AAS 09-367

ON UNDERWEIGHTING LIDAR MEASUREMENTS

Renato Zanetti,* Kyle J. DeMars,† and Robert H. Bishop‡

Underweighting is an *ad hoc* technique to reduce the Kalman filter update in order to compensate for unaccounted second order terms in the Taylor series expansion of the filter's residual. Existing underweighting techniques are revisited, these techniques heavily rely on trial and error to finalize the design. For the case of underweighting lidar measurements, a new scheme is introduced to aid the tuning of the filter, obtaining a viable underweighting coefficient bounding the second order terms.

INTRODUCTION

The extended Kalman filter[1] (EKF) is a nonlinear approximation of the optimal linear Kalman filter.[2,3] In the presence of measurements that are nonlinear functions of the state, the EKF algorithm expands the filter's residual (difference between the actual measurement and the estimated measurement) in a Taylor series centered at the *a priori* state estimate. The EKF truncates the series to first order, but second order filters also exist.[4,5] It is well known that in the presence of highly accurate measurements the contribution of the second order terms is essential when the *a priori* estimation error covariance is large.[5,6] Possible solutions include implementing a second order Gaussian filter[4,5] or an Unscented Kalman filter (UKF).[7] The UKF is a nonlinear extension to the Kalman filter capable of retaining the second moments (or higher) of the estimation error distribution. Historically, second order filters are not used because of their computational cost. The Space Shuttle, for example, utilizes an *ad hoc* technique known as underweighting.[8,9]

The Space Shuttle underweighting scheme by Bill Lear has the advantages of being simple, easily tunable, and having a long history of flight. The Gaussian second order filter and the UKF have the advantage of exactly retaining the second order moments; however, they are computationally more expensive, and they are dependent on the error distribution. As the name suggests, the Gaussian second order filter relies on the assumption that all noises are Gaussian. Optimally tuning the UKF coefficients also depends on the distribution of the noises. Lear's scheme uses a single scalar coefficient and tuning is necessary in order to achieve good performance.

The commonly implemented method for the underweighting of measurements for human space navigation was introduced by Lear[10] for the Space Shuttle navigation system. While the use of second-order filters for human spaceflight navigation was discussed as early as 1964, it was not until 1971 in anticipation of Shuttle flights that the assumption and use of a linearized relationship between the state estimate and the measurement was found to be inadequate.[11] In response, Lear

*Senior Member of the Technical Staff, GN&C Manned Space Systems, The Charles Stark Draper Laboratory, 17629 El Camino Real, Suite 470, Houston, Texas, 77058. E-mail: rzanetti@draper.com

†Ph.D. Student, Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, Texas 78712. E-mail: demarskj@mail.utexas.edu

‡Professor, Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, Texas 78712. E-mail: rhbishop@mail.utexas.edu

and others developed relationships which accounted for the second-order effects in the measurements.[8–10] It was noted that in situations involving large state errors and very precise measurements application of the standard extended Kalman filter mechanization leads to conditions in which the state estimation error covariance decreases more rapidly than the actual state errors. Consequently the extended Kalman filter begins to ignore new measurements even when the measurement residual is relatively large. Underweighting was introduced to slow down the convergence of the state estimation error covariance thereby addressing the situation in which the error covariance becomes overly optimistic with respect to the actual state errors. The original work on the application of second-order correction terms led to the determination of the underweighting method by trial-and-error.[10] More recently, Perea, et al. have studied the effects of nonlinearity in sensor fusion problems with application to relative navigation.[12] In doing so, they defined a so-called "bump-up" factor which is exactly the underweighting factor introduced in Lear.[10] However, the bump-up factor is allowed to persistently affect the Kalman gain which directly influences the obtainable steady-state covariance. Effectively, the ability to remove the underweighting factor was not introduced.

This work reviews the role of underweighting and its historical introduction. A new method for determining the underweighting factor is also introduced, together with an automated method for deciding when the underweighting factor should and should not be applied. By using the Gaussian approximation and bounding the second order contributions, suggested values for the coefficient are easily obtained. The proposed technique has the advantage of Lear's scheme simplicity combined with the theoretical foundation of the Gaussian second order filter. The result is a simple algorithm to aid the design of the underweighted EKF.

NEED FOR UNDERWEIGHTING

The update equations used to incorporate measurements in the extended Kalman filter are given by the state update as[1,13] and provide optimal blending of two quantities: the prior estimate $\hat{\mathbf{x}}_k$, and the measurement $\mathbf{y}_k$. These two quantities are combined to obtain the posterior estimate $\hat{\mathbf{x}}_k^+$

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k + \mathbf{K}_k[\mathbf{y}_k - \mathbf{h}(\hat{\mathbf{x}}_k)]\,, \tag{1}$$

and the state estimation error covariance update as

$$\mathbf{P}_k^+ = [\mathbf{I} - \mathbf{K}_k\mathbf{H}(\hat{\mathbf{x}}_k)]\mathbf{P}_k[\mathbf{I} - \mathbf{K}_k\mathbf{H}(\hat{\mathbf{x}}_k)]^{\mathrm{T}} + \mathbf{K}_k\mathbf{R}_k\mathbf{K}_k^{\mathrm{T}}\,, \tag{2}$$

where

$$\mathbf{H}(\hat{\mathbf{x}}_k) := \left[\left.\frac{\partial \mathbf{h}(\mathbf{x}_k)}{\partial \mathbf{x}_k}\right|_{\mathbf{x}_k = \hat{\mathbf{x}}_k} \right].$$

The Kalman gain, $\mathbf{K}_k$, is chosen so as to minimize the norm of the *a posteriori* estimation error, and is given by

$$\mathbf{K}_k = \mathbf{P}_k\mathbf{H}^{\mathrm{T}}(\hat{\mathbf{x}}_k)[\mathbf{H}(\hat{\mathbf{x}}_k)\mathbf{P}_k\mathbf{H}^{\mathrm{T}}(\hat{\mathbf{x}}_k) + \mathbf{R}_k]^{-1}\,. \tag{3}$$

If the gain given in Eq. (3) is applied to the state estimation error covariance of Eq. (2), then the update equation can be rewritten after some manipulation as

$$\mathbf{P}_k^+ = [\mathbf{I} - \mathbf{K}_k\mathbf{H}(\hat{\mathbf{x}}_k)]\mathbf{P}_k\,. \tag{4}$$

This is also seen to be equivalent to

$$\mathbf{P}_k^+ = \mathbf{P}_k - \mathbf{K}_k[\mathbf{H}(\hat{\mathbf{x}}_k)\mathbf{P}_k\mathbf{H}^{\mathrm{T}}(\hat{\mathbf{x}}_k) + \mathbf{R}_k]\mathbf{K}_k^{\mathrm{T}}\,. \tag{5}$$

From Eqs. (1) and (5), it is seen that lowering the Kalman gain leads to a smaller update in both the state estimate and in the state estimation error covariance. Generally we will denote $\mathbf{H}_k := \mathbf{H}(\hat{\mathbf{x}}_k)$.

It is well known that the posterior error covariance $\mathbf{P}_k^+$ is not "greater" than the prior error covariance $\mathbf{P}_k$. This fact naturally arises from the fact that the Kalman filter is optimal, and that if no better estimate was available the posterior estimate would match the prior estimate. When comparing two matrices it is not obvious what "greater" means. The Kalman filters optimizes the trace of the covariance. The trace of the covariance is not a norm; however, in the subset of symmetric positive and semi-positive definite matrices, it satisfies the three conditions of a generalized matrix norm (the fourth property, the Schwarz inequality does not hold).

Mathematically the fact that $\mathbf{P}_k^+$ is not greater than $\mathbf{P}_k$ follows immediately from the covariance update of Eq. (5)

$$\operatorname{trace} \mathbf{P}_k^+ \leq \operatorname{trace} \mathbf{P}_k. \tag{6}$$

It is also possible to show that the same property holds true for the spectral norm (the matrix 2 norm $\|\cdot\|$). The notation $\mathbf{P}_k^+ \leq \mathbf{P}_k$ indicates that $\mathbf{P}_k^+ - \mathbf{P}_k$ is negative semi-definite. From the fact that

$$\mathbf{K}_k[\mathbf{H}(\hat{\mathbf{x}}_k)\mathbf{P}_k\mathbf{H}^{\mathrm{T}}(\hat{\mathbf{x}}_k) + \mathbf{R}_k]\mathbf{K}_k^{\mathrm{T}} \leq 0 \tag{7}$$

it follows that

$$\mathbf{P}_k^+ - \mathbf{P}_k \leq -\alpha\mathbf{I}, \qquad \text{for some } \alpha \leq 0 \tag{8}$$

Therefore

$$\mathbf{P}_k^+ \leq \mathbf{P}_k - \alpha\mathbf{I} \quad \rightarrow \quad \lambda(\mathbf{P}_k^+) \leq \lambda(\mathbf{P}_k) - \alpha \quad \rightarrow \quad \|\mathbf{P}_k^+\| \leq \|\mathbf{P}_k\|. \tag{9}$$

Using the same arguments, it is possible to show that the posterior estimated-measurement error covariance $\mathbf{H}_k\mathbf{P}_k^+\mathbf{H}_k^{\mathrm{T}}$ is not greater than the prior $\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}$. Again, both the spectral norm and the trace can be used, and the inequality easily follows from

$$\mathbf{H}_k\mathbf{P}_k^+\mathbf{H}_k^{\mathrm{T}} = \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} - \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}. \tag{10}$$

Manipulating the previous expression

$$\begin{aligned}\mathbf{H}_k\mathbf{P}_k^+\mathbf{H}_k^{\mathrm{T}} &= \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k) \qquad (11)\\ &\quad - \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}\\ &= \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}(\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k)^{-1}\mathbf{R}_k. \qquad (12)\end{aligned}$$

If $\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} \gg \mathbf{R}_k$ then

$$\mathbf{H}_k\mathbf{P}_k^+\mathbf{H}_k^{\mathrm{T}} \simeq \mathbf{R}_k. \tag{13}$$

This last equation is of fundamental importance and is the motivation behind underweighting.

Second Order Update

The linear update is assumed to have the form

$$\hat{\mathbf{x}}_k^+ = \hat{\mathbf{x}}_k + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k) \tag{14}$$

which gives the posterior estimation error to be

$$\mathbf{e}_k^+ = \mathbf{e}_k + \mathbf{K}_k(\mathbf{y}_k - \hat{\mathbf{y}}_k) \simeq \mathbf{e}_k + \mathbf{K}_k\left[\mathbf{h}(\hat{\mathbf{x}}_k) + \mathbf{H}_k\mathbf{e}_k + \mathbf{b}_k + \boldsymbol{\eta}_k - \hat{\mathbf{y}}_k\right], \tag{15}$$

where $\mathbf{b}_k = \frac{1}{2}\,\mathcal{H}(\mathbf{h}_k, \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}})$ is the second order term of the Taylor series expansion of $\mathbf{h}$. The i^{th} component of $\mathbf{b}_k$ is given by

$$b_i(t_k) = \frac{1}{2}\,\mathbf{e}_k^{\mathrm{T}}\frac{\partial h_i(t_k)}{\partial \mathbf{x}_k \partial \mathbf{x}_k}\mathbf{e}_k = \frac{1}{2}\,\mathrm{trace}(\frac{\partial h_i(t_k)}{\partial \mathbf{x}_k \partial \mathbf{x}_k}\mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}) = \frac{1}{2}\,\mathcal{H}_i(\mathbf{h}_k, \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}), \tag{16}$$

where $h_i(t_k)$ is the i^{th} component of $\mathbf{h}_k$. To keep the filter unbiased the measurement estimate is chosen as $\hat{\mathbf{y}}_k = \mathbf{h}(\hat{\mathbf{x}}_k) + \hat{\mathbf{b}}_k$, where $\hat{\mathbf{b}}_k = 1/2\,\mathcal{H}(\mathbf{h}_k, \mathbf{P}_k)$.

The residual is given by

$$\boldsymbol{\epsilon}_k = \mathbf{y}_k - \hat{\mathbf{y}}_k = \mathbf{H}_k\mathbf{e}_k + 1/2\,\mathcal{H}(\mathbf{h}_k, \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}}) - 1/2\,\mathcal{H}(\mathbf{h}_k, \mathbf{P}_k) + \boldsymbol{\eta}_k, \tag{17}$$

where $\boldsymbol{\eta}_k$ is zero-mean measurement noise with covariance $\mathbf{R}_k$. The second order residual covariance is given by

$$\mathbf{W}_k = \mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{B}_k + \mathbf{R}_k \tag{18}$$

where

$$\mathbf{B}_k \triangleq 1/4\,\mathrm{E}\left\{\mathcal{H}(\mathbf{h}_k, \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}})\,\mathcal{H}(\mathbf{h}_k, \mathbf{e}_k\mathbf{e}_k^{\mathrm{T}})^{\mathrm{T}}\right\} - 1/4\,\mathcal{H}(\mathbf{h}_k, \mathbf{P}_k)\,\mathcal{H}(\mathbf{h}_k, \mathbf{P}_k)^{\mathrm{T}}. \tag{19}$$

Under the Gaussian approximation, the ij^{th} component of $\mathbf{B}_k$ is given by

$$B_{ij} = \frac{1}{2}\,\mathrm{trace}(\mathbf{H}'_j\mathbf{P}_k\,\mathbf{H}'_i\mathbf{P}_k). \tag{20}$$

If the contribution of the prior estimation error $\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}$ to the residuals covariance is much larger than the contribution of the measurement error $\mathbf{R}_k$, from Eq. (13) the EKF algorithm will produce $\mathbf{H}_k\mathbf{P}_k^{+}\mathbf{H}_k^{\mathrm{T}} \simeq \mathbf{R}_k$. But if $\mathbf{B}_k$ is of comparable magnitude of $\mathbf{R}_k$ then $\mathbf{H}_k\mathbf{P}_k^{+}\mathbf{H}_k^{\mathrm{T}} \simeq \mathbf{R}_k + \mathbf{B}_k$. Therefore a large underestimation of the posterior covariance can occur in the presence of nonlinearities when the estimated-measurement error covariance is much larger than the measurement error covariance.

UNDERWEIGHT

Underweighting is the process of modifying the residual covariance to reduce the update and compensate the second order effects. The most straight forward underweighting scheme is to add an underweighting factor $\mathbf{U}_k$ as

$$\mathbf{K}_k = \mathbf{P}_k\mathbf{H}_k^{\mathrm{T}}[\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^{\mathrm{T}} + \mathbf{R}_k + \mathbf{U}_k]^{-1}\,. \tag{21}$$

A symmetric, positive-definite underweighting factor will therefore increase the “denominator” of the Kalman gain, thus lowering the applied gain. the obvious choice is to select $\mathbf{U}_k = \mathbf{B}_k$, which is, for example, the current design for the Orion vehicle.[14] The advantage of this procedure is its theoretical foundation. The disadvantages include higher computational costs to calculate the second order partials, and the hypothesis of Gaussianity of the estimation error. While measurement noise is usually well represented as Gaussian, process noise usually is very much not Gaussian. Therefore in practical applications, the estimation error is not always easily represented as Gaussian, hence matrix $\mathbf{B}_k$ needs to be “tuned” to compensate for the Gaussian approximation. The process of tuning a positive definite matrix is less obvious than tuning a single scalar parameter.

Scaling the measurement error covariance

Another possible underweighting scheme is to scale the measurement noise, i.e., choosing

$$\mathbf{U}_k = k\mathbf{R}_k, \tag{22}$$

where k is positive. This approach has been successfully used;[15] however, it is not recommended from both a conceptual and a practical reason. Underweighting is necessary because of the second order terms of the Taylor series expansion of the measurement function. This expansion is in terms of the prior estimation error. Therefore it seems much more natural to express underweighting as a function of the prior estimation error covariance. It is reasonable to express the second order contribution as a percentage of the first order contribution. On the other hand the second order terms are not correlated at all to the measurement error covariance; therefore, choosing a constant coefficient to scale $\mathbf{R}_k$ seems less practical and will probably lead to a more complicated tuning procedure.

Lear's Method

The purpose of the underweighting matrix $\mathbf{U}_k$ is to account for the higher order terms of the Taylor series expansion of the nonlinear measurement function. A logical choice is to make $\mathbf{U}_k$ a percentage of the first order term

$$\mathbf{U}_k = k\mathbf{H}_k\mathbf{P}_k\mathbf{H}_k^T. \tag{23}$$

The Space Shuttle employs underweighting when $\sqrt{\operatorname{trace}\mathbf{P}_k} > \alpha$, where only the position states are included in the trace. The positive scalar α is a design parameter. For the Space Shuttle, k is selected to be 0.2 and α is selected to be 1000 meters.[10]

TUNING AIDS

In this section a technique to aid the tuning of the underweighting coefficient is presented. Although the Gaussian assumption does not necessarily hold, it can still be used as an aid to tune the coefficient k. On one side there is the second order residual covariance $\mathbf{W}_s$

$$\mathbf{W}_s = \mathbf{H}\mathbf{P}\mathbf{H}^\mathrm{T} + \mathbf{B} + \mathbf{R} \tag{24}$$

on the other the underweighed residual covariance

$$\mathbf{W}_u = (1+k)\mathbf{H}\mathbf{P}\mathbf{H}^\mathrm{T} + \mathbf{R}, \tag{25}$$

where all subscripts k were dropped for ease of notation.

Since all the quantities are symmetric semi-positive definite, the Kalman filter metric can be used, and the desired effect is to have

$$\operatorname{trace}\mathbf{W}_u \geq \operatorname{trace}\mathbf{W}_s, \tag{26}$$

from which we obtain immediately that

$$k \operatorname{trace}\mathbf{H}\mathbf{P}\mathbf{H}^\mathrm{T} \geq \operatorname{trace}\mathbf{B}. \tag{27}$$

Therefore a practical choice for the underweighting coefficient k is

$$k = \operatorname{trace}\mathbf{B} / \operatorname{trace}\mathbf{H}\mathbf{P}\mathbf{H}^\mathrm{T}. \tag{28}$$

Clearly this approach does not have great advantages over choosing $\mathbf{U} = \mathbf{B}$, since the second order matrix $\mathbf{B}$ needs to be calculated. However this approach is a valuable design aid, since it can be used in aiding the choice of a constant k. Underweighting should be applied when the second order contributions are comparable to the measurement noise, i.e. when

$$k \,\mathrm{trace}\, \mathbf{HPH}^{\mathrm{T}} \geq z \,\mathrm{trace}\, \mathbf{R} \quad \Rightarrow \quad \mathrm{trace}\, \mathbf{HPH}^{\mathrm{T}} \geq (z/k) \,\mathrm{trace}\, \mathbf{R} \tag{29}$$

were $0 < z < 1$ is a coefficient to be specified.

Often a bound on the Hessian of the measurement function exists, therefore it is not necessary to compute the second order term of the Taylor series expansion. It is known that the j^{th} term on the diagonal of $\mathbf{B}$ is bounded by[16]

$$B_{jj} \leq \frac{1}{2}(\|\mathbf{H}'_j\| \,\mathrm{trace}\, \mathbf{P})^2, \tag{30}$$

therefore

$$\mathrm{trace}\, \mathbf{B} \leq \frac{1}{2}(\mathrm{trace}\, \mathbf{P})^2 \sum_j \|\mathbf{H}'_j\|^2 \leq \frac{c}{2}(\mathrm{trace}\, \mathbf{P})^2, \tag{31}$$

where c is an upper bound of $\sum_j \|\mathbf{H}'_j\|^2$. It follows immediately that the value of k is related to

$$k = \frac{c}{2} \frac{(\mathrm{trace}\, \mathbf{P})^2}{\mathrm{trace}\, \mathbf{HPH}^{\mathrm{T}}}. \tag{32}$$

A good rule of thumbs in determining when underweighting should be applied is

$$\frac{c}{2}(\mathrm{trace}\, \mathbf{P})^2 > z \,\mathrm{trace}\, \mathbf{R}, \tag{33}$$

were $0 < z < 1$ is a coefficient to be specified. Since $\mathbf{P}$ is positive semi-definite, the previous condition is equivalent to

$$\mathrm{trace}\, \mathbf{P} > (\frac{2z}{c} \,\mathrm{trace}\, \mathbf{R})^{\frac{1}{2}}. \tag{34}$$

Notice that Eq. (34) is very similar to Lear's rule on when to apply underweight.

Equation (32) is derived using upper bounds. These upper bound could be artificially loose if some precautions are not taken. An artificially loose upper bound could negate the contribution of this analysis, which is to obtain a good estimate of what k should be in a single run. The most obvious precaution is to only include in the trace of $\mathbf{P}$ the states that actually contribute to the measurement. If the partial of the measurement with respect to a state is zero that state should not be included when the trace is calculated. In a spacecraft rendezvous scenario, relative measurements are often very precise, while the inertial knowledge of the two vehicles is less accurate. Therefore taking the trace of the absolute uncertainties would lead to a very loose bound, while the trace of the relative position covariance should be taken.

Simple Lidar Example

Consider a two-spacecraft system with a relative position $\mathbf{r}_{rel}$ and relative velocity $\mathbf{v}_{rel}$. The state is given by $\mathbf{r}_{rel}$, $\mathbf{v}_{rel}$, and the attitude of the chaser vehicle

$$\mathbf{x} = \begin{bmatrix} \mathbf{r}_{rel}^{\mathrm{T}} & \mathbf{v}_{rel}^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}}.$$

The range measurement is given by the range between the two vehicles $\rho = \|\mathbf{r}_{rel}\|$. The Jacobian (with respect to position only) is given by

$$\mathbf{H} = \frac{1}{\rho}\mathbf{r}_{rel}^{\mathrm{T}}.$$

The Hessian (with respect to position only) is given by

$$\mathbf{H}' = \frac{1}{\rho}\left(\mathbf{I} - \mathbf{r}_{rel}\mathbf{r}_{rel}^{\mathrm{T}}/\rho^2\right) = -\frac{1}{\rho^3}[\mathbf{r}_{rel}\times]^2.$$

Assume that the standard deviation of the range measurement error is $0.1m$, that the lidar turns on when the two vehicles are 1 kilometer apart, and that the filter is initialized with GPS position estimate of the chaser and a ground update of the target's position. The GPS accuracy is assumed 10 meters per axis 1σ, the ground update is assumed to have accuracy 20 meters per axis 1σ, the two quantities are assumed uncorrelated and the relative position covariance is given by

$$\mathbf{P}_{rr} = 10^2\mathbf{I}_{3\times3} + 20^2\mathbf{I}_{3\times3} = 500\mathbf{I}_{3\times3}\ m^2. \tag{35}$$

Using the second order filter we obtain

$$B = \frac{1}{2}(500)^2\,\mathrm{trace}(\mathbf{H}'\mathbf{H}') = \frac{1}{2}250,000\frac{2}{\rho^2} = 0.25\ m^2. \tag{36}$$

since $R = 0.01\ m^2$ it is clear that the second order effects are quite important. Also

$$\mathbf{HPH}^{\mathrm{T}} = 500\ m^2$$

The second order residual covariance is

$$W_s = \mathbf{HPH}^{\mathrm{T}} + B + R = 500.26.$$

Using the second order filter, the estimate of the posterior relative position uncertainty along the direction connecting the two vehicles is

$$\mathbf{HP}^{+}\mathbf{H}^{\mathrm{T}} = \mathbf{HPH}^{\mathrm{T}} - (\mathbf{HPH}^{\mathrm{T}})^2/W_s^{-1} = 1.008\ m^2.$$

Using the EKF and its first order residual covariance one obtains

$$\mathbf{HP}^{+}\mathbf{H}^{\mathrm{T}} \simeq 0.01\ m^2,$$

which could create problems to the filter by underestimating the posterior error covariance.

The norm of the Hessian does not need to be bounded because is known analytically $\|\mathbf{H}'\| = 2/\rho$. (Matrix $\mathbf{H}'$ is symmetric with a zero eigenvalue and a repeated eigenvalue at $-1/\rho$.) A good choice for the underweighting coefficient is

$$k = \frac{1}{2\rho^2}\frac{(\mathrm{trace}\,P)^2}{\mathbf{HPH}^{\mathrm{T}}} = \frac{0.5}{10^6}\frac{[3\,(500)]^2}{500} \simeq 0.00225.$$

The underweighted residual covariance is

$$W_u = (1+k)\mathbf{HPH}^{\mathrm{T}} + R = 501.135.$$

Using the underweighted EKF, the estimate of the posterior uncertainty in relative position along the direction connecting the two vehicles is

$$\mathbf{HP}^{+}\mathbf{H}^{\mathrm{T}} = \mathbf{HPH}^{\mathrm{T}} - (\mathbf{HPH}^{\mathrm{T}})^2/W_u^{-1} \simeq 1.13\ m^2.$$

NUMERICAL RESULTS

This section presents results from a numerical simulation. For this example the filter is initialized with the same quantities as those of the previous section. Besides relative position and velocity, the filter's state includes the absolute position and velocity of the target, accelerometer bias, chaser's attitude, and lidar bias states. The lidar's range noise and bias are assumed to be range dependent, varying linearly from 0.01 meters at docking to 0.1 meters at a relative distance of 100 meters. For any range grater than 100 meters the noise and bias are fixed to the maximum value of 0.1 meters. Azimuth and elevation bias and noise are both set to 0.1 degrees 1σ. At a relative distance of 50 meters the lidar loses track of the reflector. It is assumed that the lidar takes 2 minutes to scan its entire field of view (FOV), it is also assumed that the lidar scans its FOV twice before reacquiring. Therefore for 4 minutes the lidar stops providing measurements. After 4 minutes the lidar resumes providing measurements. Figures 1 and 2 show the performance of the filter without underweight for a set of 100 Monte Carlo runs. The blue lines are the 100 samples of the estimation error, the black lines are the 100 instances of the 3σ filter's standard deviation. Figure 3 zooms in the position error after loss of track. It can be seen that when the filter reacquires the uncertainty rapidly decreases. However the error does not decrease fast enough to follow the covariance. Therefore when the next measurement becomes available the predicted covariance of the residual is very small, and the measurement is rejected because the the residual is more than 5 times its predicted standard deviation. In the absence of residual editing, i.e. if the measurements are accepted regardless of how big the residuals are, it is likely that the filter would diverge all together. When measurement are rejected, the filter propagates only and the uncertainty increases.

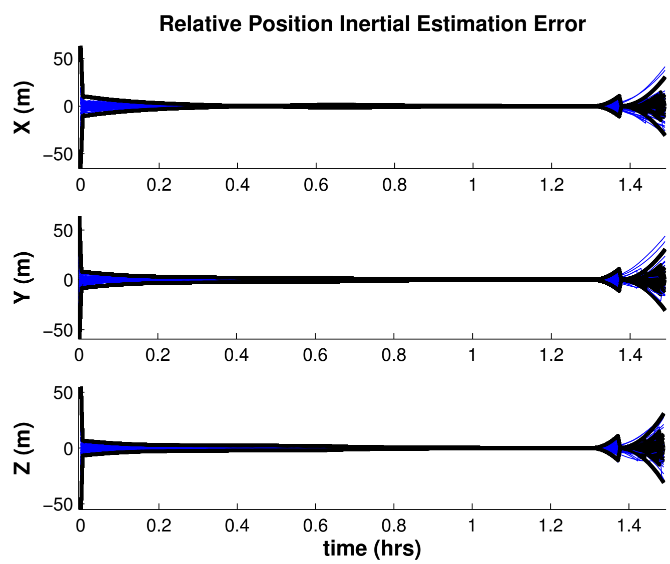

Figure 1. Relative Position Performance without Underweight

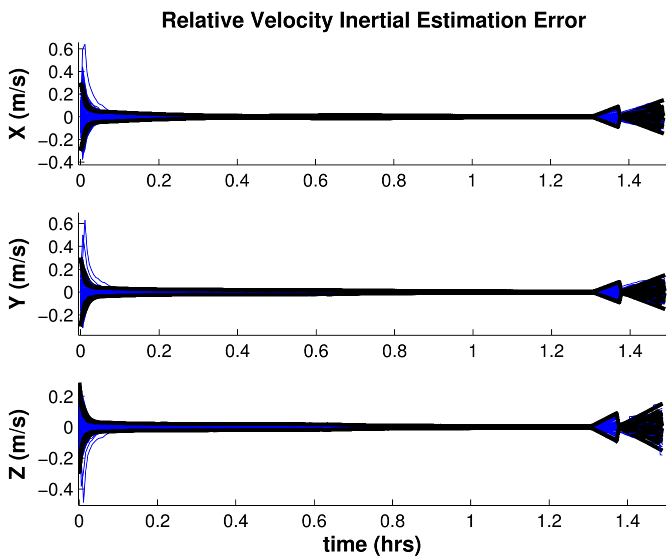

Figure 2. Relative Velocity Performance without Underweight

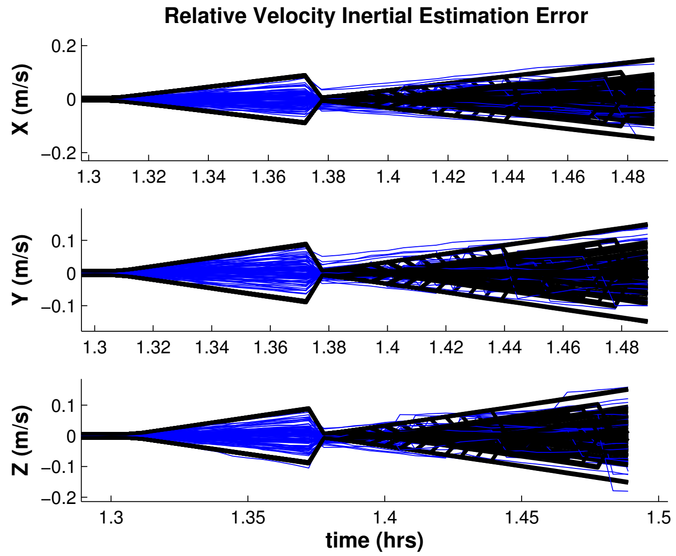

Figure 3. Relative Position Performance without Underweight

Figures 4 and 5 show the results when underweight is applied. The underweight coefficient in chosen using Eq. (32) and it is apply when the condition of Eq. (33) are met. Constant z in Eq. (33) is chosen as 0.1. The underweight coefficient is calculated for the range measurement which is the measurement that needs underweighting the most. All three components of the lidar measurement are processed together with the same underweighting coefficient. It can be seen that the proposed method allows the filter to re-converge after the lidar returns to provide measurements.

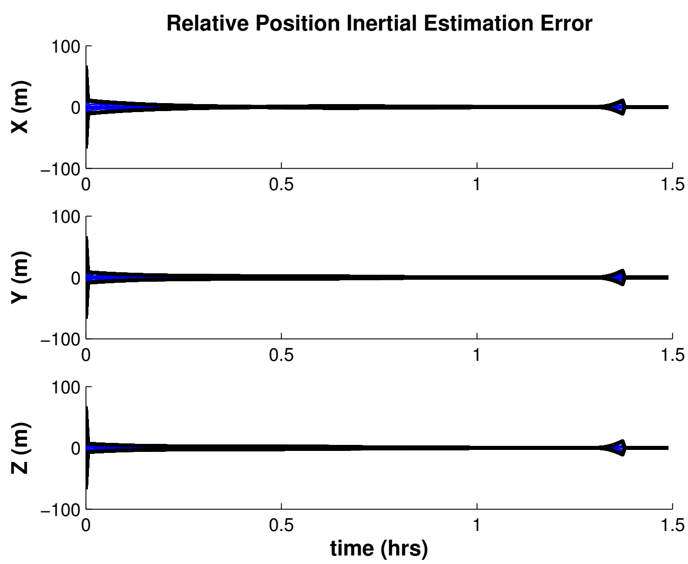

Figure 4. Relative Position Performance with Underweight

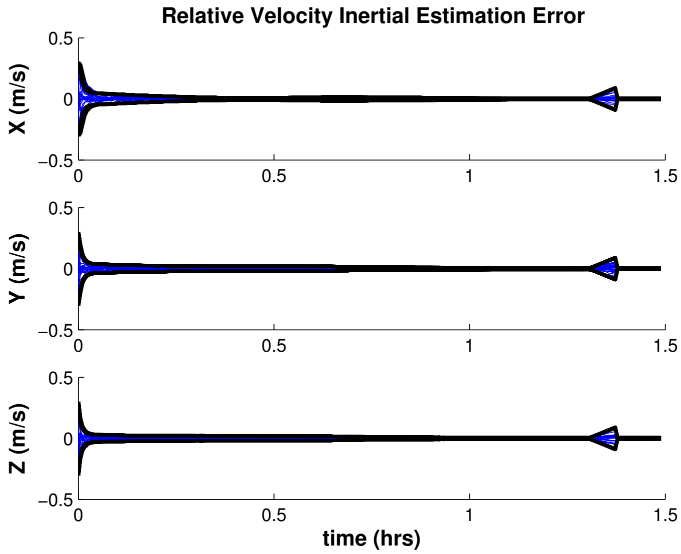

Figure 5. Relative Velocity Performance with Underweight

CONCLUSIONS

In this work the purpose of underweighting is reviewed. The original underweighting scheme by Lear is introduced. Other existing schemes are discussed. Techniques to aid the choice of the tuning parameters of Lear's underweighting schemes are introduced. A numerical example showing the need for underweighting and the performance of the proposed method is illustrated. The numerical results suggest that the proposed solution is a viable method to tune the underweighted extended Kalman filter in the presence of lidar measurements.

REFERENCES

[1] A. Gelb, ed., *Applied Optimal Estimation*. The MIT Press, Cambridge, MA, 1974.

[2] R. E. Kalman, "A New Approach to Linear Filtering and Prediction Problems," *Transactions of the ASME – Journal of Basic Engineering*, Vol. 82, 1960, pp. 35–45.

[3] R. E. Kalman and R. S. Bucy, "New Results in Linear Filtering and Prediction," *Journal of Basic Engineering*, 1961, pp. 83–95.

[4] A. H. Jazwinski, *Stochastic Processes and Filtering Theory*. Dover, 2007.

[5] P. S. Maybeck, *Stochastic Models, Estimation, and Control*, Vol. 2. Academic Press, 1982.

[6] P. J. Huxel and R. H. Bishop, "Fusing Inertial and Relative Measurements for Inertial Navigation in the Presence of Large State Error Covarianaces," *Proceedings of the 16th AAS/AIAA Spaceflight Mechanics Meeting*, January 2006.

[7] S. J. Julier and J. K. Uhlmann, "A New Extension of the Kalman Filter to Nonlinear Systems," *Int. Symp. Aerospace/Defense Sensing, Simul. and Controls*, Orlando, FL, 1997.

[8] W. M. Lear, "Kalman Filtering Techniques," Tech. Rep. JSC-20688, NASA Johnson Space Center, Houston, TX, 1985.

[9] B. A. Kriegsman and Y. C. Tau, "Shuttle Navigation System for Entry and Landing Mission Phases," *Journal of Spacecraft*, Vol. 12, April 1975, pp. 213–219.

[10] W. M. Lear, "Multi-Phase Navigation Program for the Space Shuttle Orbiter," Internal Note No. 73-FM-132, NASA Johnson Space Center, Houston, TX, 1973.

[11] E. R. Schiesser, "Measurement Underweighting," E-mail to the author, July 2009.

[12] L. Perea *et al.*, "Nonlinearity in Sensor Fusion: Divergence Issues in EKF, modified truncated SOF, and UKF," *AIAA Guidance, Navigation and Control Conference and Exhibit*, Hilton Head, SC, August 20–23, 2007.

[13] M. S. Grewal and A. P. Andrews, *Kalman Filtering: Theory and Practice*. John Wiley & Sons, 1993.

[14] C. Hanak and R. Zanetti, "Relative Navigation of the Orion Vehicle," *Proceedings of the F. Landis Markley Astronautics Symposium held June 30 – July 2, 2008, Cambridge, MD*, Vol. 132 of *Advances in the Astronautical Sciences*, 2008, pp. 793–811. AAS 08-306.

[15] S. C. Jenkins and D. K. Geller, "State Estimation and Targeting for Autonomous Rendezvous and Proximity Operations," *Proceedings of the AAS/AIAA Astrodynamics Specialists Conference*, Mackinac Island, MI, August 19–23 2007.

[16] P. J. Huxel and R. H. Bishop, "Navigation Algorithms and Observability Analysis for Formation Flying Missions," *Journal of Guidance, Control, and Dynamics*, Vol. 32, July–August 2009, pp. 1218–1231.