

Optimal Guidance of the Isotropic Rocket in a Partially Uncertain Flow Field

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Abstract—In this paper, we address the problem of characterizing the optimal strategy for guiding the so-called Isotropic Rocket to a fixed target in the presence of a partially known flowfield. We reformulate the guidance problem into an equivalent two-player pursuit evasion game, where the evader has a modified vectogram. First, we characterize the optimal strategies for both players, assuming that the evader is capturable. Then, we derive the necessary condition for capture, and visualize the capturability envelope using numerical simulations in which we compute the level sets of the optimal value function (isochrones). In addition, we consider a more general case of the game, where the pursuer is affected by friction (or drag). In that case, we identify the capture zones from geometric properties of the isochrones.

I. INTRODUCTION

Pursuit-evasion scenarios are ubiquitous in natural and man-made interactions. The most direct examples are found in wildlife and warfare; however, the analogy can be extended to nearly any situation where two or more parties have conflicting interests. In this paper, we study the optimal guidance problem for the Isotropic Rocket to a fixed target in the presence of a partially known flowfield, which is an extension of [1]. This problem can be interpreted as a two-player zero-sum linear pursuit-evasion game (PEG) that is a variation of the standard Isotropic Rocket problem [2]. In the standard problem, the evader has single integrator dynamics with constant speed. In the problem treated herein, we assume that the evader's velocity can be decomposed into two components: one that is controlled by the evader, and the other, a known constant component (analogous to known constant wind). Our objective is to determine the time of capture and capture condition for the modified game, and to investigate whether and how partial knowledge of the evader velocity can be advantageous to the pursuer, when both parties are engaged in optimal play.

Literature Review: Differential games have been well studied since their conception. However, the plethora of possible variations in games provide opportunities for continued exploration. The seminal work of Isaacs [2] introduces at large the concept of differential games, and proposes a number of pursuit-evasion games (PEGs), including the Isotropic Rocket problem. Isaacs provides a framework to analyze and geometrically visualize the game, and presents detailed analysis of the game of *kind*, and optimal feedback strategies for the game of *degree*. A different approach to differential

games including PEGs, built upon variational and optimal control theory is proposed in [3]. Alternative approaches to address PEGs are presented in [4]–[6]. In particular, [6] introduces the concept of the stroboscopic strategy as an alternative to the feedback strategies proposed by Isaacs. A comparison of feedback and stroboscopic strategies can be found in [7]. In [4], linear PEGs of a particular class, including the Isotropic Rocket problem, are treated as differential games with a terminal cost function. Linear-quadratic games are addressed in [5] and [8]. An optimal feedback strategy and the conditions for existence of a saddle point are presented in [5], while [8] deals with uncertain dynamic models, and provides conditions for existence of closed loop strategies. Furthermore, [9] investigates the optimality of Isaacs' solution to the Isotropic Rocket problem and presents geometrical results on the game of kind. Finally, a tutorial-like exposition of PEGs can be found in [10].

In [11], a three-dimensional surveillance game of kind is analyzed, drawing analogy with the Isotropic Rocket problem. The problem of minimum time guidance of the Isotropic Rocket in the presence of known winds and fixed target location is addressed in [1] using principles of optimal control theory. The interception of a moving target by multiple pursuers in wind, utilizing a continuously updated Voronoi-like partition of the state space, is presented in [12].

Differential games are a favored tool to model problems with uncertainties [13]–[18]. In particular, the problem of missile guidance with model uncertainties is addressed as a two-person game in [17]. In [18], the problem of aircraft defense against a homing missile is formulated as a linear quadratic differential game, and optimal strategies are developed for all players.

Original Contributions: In this paper, we treat the optimal guidance of the Isotropic Rocket in a partially uncertain flowfield, as a two-person zero sum differential game with free terminal time. Given the initial conditions, we employ the standard framework of Isaacs to theoretically characterize the envelope of capturability (or controllability in the presence of uncertainty), and devise an optimal strategy for guidance of the Isotropic Rocket within that envelope. Furthermore, we present the solution to the problem in the presence of friction, and highlight the difference in the qualitative nature of the solution.

Organization of the paper: The problem formulation is presented in Section II, and Section III contains a detailed analysis of the optimal strategy and the capturability conditions for the modified Isotropic Rocket problem. Numerical simulations and geometric visualization of the solution to the game are given in Section IV, followed by concluding

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remarks in Section V.

II. PROBLEM FORMULATION

Consider the problem of optimal guidance of an Isotropic Rocket to the origin in a partially unknown flowfield, in the two-dimensional Euclidean plane. The dynamic equations corresponding to this system are:

$$\begin{aligned} \dot{x}_R &= u_R + \omega_x + \bar{\omega}_x, & \dot{y}_R &= v_R + \omega_y + \bar{\omega}_y, \\ \dot{u}_R &= F \sin(\phi), & \dot{v}_R &= F \cos(\phi), \end{aligned} \quad (1)$$

where $[x_R, y_R]^T$ is the position vector of the rocket, and $[u_R, v_R]^T$ is its velocity, at time t . The initial state at time $t = 0$ is denoted by $[x_{R0}, y_{R0}, u_{R0}, v_{R0}]^T$. The uncertain component of the flow field is $\omega = [\omega_x, \omega_y]^T$ and the known component is $\bar{\omega} = [\bar{\omega}_x, \bar{\omega}_y]^T$. The problem is to find the control input $\phi(x_R, y_R, u_R, v_R)$ that drives the rocket to the origin in the minimum possible time, in the presence of the most adverse flowfield.

The equivalent zero-sum two-player differential game is stated as follows. Consider two agents, a pursuer P and an evader E , in the two-dimensional Euclidean plane \mathbb{R}^2 . At each instant of time t , the position vectors of P and E are denoted by $[x_P, y_P]^T$ and $[x_E, y_E]^T$ respectively. The magnitude of the pursuer's acceleration is equal to F which is a constant, and the control input of P is the direction ϕ of its acceleration. Our problem departs from Isaac's Isotropic Rocket problem [2] in that the evader's velocity can be decomposed into two components, of which one is controllable and has a constant magnitude w . E can control the heading ψ of this component. It is notable that the evader has single integrator dynamics, so it can change its direction of motion abruptly if required, that is, it can "swerve". The pursuer has second order dynamics, and is unable to execute a similar maneuver.

The uncontrollable component of the evader's velocity is always known to the pursuer, and is constant in time and space. This known component is denoted by $\bar{w} = [\bar{w}_x, \bar{w}_y]^T$. The resulting modified vectograms for the evader are illustrated in Figure 1. The base case of the Isotropic Rocket problem which is treated in [2] can be recovered by setting $\bar{w} = 0$. The variation from the base case in the present problem could be viewed as P having an informational advantage in the game over E , or that E is constrained in his movement by an uncontrollable constant component.

The state of the game at a given time instant t is denoted by $\mathbf{x} = [x_P, y_P, u_P, v_P, x_E, y_E]^T$. The dynamics of the two agents in \mathbb{R}^6 are described by the following equations:

$$\begin{aligned} \dot{x}_P &= u_P & \dot{y}_P &= v_P, \\ \dot{u}_P &= F \sin(\phi) & \dot{v}_P &= F \cos(\phi), \\ \dot{x}_E &= w \sin(\psi) + \bar{w}_x, & \dot{y}_E &= w \cos(\psi) + \bar{w}_y. \end{aligned} \quad (2)$$

Note that $\mathbf{w} = -\omega$ and $\bar{\mathbf{w}} = -\bar{\omega}$. For convenience, we will denote the vector field of the system given in (2) by $\mathbf{f}(\mathbf{x}, \phi, \psi)$. Let $\mathbf{r} := [x, y]^T$, where $x := x_E - x_P$ and $y := y_E - y_P$, be the relative position vector of E with respect to P . Capture occurs when $r \leq l$, where $r := \|\mathbf{r}\|$.

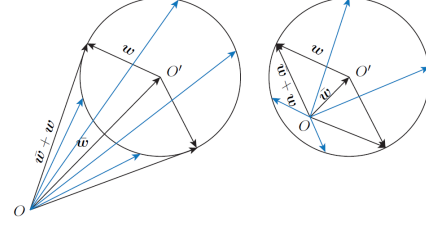


Fig. 1. Modified vectograms for the evader. The vectogram when $\|\bar{\mathbf{w}}\| > \|\mathbf{w}\|$ is illustrated on the left, while the one for $\|\bar{\mathbf{w}}\| < \|\mathbf{w}\|$, is on the right. O is the actual position of the evader, and O' is the center of its circular vectogram displaced due to the flowfield.

The payoff of the game $\mathcal{P}(\mathbf{x}, \phi, \psi)$ is the time of capture resulting from inputs (ϕ, ψ) applied to the game with initial condition \mathbf{x} . Hence, the problem at hand qualifies as a game of *degree*, although the constraint for capturability which is usually pertaining to a game of *kind* is also discussed in this paper. For simplicity, we assume that there is no initial or terminal cost.

The time of capture is the quantity that the pursuer seeks to minimize and the evader seeks to maximize. Consequently, the value of the game $\mathcal{V}(\mathbf{x})$ is defined by the equation

$$\mathcal{V}(\mathbf{x}) = \min_{\phi} \max_{\psi} \mathcal{P}(\mathbf{x}, \phi, \psi), \quad (3)$$

provided that this minmax exists. In the expression for the value function of this problem, the terms involving the control inputs of the two players are independent of each other. Hence, one could say that max and min operators "commute." The value function thus defined in equation (3) yields the following equation for the game, called the *main equation*:

$$1 + \min_{\phi} \max_{\psi} [\langle \nabla_{\mathbf{x}} \mathcal{V}(\mathbf{x}), \mathbf{f}(\mathbf{x}, \phi, \psi) \rangle] = 0, \quad (4)$$

where $\nabla_{\mathbf{x}} \mathcal{V}$ denotes the gradient of the value function with respect to the state \mathbf{x} . The problem is to find the optimal strategies $\bar{\phi}(\mathbf{x})$ and $\bar{\psi}(\mathbf{x})$ that satisfy equation (4).

III. THEORETICAL ANALYSIS

In this section, we will derive the condition for capturability, and the optimal strategies to be employed when it is satisfied. We will demonstrate that the case where the evader has a modified vectogram is easily reduced to the base case of the standard problem by performing a simple variable transform. We will also investigate how the presence of friction for the pursuer affects these results.

A. Optimal strategy within the region of capturability

From equations (2) and (4), the main equation can be written as

$$\begin{aligned} 1 + u_P \mathcal{V}_{x_P} + v_P \mathcal{V}_{y_P} + \min_{\phi} [F \sin(\phi) \mathcal{V}_{u_P} + F \cos(\phi) \mathcal{V}_{v_P}] + \\ \max_{\psi} [w \sin(\psi) \mathcal{V}_{x_E} + w \cos(\psi) \mathcal{V}_{y_E}] + \bar{w}_x \mathcal{V}_{x_E} + \bar{w}_y \mathcal{V}_{y_E} = 0. \end{aligned}$$

Let $\rho_P := \sqrt{\mathcal{V}_{u_P}^2 + \mathcal{V}_{v_P}^2}$ and $\rho_E := \sqrt{\mathcal{V}_{x_E}^2 + \mathcal{V}_{y_E}^2}$. From [2], we have

$$\begin{aligned}\sin(\bar{\phi}) &= -\mathcal{V}_{u_P}/\rho_P, & \cos(\bar{\phi}) &= -\mathcal{V}_{v_P}/\rho_P \\ \sin(\bar{\psi}) &= \mathcal{V}_{x_E}/\rho_E, & \cos(\bar{\psi}) &= \mathcal{V}_{y_E}/\rho_E.\end{aligned}\quad (5)$$

Then the main equation becomes

$$u_P \mathcal{V}_{x_P} + v_P \mathcal{V}_{y_P} - F \rho_P + w \rho_E + \bar{w}_x \mathcal{V}_{x_E} + \bar{w}_y \mathcal{V}_{y_E} + 1 = 0.$$

Next we define the so-called *retrogressive* path equations [2] using equations (2) and (5). Here, the integration is performed backward in time, starting from the terminal point of the game. Henceforth, the variable of integration will be $\tau = T - t$. There are 12 equations (6 for the states, 6 for partial derivatives of the value function), which are given as follows:

$$\begin{aligned}\dot{x}_j &= -f_j(\mathbf{x}, \bar{\phi}, \bar{\psi}), & \dot{\mathcal{V}}_{u_P} &= \mathcal{V}_{x_P}, \\ \dot{\mathcal{V}}_{x_P} &= \dot{\mathcal{V}}_{y_P} = \dot{\mathcal{V}}_{x_E} = \dot{\mathcal{V}}_{y_E} = 0, & \dot{\mathcal{V}}_{v_P} &= \mathcal{V}_{y_P}.\end{aligned}\quad (6)$$

We define the terminal states $\mathbf{x}(T)$ at capture, in terms of five parameters s_i , $i = 1, \dots, 5$, keeping in mind that at the termination of the game, $\mathbf{r} \in \{\mathbf{z} \in \mathbb{R}^2 : \|\mathbf{z}\| \leq l\}$. The terminal states are:

$$\begin{aligned}x_P &= s_1, & y_P &= s_2, & u_P &= s_3, \\ v_P &= s_4, & x_E &= s_1 + l \sin(s_5), & y_E &= s_1 + l \cos(s_5).\end{aligned}\quad (7)$$

The equations in (7) correspond to a parametrization of the terminal surface, and provide initial conditions to integrate the equations in (6) backwards. At the terminal surface, we know that $r = \|\mathbf{r}\| = l$, and the time derivative of r yields:

$$\begin{aligned}\dot{r} &= \sin(s_5)(w \sin(\psi) + \bar{w}_x - u_P) \\ &\quad + \cos(s_5)(w \cos(\psi) + \bar{w}_y - v_P).\end{aligned}\quad (8)$$

The part of the terminal surface in which capture can occur is called the Usable Part (UP) which is defined as follows:

$$\max_{\psi \in [0, 2\pi]} \dot{r} < 0. \quad (9)$$

At the terminal surface, the value function equals zero. By differentiating \mathcal{V} with respect to each of the five parameters s_i , we can get the rest of the initial conditions that are needed to solve the equations in (6). For instance, for the first parameter,

$$\frac{\partial \mathcal{V}}{\partial s_1} = \mathcal{V}_{x_P} + \mathcal{V}_{x_E} = 0. \quad (10)$$

Similarly,

$$\begin{aligned}\mathcal{V}_{y_P} + \mathcal{V}_{y_E} &= 0, & \mathcal{V}_{u_P} &= \mathcal{V}_{v_P} = 0, \\ \mathcal{V}_{x_E} \sin(s_5) - \mathcal{V}_{y_E} \cos(s_5) &= 0.\end{aligned}\quad (11)$$

We define a parameter λ such that $\mathcal{V}_{x_E} = \lambda \sin(s_5)$ and $\mathcal{V}_{y_E} = \lambda \cos(s_5)$. We know from (11) that $\mathcal{V}_{x_P} = -\mathcal{V}_{x_E}$ and $\mathcal{V}_{y_P} = -\mathcal{V}_{y_E}$.

Integrating the equations in (6) with the initial conditions from equations in (11), we get expressions for the partial derivatives of the value function. From the partial derivatives

of \mathcal{V} , we obtain the optimal strategies of the two players in terms of the parameters of the terminal surface. The partial derivatives are given as follows:

$$\mathcal{V}_{u_P} = -\lambda \tau \sin(s_5), \quad \mathcal{V}_{v_P} = -\lambda \tau \cos(s_5). \quad (12)$$

The expressions for the other partial derivatives are similarly obtained. From equations (5) and (12), we have,

$$\bar{\phi} = \bar{\psi} = s_5. \quad (13)$$

Now we integrate the rest of the equations in (6), to obtain

$$\begin{aligned}x_E &= s_1 + l \sin(s_5) - w \tau \sin(s_5) - \bar{w}_x \tau, \\ y_E &= s_1 + l \cos(s_5) - w \tau \cos(s_5) - \bar{w}_y \tau, \\ x_P &= s_1 - s_3 \tau + \frac{F \tau^2}{2} \sin(s_5), \\ y_P &= s_1 - s_4 \tau + \frac{F \tau^2}{2} \cos(s_5).\end{aligned}\quad (14)$$

Let $Q(\tau) := l - w \tau + \frac{F \tau^2}{2}$. From equations (14), the optimal strategy for the players is given by

$$\begin{aligned}\sin(s_5) &= \frac{x_E - x_P - (u_P - \bar{w}_x) \tau}{Q(\tau)}, \\ \cos(s_5) &= \frac{y_E - y_P - (v_P - \bar{w}_y) \tau}{Q(\tau)}.\end{aligned}\quad (15)$$

B. Analogy to base case by means of a variable transform

Recalling that $\mathbf{r} = [x_E - x_P, y_E - y_P]^T$ and defining the *effective* velocity vector of P as $\bar{\mathbf{u}} := [u_P - \bar{w}_x, v_P - \bar{w}_y]^T$, we can condense equations (15) into the following equation:

$$\mathbf{r}^2 - 2(\mathbf{r} \cdot \bar{\mathbf{u}}) \tau + \bar{\mathbf{u}}^2 \tau^2 = Q^2(\tau). \quad (16)$$

Equation (16) gives the value function for the entire state space under the assumption that the conditions for capture are satisfied. In particular, under this assumption, the smallest positive root of equation (16) is the time of capture. Alternatively, the time taken for termination of the game with successful capture of E by P , starting from an arbitrary initial condition, is the least positive root of equation (16). The base case corresponds to the known component $\bar{\mathbf{w}}$ being identically zero. The optimal game strategy for the case with the modified evader dynamics is very similar to the base case found in [2], because the variable change of $\bar{u}_P = u_P - \bar{w}_x$ and $\bar{v}_P = v_P - \bar{w}_y$ preserves the form of the dynamic equations.

C. Condition for capturability

The criterion for capturability is derived in a reduced space where the state vector is taken to be $\mathbf{x}_r = [x, y, u_P, v_P]^T$. In the reduced space, the pursuer is always at the origin and x and y are the position coordinates of E with respect to P . The capturability conditions for the base case with $\bar{\mathbf{w}} = \mathbf{0}$ are derived in a three-dimensional reduced space in [2]. By employing a similar procedure and using the analogy between the base case and our case of interest, we can characterize the condition for capturability, based solely on

the input parameters to the game. In particular, the dynamic equations in the reduced space are now given as follows:

$$\begin{aligned}\dot{x} &= w \sin(\psi) + \bar{w}_x - u_P, & \dot{u}_P &= F \sin(\phi), \\ \dot{y} &= w \cos(\psi) + \bar{w}_y - v_P, & \dot{v}_P &= F \cos(\phi).\end{aligned}\quad (17)$$

The terminal states of the game, which define the terminal surface, are given by equation (7). The Usable Part is defined by equation (9). The Boundary of the Usable Part (BUP) is determined by the equation $\max_{\psi \in [0, 2\pi]} \dot{r} = 0$, at $r = l$, and is given by

$$(s_3 - \bar{w}_x) \sin(s_1) + (s_4 - \bar{w}_y) \cos(s_1) - w = 0. \quad (18)$$

In place of the partial derivatives of the value function in the preceeding analysis, we will use the costate variables $\nu_j, j = 1, \dots, 4$, which represent the normal vector to the terminal surface at $\mathbf{x}_r = [x, y, u_P, v_P]^T$. Then, the main equation is given by

$$\min_{\phi} \max_{\psi} \sum_i f_i(\mathbf{x}_r, \phi, \psi) \nu_i = 0, \quad (19)$$

whereas the *retrogressive* path equations are given by

$$\dot{\mathbf{x}}_j = -f_j(\mathbf{x}_r, \bar{\phi}, \bar{\psi}), \quad \dot{\nu}_j = \sum_i \nu_i f_{ij}(\mathbf{x}_r, \bar{\phi}, \bar{\psi}), \quad (20)$$

where $i, j = 1, \dots, 4$. The terminal conditions in equation (7) become the initial conditions for the state variables when integrating equation (20). Furthermore, the initial conditions for $\nu_i, i = 1, \dots, 4$, are given by

$$\nu_1 = \sin(s_1), \quad \nu_2 = \cos(s_1), \quad \nu_3 = 0, \quad \nu_4 = 0. \quad (21)$$

These initial conditions are chosen such that $\nu_j, j = 1, \dots, 4$, are normal to the terminal surface and satisfy equation (20) at the terminal surface. Integrating the retrogressive path equations, we get the expressions for the state variables at time τ as follows:

$$\begin{aligned}x &= \sin(s_1) (l - w\tau - F\tau^2/2) + (s_3 - \bar{w}_x)\tau, \\ y &= \cos(s_1) (l - w\tau - F\tau^2/2) + (s_4 - \bar{w}_y)\tau, \\ u_P &= s_3 - F\tau \sin(s_1), \quad v_P = s_4 - F\tau \cos(s_1).\end{aligned}\quad (22)$$

1) The barrier: The barrier in this game is defined as the *semipermeable* manifold formed by the envelopes of constant time manifolds (isochrones) which are described by equation (16). If we assume that the condition for capturability is satisfied, the barrier corresponds to a discontinuity in the value function and the strategy of pursuit, and it is not crossed in optimal play. For this problem, the projections of the isochrones on the $x - y$ plane are circular, and $Q(\tau)$ is the radius of the isochrone for time τ . Henceforth in this paper, the term isochrones will denote the cross-section of the isochronic manifolds at a particular velocity, usually the initial velocity of the pursuer.

2) Escape condition from the structure of the barrier: Let the components of the effective velocity vector of P be $\bar{u}_P := u_P - \bar{w}_x$ and $\bar{v}_P := v_P - \bar{w}_y$, where $\bar{\mathbf{u}} := [\bar{u}_P, \bar{v}_P]^T$. The condition for E to escape is that the two parts of the barrier on either side of the axis of symmetry will intersect. Alternatively, there exists $\tau_0 > 0$ such that at time $\tau = \tau_0$, \mathbf{r} and $\bar{\mathbf{u}}$ are collinear. One should note here that the effective

velocity vector corresponds to the axis of symmetry of the barrier. The intersection of the barrier manifolds means that there is a portion of the state space such that, if the game is initiated from there, P cannot win provided that E plays optimally [2]. However, in this paper we confine our analysis to cases where capture is unavoidable. From the collinearity condition between \mathbf{r} and $\bar{\mathbf{u}}$, we have

$$x/y = \bar{u}_P/\bar{v}_P. \quad (23)$$

Rewriting the equations (22) in terms of $Q(\tau)$ as defined earlier, and substituting into equation (23), we have after simplification:

$$\frac{F\tau^2}{2} + l - w\tau = 0, \text{ or } Q(\tau) = 0. \quad (24)$$

Equation (24) is the condition for the barrier manifolds to intersect such that there is a zone of definite capture and a cut-off zone where escape is possible. The condition that the barrier manifolds intersect directly implies that the radius of the isochrone is zero at some time τ_0 . For this to be possible, we require that equation (24) has a real positive root, for which it is necessary that $w^2 \geq 2Fl$.

On the other hand, the sufficient condition for capture to certainly occur for all initial positions of E is

$$w^2 < 2Fl. \quad (25)$$

It is interesting to note that the condition given in (25) is the same as the capture condition for the base case. By contrast, the axis of symmetry of the isochrone diagram is rotated depending on the magnitude and direction of the known component $\bar{\mathbf{w}}$.

D. Case with friction

We now consider another variation of the standard Isotropic Rocket problem in which the evader has a modified vectogram and the pursuer's velocity is affected by a negative feedback term, which acts as a friction force or drag. The implicit assumptions here are that $w < \frac{F}{k}$ and $\|\bar{\mathbf{w}}\| < \frac{F}{k}$, since $\frac{F}{k}$ is the limiting speed of P [2]. For capture to occur, P must have an advantage in speed over E . The new dynamic equations are:

$$\begin{aligned}\dot{x}_P &= u_P, & \dot{y}_P &= v_P, \\ \dot{u}_P &= F \sin(\phi) - ku_P, & \dot{v}_P &= F \cos(\phi) - kv_P, \\ \dot{x}_E &= w \sin(\psi) + \bar{w}_x, & \dot{y}_E &= w \cos(\psi) + \bar{w}_y.\end{aligned}\quad (26)$$

By using a procedure similar to the one described in Section III-A, we obtain the retrogressive state variable expressions as follows:

$$\begin{aligned}x_P &= s_1 - s_3 \left(\frac{e^{k\tau} - 1}{k} \right) + F \sin(s_5) \left(\frac{e^{k\tau} - k\tau - 1}{k^2} \right), \\ y_P &= s_1 - s_4 \left(\frac{e^{k\tau} - 1}{k} \right) + F \cos(s_5) \left(\frac{e^{k\tau} - k\tau - 1}{k^2} \right), \\ x_E &= s_1 + l \sin(s_5) - w\tau \sin(s_5) - \bar{w}_x\tau, \\ y_E &= s_2 + l \cos(s_5) - w\tau \cos(s_5) - \bar{w}_y\tau.\end{aligned}\quad (27)$$

Here, let $Q(\tau)$ be defined as follows:

$$Q(\tau) = l - w\tau + F \left(\frac{e^{-k\tau} + k\tau - 1}{k^2} \right). \quad (28)$$

From equations (27), we get the optimal strategy for both players in the presence of friction:

$$\begin{aligned} \sin(s_5) &= \frac{k(x_E - x_P + \bar{w}_x\tau) - u_P(1 - e^{-k\tau})}{kQ(\tau)}, \\ \cos(s_5) &= \frac{k(y_E - y_P + \bar{w}_y\tau) - v_P(1 - e^{-k\tau})}{kQ(\tau)}. \end{aligned} \quad (29)$$

Now we consider the condition for capturability when friction is present in the pursuer's dynamics. By using similar arguments to the analysis presented in Section III-C, we arrive at the condition for capture as $Q(\tau) > 0$ for all $\tau \geq 0$, where $Q(\tau)$ is defined in equation (28). Simplifying, we get the condition for capture as:

$$\frac{F}{wk} (1 - e^{-\gamma}) < 1, \quad \gamma := k \frac{w - lk}{F - wk}. \quad (30)$$

We observe that the capturability condition for the Isotropic Rocket is independent of the known component \bar{w} . However, under the condition of capturability, the solution to the differential game shows qualitative and quantitative differences from the base case, as will be evidenced in the following section.

IV. NUMERICAL SIMULATIONS

In this section, the theoretical results from Section III are illustrated with the aid of numerical simulations. For all simulations in this section, we assume that the condition for capturability holds. The relevance of the barrier described in Section III-B to the resultant trajectory of P can be observed by using different initial positions of the evader E . The initial position of E relative to the barrier indicates the type of pursuit trajectory followed by P . For initial locations of E that are not across the barrier from P , the pursuit trajectory is directly towards E at all times, and the game ends with the capture of E . For locations of E that are across the barrier from P , the trajectory of the former involves a “swerve” maneuver, and to counter this, P initially moves away from E , executes a turn and then finally captures E . We know that the barrier corresponds to a discontinuity in the value function for the game. This practically means that there exist initial locations of E that are close to each other, but on either side of the barrier, for which we find a large discrepancy in the times of capture.

The following numerical values are used throughout this section: $F = 1$ and $l = 0.2$. The initial position of the pursuer P is taken to be the origin. The values of the other parameters are varied to study different conditions of interest.

A. Case without friction

First, we study the case without friction, that is, $k = 0$. The initial velocity of P is $[1.1400, -1.0450]^T$ and $w = 0.5$. These values satisfy the capturability condition. In this section, we compare cases with varying magnitude of the

known component \bar{w} . The variation in the barrier shape with the magnitude of \bar{w} is of interest because the barrier shape dictates the capture trajectory along with the time of capture in each case.

There are three possible combinations for w and $\|\bar{w}\|$:

- (1) $\bar{w} = \mathbf{0}$: This is the case where there is no known component in the evader's dynamics, and the resulting isochrone diagram for initial velocity of the pursuer is shown in Figure 2(a). This is the base case of the Isotropic Rocket game formulated and studied by Isaacs.
- (2) $\bar{w} \neq \mathbf{0}$ and $w \neq 0$: The evader's modified vectograms are shown in Figure 1. When $\|\bar{w}\| \leq w$, the corresponding isochrone diagram is as illustrated in Figure 2(b). The new axis of symmetry can be obtained by a rotation from that of Figure 2(a), and there is also elongation of the barrier. As is seen when $\|\bar{w}\| > w$, the elongation of the barrier is more pronounced as $\|\bar{w}\|$ increases. The isochrone diagram for this case is illustrated in Figure 2(c). The elongation of the barrier as $\|\bar{w}\|$ increases is reflected as greater difference in times of capture for evader locations initially across the barrier from each other.
- (3) $w = 0$: In this case, there is no ambiguity in the evader's movement. The problem is reduced to an optimal control problem for the pursuer [1], and the barrier in Figure 2(d) is shortened, and unlike the other cases, there is no reduction in the radius of the isochrones from the terminal circle. This is consistent with the fact that the condition for escape can never be satisfied in this case. Again, the axis of symmetry can be obtained by a rotation from that of Figure 2(a). For all cases, the angle of rotation of the axis of symmetry is the angle between the velocity vector \mathbf{u} and the effective velocity vector $\bar{\mathbf{u}}$.

B. Case with friction, $k \neq 0$

Figures 2(e) and 2(f) illustrate the isochrone diagrams for $\bar{w} = [0.7, 0.7]^T$ and $\bar{w} = [0, 0]^T$ respectively, with $k = 0.5$. When both k and $\|\bar{w}\|$ are not zero, the trace of the centers of circles in the isochrone cross section at the initial velocity of P is no longer a straight line. This is the only case where we observe a qualitative difference in the solution to the differential game.

V. CONCLUSION

In this paper, we have studied the optimal guidance problem for the Isotropic Rocket in a partially known flowfield. Two cases have been discussed: one without friction and the other where the rocket is affected by a friction force or drag. The condition for capturability (successful guidance to the origin) has been derived, and the optimal strategies for the corresponding Isotropic Rocket pursuit-evasion game have been presented. The guidance problem of the Isotropic Rocket to a fixed target in a partially uncertain flowfield is of high relevance to problems where the dynamic model contains uncertainties, or there are uncontrollable external

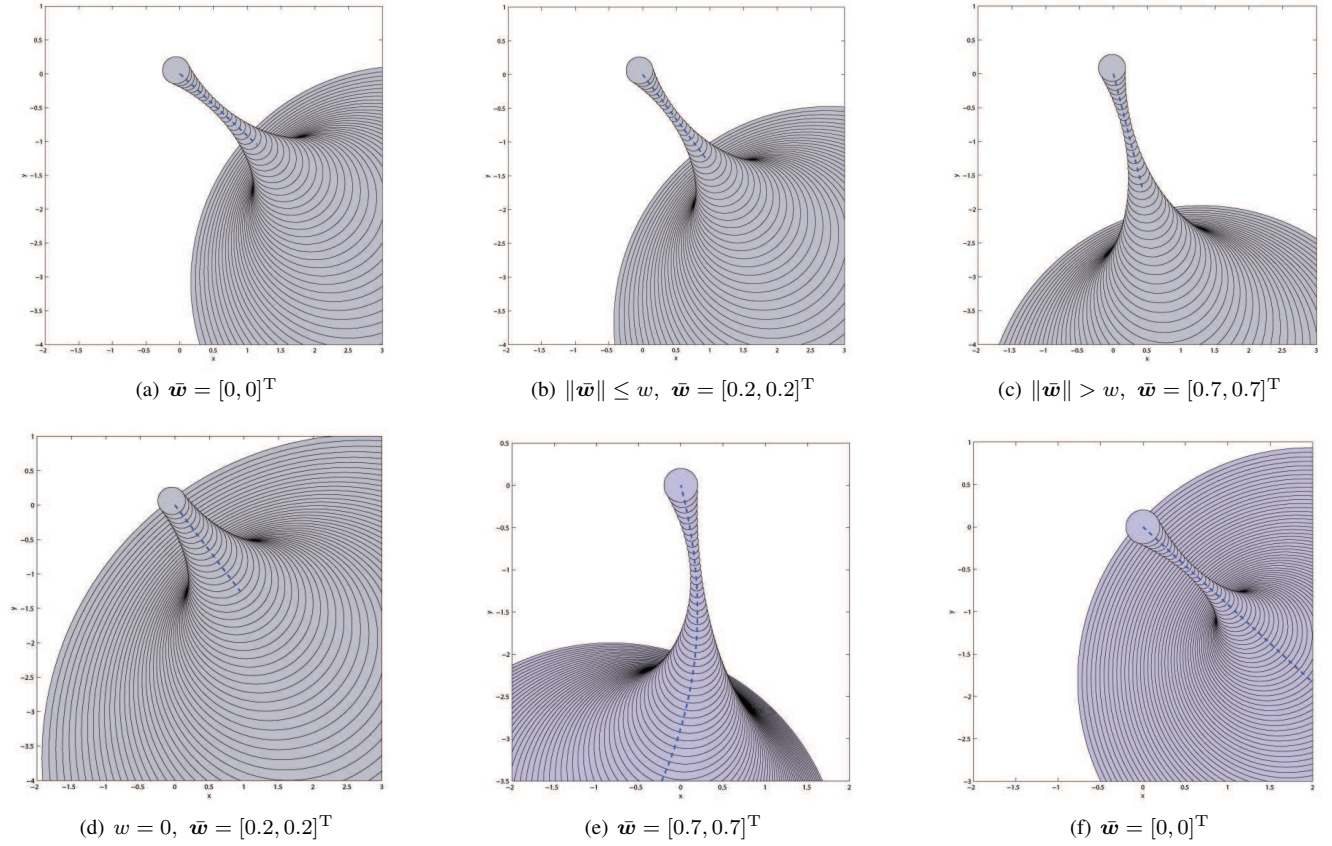


Fig. 2. The isochrone cross sections corresponding to the initial velocity of P are shown in figures (a), (b), (c) and (d), for $k = 0$, and for different times τ where $\tau \in [0, 3]$. Isochrone cross sections in the presence of friction ($k = 0.5$) are shown in (e) and (f), for different times τ where $\tau \in [0, 5]$. The trace of the centers of the isochronic circles is shown as a blue dashed line.

components. These elements of the control problem can be modeled as the secondary (opposing) player in a PEG, who is at best passive, or at worst, adversarial to the primary player that is of interest to us. One such instance of a control problem is capture of orbital debris where the debris (evader) is non-cooperative, or affected by constant winds in the upper atmosphere. In future, we will extend this work to study cases where the known component of the evader's velocity has a time-varying profile.

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