The University of Texas at Austin
Cockrell School of Engineering

Human Centered Robotics Laboratory

Safety Control Synthesis with Input Limits

A Hybrid Approach

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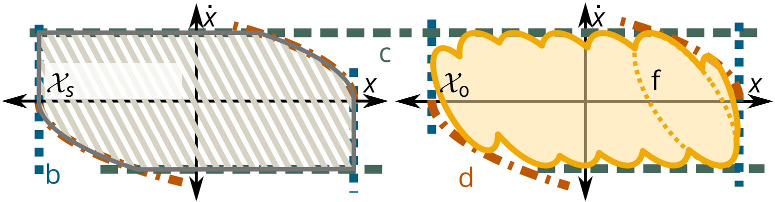
Introduction

A system $\Sigma_0 : \dot{x} = f(x) + g(x)u$

- state constraint $x \in \mathcal{X}$
- input constraint $u \in \mathcal{U}$

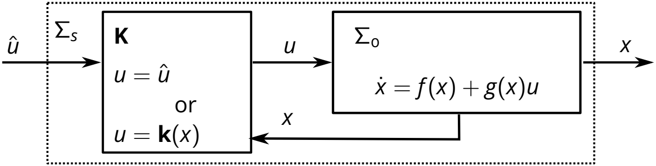


We develop a controlled safe region $\mathcal{X}_0$ with a backup controller $\mathbf{k}(x)$ which provides

- a safety guarantee for all initial states in a region $\mathcal{X}_0 \subseteq \mathcal{X}_s$,
- a close approximation of the exact safe region $\mathcal{X}_s$.

Hybrid Safety Controller

A hybrid safety controller **K** in feedback with the original system Σ_o produces a safe system Σ_s.



Hybrid Safety Controller K

- chooses either to be completely transparent ($u = \hat{u}$) or apply the backup controller input $u = \mathbf{k}(x)$.
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Hybrid Safety Controller

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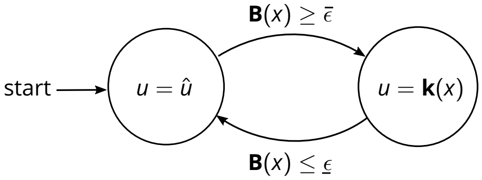

Hybrid Safety Controller $\mathbf{K}$

- chooses either to be completely transparent ($u = \hat{u}$) or apply the known-to-be-safe input $u = \mathbf{k}(x)$.
- is tuned by two thresholds $\bar{\epsilon}$ and $\underline{\epsilon}$.

Background

Verification	Stability	State Limits	Input Limits
Lyapunov Function	✓	✗	✗
Barrier Certificate [1]	✓	✓	✗
Synthesis	**Stability**	**State Limits**	**Input Limits**
Barrier Lyapunov Function [2]	✓	✓	✗
Control Lyapunov Function [3]	✓	✗	✗
Control Barrier Function [4]	✓	✓	✓✗
Barrier Pair	✓	✓	✓

[1] Stephen Prajna and Ali Jadbabaie. Safety verification of hybrid systems using barrier certificates. In: *Hybrid Systems: Computation and Control*. Vol. 2993. Lecture Notes in Computer Science. 2004, pp. 477–492

[2] Aaron D Ames, Xiangru Xu, Jessy W Grizzle, and Paulo Tabuada. Control barrier function based quadratic programs for safety critical systems. In: *IEEE Transactions on Automatic Control* 62.8 (2017), pp. 3861–3876

[3] Eduardo D Sontag. A 'universal' construction of Artstein's theorem on nonlinear stabilization. In: *Systems & Control Letters* 13.2 (1989), pp. 117–123

[4] Yan-Jun Liu and Shaocheng Tong. Barrier Lyapunov functions-based adaptive control for a class of nonlinear pure-feedback systems with full state constraints. In: *Automatica* 64 (2016), pp. 70–75

Barrier Pair

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Control Barrier Function	✓	✓	✓✗
Barrier Pair	✓	✓	✓

A *Barrier Pair* is a pair of functions $(B,\ k)$ with two properties:

- $-1 < B(x) \leq 0, u = k(x) \implies \dot{B}(x) < 0,$
- $B(x) \leq 0 \implies x \in \mathcal{X},\ k(x) \in \mathcal{U}.$

LMI Sub-problem

Consider a linear differential inclusion [5] (LDI) model which approximates Σ_0 near an equilibrium—a robust linearization.

Contrained Polytopic LDIs

Linearized Dynamics

$$\dot{x} \in \mathbf{Co}\{A_l(x - x_e) + B_l(u - u_e),\ l = 1, \ldots, L\}.$$

Region of Validity

$$\begin{aligned} \forall \quad & x \in \{x \ : \ |a_i^T(x - x_e)| \leq \alpha_i,\ i = 1, \ldots, n_a\} \subseteq \mathcal{X}, \\ & u \in \{u : \ |b_i^T(u - u_e)| \leq \beta_i,\ i = 1, \ldots, n_b\} \subseteq \mathcal{U}. \end{aligned}$$

[5] Stephen Boyd, Laurent El Ghaoui, Eric Feron, and Venkataramanan Balakrishnan. *Linear Matrix Inequalities in System and Control Theory*. SIAM, 1994

LMI Sub-problem

A set of LMI constraints determine a positive definite matrix Q and full state feedback matrix K such that—defining

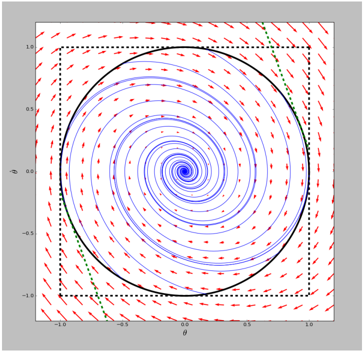
- $B(x) \triangleq V(x) - 1 = (x - x_e)^T Q^{-1}(x - x_e) - 1$
- $k(x) \triangleq u_e + K(x - x_e)$

LMIs:

- Lyapunov Stability
- State and Input Constraints

Cost Function:

- $log(det(Q))$



Composition of Barrier Pairs

Proposition

For any list of barrier pairs $(B_1, k_1),\ (B_2, k_2),\ \ldots,\ (B_N, k_N)$, the pair comprising the min-barrier function

$$\mathbf{B}(x) \triangleq \min_{n=1,\ldots,N} B_n(x)$$

and (occasionally ambiguous) control input

$$\mathbf{k}(x) \triangleq k_n(x) \mid n \in \arg\min_{n=1,\ldots,N} B_n(x),$$

$(\mathbf{B}, \mathbf{k})$, is also a barrier pair.

Inverted Pendulum Example

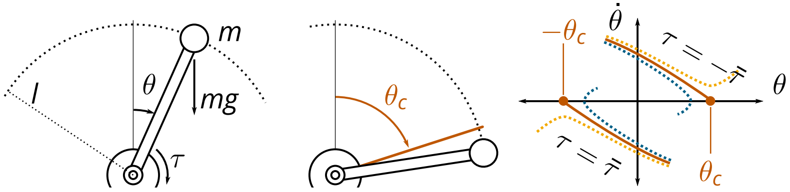
An inverted pendulum ($m = 1\,\mathrm{kg}$, $l = 1.213\,\mathrm{m}$, $g = 9.8\,\mathrm{m/s^2}$)

Dynamics

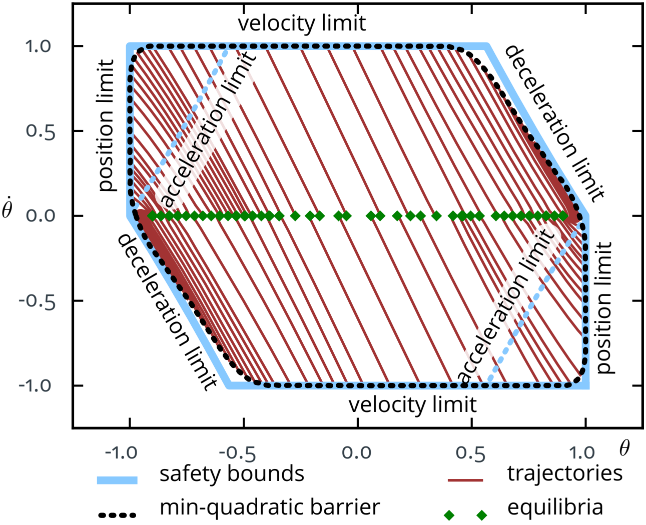
$$ml^2\ddot{\theta} = \tau + mgl \cdot \sin(\theta)$$

Safe Region

- $\mathcal{X} = \left\{ \begin{bmatrix} \theta \\ \dot{\theta} \end{bmatrix} : |\theta| \le \theta_c = 1\,\mathrm{rad},\ |\dot{\theta}| \le 1\,\mathrm{rad/s} \right\}$,
- $\mathcal{U} = \{\tau : |\tau| \le \bar{\tau} = 10\,\mathrm{N \cdot m}\}$.

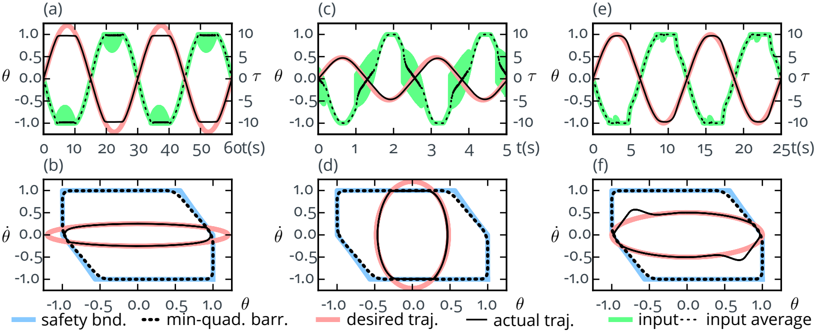


Inverted Pendulum Example



Inverted Pendulum Example

The inverted pendulum system protected by the safety controller is placed in feedback.



Dual Spring-Mass Example

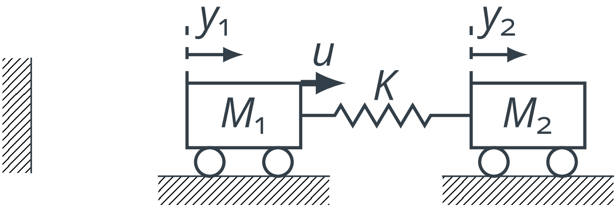
A dual spring-mass system ($M_1 = M_2 = 1, K = 1$)

Dynamics

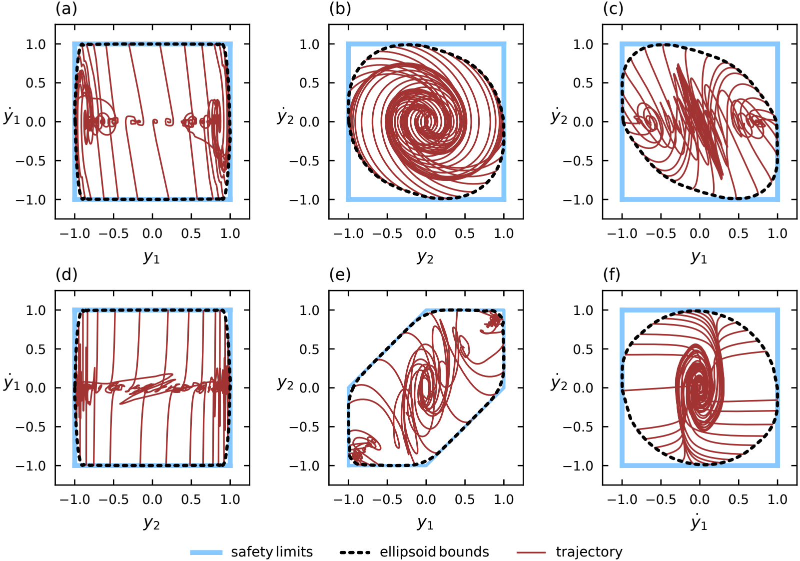
$$M_1\ddot{y}_1 = K(y_2 - y_1) + u, \quad M_2\ddot{y}_2 = K(y_1 - y_2).$$

Safe Region

- $\mathcal{X} = \{(y_1, \dot{y}_1, y_2, \dot{y}_2)^T : |y_1 - y_2| \leq 1, |y_i| \leq 1, \ |\dot{y}_i| \leq 1, \ i = 1, 2\}$,
- $\mathcal{U} = \{u : \ |u| \leq 10\}$.



Dual Spring-Mass Example



Thank you for listening!

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LMI Sub-problem

A set of LMI constraints determine a positive definite matrix Q and full state feedback matrix $K = YQ^{-1}$ such that—defining

- $B(x) \triangleq (x - x_e)^T Q^{-1}(x - x_e) - 1$
- $k(x) \triangleq u_e + K(x - x_e)$

LMI Sub-problem

$$
\begin{aligned}
\underset{Q,Y}{\text{maximize}} \quad & \log(\det(Q)) \\
\text{subject to} \quad & Q \succeq \varepsilon I \\
& a_i^T Q a_i \leq \alpha_i^2, \ \forall\, i = 1, \ldots, n_a \\
& \begin{pmatrix} Q & Y^T b_i \\ b_i^T Y & \beta_i^2 \end{pmatrix} \succeq 0, \ \forall\, i = 1, \ldots, n_b \\
& A_l Q + Q A_l^T + B_l Y + Y^T B_l^T + \varepsilon I + \lambda Q \preceq 0, \ \ \forall\, l = 1, \ldots, L
\end{aligned}
$$

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