := \begin{bmatrix} \boldsymbol{\Sigma}_{\mathrm{f}} & \mathbf{Z} \\ \mathbf{Z}^{\mathrm{T}} & \mathbf{I} \end{bmatrix} \succeq \mathbf{0}. \tag{33}$$

In the previous derivation, we have used the fact that $\boldsymbol{\Sigma}_{\mathrm{f}} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}$ is the Schur complement of $\mathbf{I}$ in $\boldsymbol{\mathcal{X}}$.

It is interesting to note that if one views the state covariance at stage $t = N$ as a measure of the dispersion of the endpoints of a representative sample of state trajectories of the close-loop system from the mean of the goal normal distribution, then the proposed relaxation leads to desirable results from a practical point of view.

IV. Conclusion

In this work, we have addressed a finite-horizon covariance control problem for discrete-time stochastic linear systems for the case when the information about the state of the system is incomplete. In the formulation of this stochastic optimal control problem, we have also enforced input and state constraints in the form of explicit upper bounds on the expected value of certain finite sums of (convex) quadratic functions of the state and / or the control input. We have shown that by restricting our attention to control policies that correspond to sequences of non-anticipative control laws that can be expressed as linear combinations of the past and present output measurements of the system, the stochastic optimal control problem can be associated with a finite-dimensional deterministic nonlinear program. The latter problem can be in turn reduced to a convex program via a simple relaxation technique. In the future, we plan to use some of the ideas and numerical techniques presented herein in order to develop numerical algorithms for the solution of covariance control problems for discrete-time stochastic *nonlinear* systems with incomplete state information and subject to input and state constraints.

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