ROTATIONS, TRANSFORMATIONS, LEFT QUATERNIONS, RIGHT QUATERNIONS?

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This paper surveys the two fundamental possible choices in representing the attitude of an aerospace vehicle: active and passive rotations. The consequences of the choice between the two are detailed for the two most common attitude parameterizations, a three-by-three orthogonal matrix and the quaternion. Successive rotations are also reviewed in this context as well as the attitude kinematic equations.

INTRODUCTION

It has been 25 years since John Junkins and Malcolm Shuster guest edited a special issue of the Journal of Astronautical Sciences on attitude representations. The issue in general, and reference [1] in particular, detail many parameterizations of the rotation group with the constant assumption that attitudes are represented as passive rotations. This work aims to detail in a clear and concise manner the difference between the active and passive representations of attitude and the implications of either choice. In a single reference, all fundamental equations are derived and the implications of each choice are discussed. As such, this paper is of interest to researchers and practitioners in the field that seek a single reference detailing the two approaches with all relevant equations derived in a consistent notation.

The attitude of a spacecraft is simply its orientation. When we think of the position of a spacecraft as a physical quantity, free of the definition of a cartesian frame and associated coordinates, we usually think of an arrow (vector) starting at an origin and terminating at the location of interest. The orientation (attitude) is similar in that it can be expressed as a rotation from a reference orientation to the current spacecraft orientation. It is well known, however, that rotations do not form a vector space. This interpretation of attitude as a rotation from the reference orientation to the current orientation is known as "active" interpretation of rotations [2]. This approach takes the point of view of an observer fixed with the reference that sees the spacecraft rotate and represents the attitude of the spacecraft with the physical rotation needed to take the reference frame into the body frame. This interpretation is therefore the most natural when the observer of the motion is fixed with the reference frame and sees the spacecraft actively moving. As such, this interpretation is perhaps most natural when describing the motion of an observed object that we are not controlling and when the direction of successive rotations are known in the inertial frame. This is the interpretation of attitude that Shuster defines as "historical" [1]. This interpretation is consistent with the study of translational dynamics, which rarely takes the opposite point of view of an observer moving with the vehicle that sees the Earth orbit around the spacecraft but in the opposite direction.

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In practical applications, the purely geometric, coordinate-less representation of vectors and rotations is abandoned for the convenience of introducing Cartesian coordinates and representing physical quantities with numbers: three for a vector, four for a quaternion, and nine for an orthogonal matrix. The reference orientation is any right-handed coordinate system, but throughout this work, the reference orientation will always be that of an inertial coordinate system, denoted as i. The conclusions of this paper are not affected by the choice of reference system. Another right-hand coordinate system is attached to the spacecraft and rotates with it, it is the body-fixed frame, or simply body frame b.

The passive interpretation of rotations takes the point of view of the rotating spacecraft, therefore the observer does not experience the actual rotation of the spacecraft, rather it sees the inertial frame rotate in the opposite direction. The passive interpretation is perhaps most natural when knowledge of the direction of successive rotations is known in the body-fixed frame and when describing the motion of an object that we are controlling. This interpretation is also convenient in the study of attitude kinematics, because the angular velocity of the spacecraft is more readily available in a body-fixed frame using Euler equation of rotational dynamics. Expressing the rotational dynamics in inertial coordinates will cause inertial properties of the spacecraft to change, while the inertia matrix is constant in the body frame. The term coordinate transformation or simply transformation is used in this work as a synonym with passive rotation.

The two approaches are clearly related, but due to the non-vectorial nature of rotations, they have subtle consequences that merit full discussion. One of these consequences is the difference between the rotation matrix and the transformation matrix. Another consequence is the definition of left quaternion used in the Space Shuttle onboard flight software [3].

ROTATIONS

Physical three-dimensional (3D) vectorial quantities, such as the position of a point, exist regardless of the definition of a coordinate system. Similarly, vectorial operations (such as vector sum, cross product, and projections/dot products) can be constructed purely geometrically without resorting to numerical coordinates. When doing calculations, however, it is usually more convenient to express these physical quantities with three scalar numbers, which are the projections of the vector into three orthogonal coordinates.

While rotations are not vectors, the rotation of an object in 3D space is a physical action that can also be described purely geometrically. Euler famously stated that each 3D rotation is uniquely defined by an axis, represented by the unit vector $\mathbf{n}$, and an angle θ. Let's denote with $\mathcal{R}(\mathbf{v})$ the physical operation of rotating a vector $\mathbf{v}$, and let's denote with $\mathbf{v}'$ the rotated vector. Let's express $\mathbf{v}$ as the sum of a component parallel to $\mathbf{n}$ and one perpendicular to it, as

$$\mathbf{v} = (\mathbf{n} \cdot \mathbf{v})\, \mathbf{n} - \mathbf{n} \times (\mathbf{n} \times \mathbf{v}) \tag{1}$$

Let's define an orthogonal triad with the the z-axis $\mathbf{z}$ coinciding with $\mathbf{n}$, the y-axis $\mathbf{y}$ aligned with $\mathbf{n} \times \mathbf{v}$, and the x-axis $\mathbf{x}$ aligned with $-\mathbf{n} \times (\mathbf{n} \times \mathbf{v})$.

$$\mathbf{v} = (\mathbf{n} \cdot \mathbf{v})\, \mathbf{z} + \sqrt{\mathbf{v} \cdot \mathbf{v} - (\mathbf{n} \cdot \mathbf{v})^2}\, \mathbf{x}$$

The component of $\mathbf{v}$ parallel to $\mathbf{z}$ (the Euler or rotation axis) remains unchanged, while the component parallel to $\mathbf{x}$ rotates counterclockwise (right–hand rule) by the rotation angle (Euler angle)

$$\begin{aligned}
\mathbf{v}' &= (\mathbf{n}\cdot\mathbf{v})\,\mathbf{z} + \cos\theta\,\sqrt{\mathbf{v}\cdot\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})^2}\,\mathbf{x} + \sin\theta\,\sqrt{\mathbf{v}\cdot\mathbf{v}-(\mathbf{n}\cdot\mathbf{v})^2}\,\mathbf{y} \\
&= (\mathbf{v}\cdot\mathbf{n})\,\mathbf{n} - \cos\theta\,\mathbf{n}\times(\mathbf{n}\times\mathbf{v}) + \sin\theta\,\mathbf{n}\times\mathbf{v} \\
&= \mathbf{n}\mathbf{n}^{\mathrm{T}}\mathbf{v} + \sin\theta\,(\mathbf{n}\times\mathbf{v}) - \cos\theta\,[\mathbf{n}\times(\mathbf{n}\times\mathbf{v})] \\
&= \mathbf{v} + \sin\theta\,(\mathbf{n}\times\mathbf{v}) + (1-\cos\theta)\,[\mathbf{n}\times(\mathbf{n}\times\mathbf{v})] \qquad (2)
\end{aligned}$$

where the following identity was used

$$\mathbf{n}\times(\mathbf{n}\times\mathbf{v}) = \mathbf{n}(\mathbf{n}\cdot\mathbf{v}) - \|\mathbf{n}\|^2\,\mathbf{v}$$

Eq. (2) expressed the rotation of a vector with a coordinate independent formula. When doing operations with components, we must express vectors $\mathbf{v}$ and $\mathbf{n}$ in the same frame, and the components of the rotated vector $\mathbf{v}'$ are therefore also expressed in the same frame. This is the most natural way to study physical rotations: an external viewer and a fixed coordinate system that looks at the rotating body. Let's use a superscript i to indicate that a vector is to be interpreted as three numbers that represent coordinates in the inertial frame rather than a coordinate-less physical vector

$$\begin{aligned}
\mathcal{R}(\mathbf{v}^i) &= \mathbf{v}^i + \sin\theta\,(\mathbf{n}^i\times\mathbf{v}^i) + (1-\cos\theta)\,[\mathbf{n}^i\times\ (\mathbf{n}^i\times\mathbf{v}^i)] \\
&= \left(\mathbf{I}_{3\times3} + \sin\theta\,[\mathbf{n}^i\times]\ + (1-\cos\theta)\,[\mathbf{n}^i\times]^2\right)\mathbf{v}^i \qquad (3) \\
&= \mathbf{R}^i\,\mathbf{v}^i \qquad (4)
\end{aligned}$$

where $\mathbf{R}^i$ is called the rotation matrix and the above equalities use the definition of the 3×3 skew-symmetric cross-product matrix $[\mathbf{w}\times]$ formed from a 3D vector $\mathbf{w}$. In this example, the rotation matrix is also referenced to the inertial frame because it must be calculated from the Euler vector $\mathbf{n}$ expressed in the same frame as the vector $\mathbf{v}$ we intend to rotate. Exactly like vectors, rotations exist regardless of frame definitions and, exactly like vectors, when doing numerical calculations we define a coordinate system to express them.

The Euler axis remains unchanged across a rotation, i.e. $\mathbf{R}\,\mathbf{n} = \mathbf{n}$. While Eq. (3) is ubiquitous in dynamics, mathematics, and computer science; in aerospace engineering we usually see a minus sign in front of $\sin\theta$. That is because we usually interpret rotations as passive.

COORDINATE TRANSFORMATIONS

The attitude (orientation) of a spacecraft can be expressed as the rotation that takes the inertial frame into the body fixed frame, we will denote with $\mathbf{R}_{i\to b}$ the rotation matrix that rotates the inertial frame into the body frame and call this representation of attitude active rotations or simply rotations. Notice that the Euler axis of this rotation has the same numerical coordinates in both the inertial and body frame, therefore $\mathbf{R}_{i\to b}$ is the same when expressed in either frame. In the active interpretation the body frame actively moves and it is sometimes referred to as the *alibi* description from Latin. In the passive, or *alias* description, the observer is fixed with the rotating body, therefore body-fixed quantities are perceived as stationary. In the passive description we concentrate on the fact that $\mathbf{v}^b$ is constant for a rigid body and write

$$\mathbf{v}^b = \mathbf{T}_i^b\,\mathbf{v}^i \qquad (5)$$

where $\mathbf{T}_i^b$ is the direction cosine matrix (DCM), also known as coordinate transformation matrix, which is the matrix that changes the coordinates of a vector from frame i to frame b. Therefore a passive rotation matrix is the same as a DCM or transformation matrix. In order to avoid confusion, we prefer to reserve the name rotation only to active ones, and we will refer to passive rotations as transformations. The use of DCMs to represent attitude is so prevalent that they are often referred to as Attitude matrices.

Let $\{\mathbf{x}_i, \mathbf{y}_i, \mathbf{z}_i\}$ be the basis of the inertial frame and $\{\mathbf{x}_b, \mathbf{y}_b, \mathbf{z}_b\}$ the basis of the body frame. Since $\mathbf{R}_{i\to b}$ is the rotation matrix that rotates the inertial frame into the body frame, we have that

$$\begin{bmatrix}\mathbf{x}_b & \mathbf{y}_b & \mathbf{z}_b\end{bmatrix} = \mathbf{R}_{i\to b} \begin{bmatrix}\mathbf{x}_i & \mathbf{y}_i & \mathbf{z}_i\end{bmatrix} \tag{6}$$

It is also clear that

$$\begin{bmatrix}\mathbf{x}_i^i & \mathbf{y}_i^i & \mathbf{z}_i^i\end{bmatrix} = \mathbf{I}_{3\times 3} \tag{7}$$

therefore

$$\mathbf{v}^b = \begin{bmatrix}(\mathbf{x}_b^i)^{\mathrm{T}}\mathbf{v}^i \\ (\mathbf{y}_b^i)^{\mathrm{T}}\mathbf{v}^i \\ (\mathbf{z}_b^i)^{\mathrm{T}}\mathbf{v}^i\end{bmatrix} = \begin{bmatrix}(\mathbf{R}_{i\to b}\,\mathbf{x}_i^i)^{\mathrm{T}} \\ (\mathbf{R}_{i\to b}\,\mathbf{y}_i^i)^{\mathrm{T}} \\ (\mathbf{R}_{i\to b}\,\mathbf{z}_i^i)^{\mathrm{T}}\end{bmatrix} \mathbf{v}^i = \begin{bmatrix}(\mathbf{x}_i^i)^{\mathrm{T}} \\ (\mathbf{y}_i^i)^{\mathrm{T}} \\ (\mathbf{z}_i^i)^{\mathrm{T}}\end{bmatrix} \mathbf{R}_{i\to b}^{\mathrm{T}}\,\mathbf{v}^i = \mathbf{R}_{i\to b}^{\mathrm{T}}\,\mathbf{v}^i = \mathbf{T}_i^b\,\mathbf{v}^i \tag{8}$$

hence

$$\mathbf{T}_i^b = \mathbf{R}_{i\to b}^{\mathrm{T}} \tag{9}$$

The DCM (or Transformation matrix or Attitude matrix or passive rotation matrix) from i to b is the transpose of the (active) rotation matrix that takes i into b. Some practitioners refer to active rotations as “rotating the vector” and passive rotations as “rotating the frame”.

If $\mathbf{n}$ and θ are the Euler axis and angle of the rotation that takes frame i into b, then

$$\begin{aligned}\mathbf{T}_i^b &= \left(\mathbf{I}_{3\times 3} + \sin\theta\,[\mathbf{n}\times]\, + (1-\cos\theta)\,[\mathbf{n}\times]^2\right)^{\mathrm{T}} && (10)\\ &= \mathbf{I}_{3\times 3} - \sin\theta\,[\mathbf{n}\times]\, + (1-\cos\theta)\,[\mathbf{n}\times]^2 && (11)\end{aligned}$$

this equation with the minus sign in front of $\sin\theta$ is most commonly used in aerospace engineering applications. When numerically calculating the nine entries of the 3×3 matrix $\mathbf{T}_i^b$ from Eq. (11); in which frame should we express the coordinates of $\mathbf{n}$? Either i or b, it does not matter since their numerical values coincide.

In summary, the attitude of a rigid body is the relative orientation of a frame fixed with the body with respect to a reference frame. Taking the convention that we start from the reference frame and go to the body frame, we can decide to describe the attitude in two different ways. First, as the physical rotation that takes the reference frame into the body frame, we parameterize this rotation with a 3×3 orthogonal matrix $\mathbf{R}_{i\to b}$. Alternatively, we can express the attitude as the DCM $\mathbf{T}_i^b$ that transforms the coordinates of a vector from inertial coordinates to body coordinates. Matrix $\mathbf{T}_i^b$ is also referred as attitude matrix, transformation matrix, and (unfortunately) even as rotation matrix, with the usually unspoken understanding that it is a passive rotation. The two matrices are related as $\mathbf{T}_i^b = \mathbf{R}_{i\to b}^{\mathrm{T}}$. Matrix $\mathbf{R}_{i\to b}$ is a rotation and it can be expressed in any frame we wish by coordinatizing its Euler axis accordingly. To calculate $\mathbf{T}_i^b$, on the other hand, the Euler axis of $\mathbf{R}_{i\to b}^{\mathrm{T}}$ must be coordinatized in the i or b frame, since this matrix is used with vectors expressed in those coordinates.

COMPOSITION OF ROTATIONS

From Euler's 1775 famous theorem stating that displacements about a fixed point can be represented with a rotation about an axis [4], an attitude representation comes natural: the so called Euler axis and angle [5, 6]*. This parametrization is the most intuitive method to represent a rotation, while many authors view the directions cosine matrix as the most fundamental attitude parametrization [7].

From Euler's theorem we known that any number of successive rotations can be expressed as a single one. Therefore there must be a composition formula that combines two successive rotations $\mathcal{R}_1$ and $\mathcal{R}_2$ into a single one $\mathcal{R}_3$

$$\mathcal{R}_3(\mathbf{v}) = \mathcal{R}_2(\mathcal{R}_1(\mathbf{v})) = (\mathcal{R}_2 \circ \mathcal{R}_1)\,(\mathbf{v}) \tag{12}$$

using the rotation matrix the composition rule is trivially found as the row-by-column matrix multiplication

$$\mathbf{v}'' = \mathbf{R}_3\mathbf{v} = \mathbf{R}_2\mathbf{v}' = \mathbf{R}_2\,(\mathbf{R}_1\mathbf{v}) = (\mathbf{R}_2\mathbf{R}_1)\,\mathbf{v} \tag{13}$$

expanding the discussion above, in doing calculations with components, $\mathbf{v}$, $\mathbf{v}'$ and $\mathbf{v}''$ must be all expressed in the same frame as must be $\mathbf{n}_1$ and $\mathbf{n}_2$; the Euler axes of the two rotations. This means that the direction of the second rotation is referenced to fixed-space (i.e. non rotating). For example a first rotation with an Euler axis coinciding with the inertial x-axis followed by a second rotation around the inertial y-axis. The Euler axis of the second rotation does not rotate together with $\mathbf{v}$ due to $\mathcal{R}_1$.

Paul [8] presents a very elegant derivation of the composition rule for Euler axis and angle. The citations in Paul's article are also valuable since they point to a few different approaches to solve the same problem. The derivation by Paul is shown in the appendix. The composition rule for Euler axis and angle was first introduced by Rodrigues [9, page 408] and its derivation assumes that the Euler axis of the second rotation $\mathbf{n}_2$ is fixed and known a priori. To reinforce this fact, we will denote it as $\mathbf{n}_{2f}$. This is the most natural point of view for a coordinate-less description of rotations and for the active description, where we observe the rotation from the reference frame. With this premises, the composition rule of the Euler axis and angle is

$$\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathbf{n}_{2f}\cdot\mathbf{n}_1 \tag{14}$$

$$\mathbf{n}_3 = \frac{\sin(\theta_1/2)\cos(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_1 + \frac{\sin(\theta_2/2)\cos(\theta_1/2)}{\sin(\theta_3/2)}\mathbf{n}_{2f} + \frac{\sin(\theta_1/2)\sin(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_{2f}\times\mathbf{n}_1 \tag{15}$$

this is the most common formula seen in mathematics, but it is not what we usually see in aerospace applications. Successive rotations in aerospace systems are usually referred to the rotating space. For example, in the Euler sequence roll-pitch-yaw, the pitch axis rotates together with the first (roll) rotation, and therefore it is not known *a priori*, but rather it is a function of the first rotation. Similarly, the third rotation along the yaw axis rotates due to the first two rotations.

Assume the second rotation is expressed with respect to the rotating space; the second Euler axis $\mathbf{n}_{2r}$ rotates with the first rotation. Therefore, we can use the matrix composition rule above but we

*At Euler's time vector notation did not exist. Euler notation consists in a rotation angle ϕ and the direction cosines of the axis $\cos\zeta$, $\cos\eta$, $\cos\theta$, all his equations are derived component by component.

must replace "$\mathbf{n}_{2f}$" with "$\mathbf{R}_1\mathbf{n}_{2r}$"

$$\begin{aligned}
\mathbf{R}_3 &= \mathbf{R}_{2f}\mathbf{R}_1 \\
&= \left(\mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{R}_1\mathbf{n}_{2r}\times]\; + (1-\cos\theta_2)\,[\mathbf{R}_1\mathbf{n}_{2r}\times]^2\right)\mathbf{R}_1 \\
&= \left(\mathbf{I}_{3\times3} + \sin\theta_2\,\mathbf{R}_1[\mathbf{n}_{2r}\times]\mathbf{R}_1^{\mathrm{T}} + (1-\cos\theta_2)\,\mathbf{R}_1[\mathbf{n}_{2r}\times]^2\mathbf{R}_1^{\mathrm{T}}\right)\mathbf{R}_1 \\
&= \mathbf{R}_1\left(\mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{n}_{2r}\times] + (1-\cos\theta_2)\,[\mathbf{n}_{2r}\times]^2\right) \\
&= \mathbf{R}_1\mathbf{R}_{2r}
\end{aligned} \tag{16}$$

therefore successive rotations referenced to the rotating frame compose in the opposite order as those reference to the fixed frame. By flipping the order of the arguments in Eq. (14) and Eq. (15), the only difference is due to the non-commutative nature of the cross product

$$\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathbf{n}_{2r}\cdot\mathbf{n}_1 \tag{17}$$

$$\mathbf{n}_3 = \frac{\sin(\theta_1/2)\cos(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_1 + \frac{\sin(\theta_2/2)\cos(\theta_1/2)}{\sin(\theta_3/2)}\mathbf{n}_{2r} - \frac{\sin(\theta_1/2)\sin(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_{2r}\times\mathbf{n}_1 \tag{18}$$

this is the most common formula seen in aerospace engineering.

In the mathematical notation of function composition $(f\circ g)(x) = f(g(x))$, the first function applied is the one on the right (g in this case). Therefore, the rotations composition rule for matrices is very pleasing when we write the second rotation referenced to fixed-space

$$\mathcal{R}_2\circ\mathcal{R}_1 \to \mathbf{R}_{2f}\mathbf{R}_1 \tag{19}$$

but when we write the second rotation referenced to the rotating-space the composition rule is in the "opposite" or "wrong" order

$$\mathcal{R}_2\circ\mathcal{R}_1 \to \mathbf{R}_1\mathbf{R}_{2r} \tag{20}$$

we do notice, however, that function composition and row-by-column multiplication are completely different concepts and their order do not need to match.

Consider two successive coordinate transformations: $\mathbf{T}_a^b$ whose Euler axis is expressed in either the the a or b frame (it has the same numerical value in either frame), and $\mathbf{T}_b^c$ whose Euler axis is expressed in either the the c or b frame (it has the same numerical value in either frame).

$$\mathbf{v}^c = \mathbf{T}_b^c\,\mathbf{v}^b = \mathbf{T}_b^c\,\mathbf{T}_a^b\,\mathbf{v}^a = \mathbf{T}_a^c\,\mathbf{v}^a \tag{21}$$

we immediately notice that the composition rule of transformations is also the row-by-column multiplication.

In multiplying $\mathbf{T}_b^c\,\mathbf{T}_a^b$ are we doing a cross product between Euler axes in different frames? Let $\mathbf{n}_1$ be the Euler axis of the first transformation $\mathbf{T}_a^b = \mathbf{T}_1 = \mathbf{R}_1^{\mathrm{T}}$ and $\mathbf{n}_2$ be the Euler axis of $\mathbf{T}_b^c = \mathbf{T}_2 = \mathbf{R}_2^{\mathrm{T}}$, so that

$$\begin{aligned}
\mathbf{R}_1^a = \mathbf{R}_1^b &= \mathbf{I}_{3\times3} + \sin\theta_1\,[\mathbf{n}_1^a\times]\; + (1-\cos\theta_1)\,[\mathbf{n}_1^a\times]^2 \\
&= \mathbf{I}_{3\times3} + \sin\theta_1\,[\mathbf{n}_1^b\times]\; + (1-\cos\theta_1)\,[\mathbf{n}_1^b\times]^2 \\
\mathbf{R}_2^b = \mathbf{R}_2^c &= \mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{n}_2^b\times]\; + (1-\cos\theta_2)\,[\mathbf{n}_2^b\times]^2 \\
&= \mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{n}_2^c\times]\; + (1-\cos\theta_2)\,[\mathbf{n}_2^c\times]^2
\end{aligned}$$

The Euler axis of $\mathbf{T}_a^c$ must be expressed in either the a or c frame, let's, for example, do the calculations in the a frame. We have that

$$\mathbf{n}_2^a = \mathbf{T}_b^a\,\mathbf{n}_2^b = \mathbf{R}_1^b\,\mathbf{n}_2^b = \mathbf{R}_1^a\,\mathbf{n}_2^b$$

and

$$\begin{aligned}
\mathbf{T}_a^c &= (\mathbf{R}_2^a\,\mathbf{R}_1^a)^{\mathrm{T}} \\
&= \left[\left(\mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{n}_2^a\times]\ + (1-\cos\theta_2)\,[\mathbf{n}_2^a\times]^2\right)\mathbf{R}_1^a\right]^{\mathrm{T}} \\
&= \left[\left(\mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{R}_1^a\mathbf{n}_2^b\times]\ + (1-\cos\theta_2)\,[\mathbf{R}_1^a\mathbf{n}_2^b\times]^2\right)\mathbf{R}_1^a\right]^{\mathrm{T}} \\
&= \left[\left(\mathbf{I}_{3\times3} + \sin\theta_2\,\mathbf{R}_1^a[\mathbf{n}_2^b\times](\mathbf{R}_1^a)^{\mathrm{T}} + (1-\cos\theta_2)\,\mathbf{R}_1^a[\mathbf{n}_2^b\times]^2(\mathbf{R}_1^a)^{\mathrm{T}}\right)\mathbf{R}_1^a\right]^{\mathrm{T}} \\
&= \left[\mathbf{R}_1^a\left(\mathbf{I}_{3\times3} + \sin\theta_2\,[\mathbf{n}_2^b\times] + (1-\cos\theta_2)\,[\mathbf{n}_{2r}\times]^2\right)\right]^{\mathrm{T}} \\
&= \left[\mathbf{R}_1^a\,\mathbf{R}_2^b\right]^{\mathrm{T}} = \mathbf{T}_b^c\,\mathbf{T}_a^b
\end{aligned}$$

therefore, while it might seem that we are doing operations with Euler axes in different frames, in reality we are not; the process of changing coordinates of the Euler axis is equivalent to flipping the order of the matrix multiplication; i.e. composing them with the second rotation referenced to the rotating space.

It is hard to imagine the need of composing DCMs where the second transformation is referenced to the fixed-space. Therefore, it is natural for subscribers of the passive interpretation of rotations to always and only assume rotating-space successive rotations. A similar argument cannot be made for active rotations. While Rodrigues and most mathematicians usually worked with fixed-space successive rotations, Euler himself invented the rotation sequence named after him in which three successive rotations were referenced to the rotating-space.

SUMMARY OF MATRIX PARAMETERIZATIONS OF ATTITUDE

Shuster [1] rightly notes that subscribers of the passive interpretation have not completed avoided the use of the external observer point of view. In particular, the Euler axis and angle are always taken as active rotations, they are never re-defined as passive. In summarizing the equations of the passive and active interpretations of the matrix representation of attitude, we will make two assumptions

1. We start from the Euler axis $\mathbf{n}$ and angle θ which are always the parameterization of the active rotation that takes the inertial frame into the body frame
2. Successive rotations are referenced to the rotating space, as most commonly assumed in aerospace applications

In the active interpretation the 3×3 orthogonal matrix $\mathbf{R}_{i\to b}$ parameterizes the rotation from inertial to body

$$\mathbf{R}_{i\to b} = \mathbf{I}_{3\times3} + \sin\theta\,[\mathbf{n}\times]\ + (1-\cos\theta)\,[\mathbf{n}\times]^2 \tag{22}$$

the composition rule of a first rotation $\mathbf{R}_{i\to b}$ followed by a second one $\mathbf{R}_{b\to c}$ referenced to the rotating-space is the row-by-column multiplication with the matrices appearing in the "opposite" order

$$\mathbf{R}_{i\to c} = \mathbf{R}_{i\to b}\,\mathbf{R}_{b\to c} \tag{23}$$

vector coordinates can be transformed as

$$\mathbf{v}^b = (\mathbf{R}_{i\to b}^{\mathrm{T}})\,\mathbf{v}^i \tag{24}$$

The action of physically rotating vectors is not usually needed but is simply given by

$$\mathbf{v}' = \mathbf{R}\mathbf{v} \tag{25}$$

If we are more interested in coordinate transformations and we work more often with successive rotations referenced to the rotating space, the active interpretation seems less practical, and most researchers and practitioners in the field favor a passive interpretation in which the 3×3 orthogonal matrix $\mathbf{T}_i^b$ is a DCM and is given by

$$\mathbf{T}_i^b = \mathbf{I}_{3\times 3} - \sin\theta\,[\mathbf{n}\times]\; + (1-\cos\theta)\;[\mathbf{n}\times]^2 \tag{26}$$

the composition rule is

$$\mathbf{T}_i^c = \mathbf{T}_b^c\,\mathbf{T}_i^b \tag{27}$$

and the change of coordinates is given by

$$\mathbf{v}^b = \mathbf{T}_i^b\,\mathbf{v}^i \tag{28}$$

QUATERNION

In introducing the composition rule for rotations [9], Olinde Rodrigues uses four parameters, which in modern notation are given by†

$$\sin\left(\frac{\theta}{2}\right)\mathbf{n}, \qquad \cos\left(\frac{\theta}{2}\right) \tag{29}$$

This four dimensional attitude representation is sometimes attributed to Euler because in Ref. [10] Euler studied orthogonal matrices, and in Section 23 he discovered how to compute a 3×3 orthogonal matrix in terms of four parameters p, q, r, and s. By choosing these four parameters as in Eq. (29), Euler's construction returns the rotation matrix. The four parameters of Eq. (29) are sometimes called Euler symmetric parameters or Euler-Rodrigues symmetric parameters. However, this representation is more commonly referred to as quaternion-of-rotation, or simply quaternion.

Three years after Rodrigues, in 1843, Sir William Rowan Hamilton presented the quaternion at the Royal Irish Academy. The first paper on quaternions appeared in the Academy's proceedings the following year [11]. Hamilton invented a new algebra in which the elements are both operators

†At Rodrigues' time vector notation did not exist yet. Similarly to Euler's notation, Rodrigues represents the Euler axis with its three direction cosines. Originally Rodrigues four symmetric parameters were given by: $\cos\frac{1}{2}\theta$, and $\sin\frac{1}{2}\theta\cos g$, $\sin\frac{1}{2}\theta\cos h$, $\sin\frac{1}{2}\theta\cos l$, where g, h, l, are the three direction angles of the Euler axis with respect to the coordinate frame. The entire Volume 5 of Journal de Mathématiques Pures et Appliquées has been digitalized and is readily available.

(rotations) and operands (vectors) [12]. More specifically he invented a skew-field, which is a field in which the multiplication is non commutative, in contrast with the regular commutative field. The first person to notice a relation between the quaternion and the four parameters by Rodrigues was Cayley [13]. Cayley discovered that by defining a quaternion via Euler-Rodrigues parameters, the resulting unitary quaternion represents a rotation and he showed that using this quaternion the vector rotation rule is exactly the rule introduced by Rodrigues.

The quaternion used here is denoted by an upper bar and has the vector component first and scalar last, $\bar{\mathbf{q}} = \begin{bmatrix}\mathbf{q}^{\mathrm{T}} & q\end{bmatrix}^{\mathrm{T}}$. If $\mathbf{n}$ and θ are the Euler axis and angle parameterization of a rotation, the corresponding quaternion parameterization is given by

$$\bar{\mathbf{q}} = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \tag{30}$$

From the trigonometric identities

$$\begin{aligned} \sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \\ \cos\theta &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \\ 1 &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2} \end{aligned}$$

we can rewrite Eq. (3) as

$$\mathbf{R} = \mathbf{I}_{3\times 3} + 2\sin(\theta/2)\cos(\theta/2)\,[\mathbf{n}\times] + 2\sin^2(\theta/2)[\mathbf{n}\times]^2 \tag{31}$$

or

$$\mathbf{R} = \mathbf{R}(\bar{\mathbf{q}}) = \mathbf{I}_{3\times 3} + 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2 \tag{32}$$

Once again, Eq. (32) is the most commonly seen equation in mathematics to obtain a rotation matrix from a quaternion, but it is not what we usually see in aerospace engineering where transformations are used and a minus sign precedes the $2q[\mathbf{q}\times]$ term.

To perform a sequence of two rotations $\mathbf{R}_1$ followed by $\mathbf{R}_{2f}$ (referenced to the fixed space), the total rotation is

$$\mathbf{R}_3 = \mathbf{R}_{2f}\,\mathbf{R}_1 = \mathbf{R}(\bar{\mathbf{q}}_{2f})\mathbf{R}_1(\bar{\mathbf{q}}_1) = \mathbf{R}(\bar{\mathbf{q}}_{2f} \circledast \bar{\mathbf{q}}_1) \tag{33}$$

The Hamiltonian quaternion product $\circledast$ is defined as [14]

$$\bar{\mathbf{q}}_2 \circledast \bar{\mathbf{q}}_1 = \begin{bmatrix} q_1\mathbf{q}_2 + q_2\mathbf{q}_1 + \mathbf{q}_2 \times \mathbf{q}_1 \\ q_1 q_2 - \mathbf{q}_2 \cdot \mathbf{q}_1 \end{bmatrix} \tag{34}$$

which can be derived immediately from Rodrigues' composition formula for fixed-space successive rotations (Eqs. (14) and (15)). Using Hamilton's quaternion multiplication, quaternions multiply in the same order as rotation matrices when the successive rotations are referenced to the fixed space, which is the norm in the historical study of rotations and in many disciplines outside of aerospace.

The historical development of rotations sees them as active and with successive rotations referenced to the fixed space. In this context, rotation matrices compose in the most intuitive order and Hamilton's quaternion multiplication matches this order of the operands. In the more recent aerospace literature, rotations are usually interpreted as passive and successive rotations are most often referenced to the rotating space. Therefore they compose in opposite order. Shuster introduced a modified quaternion multiplication

$$\bar{\mathbf{q}}_1 \otimes \bar{\mathbf{q}}_2 = \bar{\mathbf{q}}_2 \circledast \bar{\mathbf{q}}_1 = \begin{bmatrix} q_1\mathbf{q}_2 + q_2\mathbf{q}_1 - \mathbf{q}_1 \times \mathbf{q}_2 \\ q_1 q_2 - \mathbf{q}_1 \cdot \mathbf{q}_2 \end{bmatrix} \tag{35}$$

which is equivalent to Eqs. (17) and (18); the composition formula for rotating-space successive rotations. Using Shuster's quaternion multiplication, quaternions multiply in the same order as transformation matrices when the successive rotations are referenced to the rotating space.

In his 1993 survey paper [1], with regards to the quaternion composition formula, Shuster writes: "the Euler-Rodrigues symmetric parameters for successive rotations have been written in the same order as the rotation matrices. This has not always been the convention followed (...). It was once the convention to write the composition of matrices also in the opposite order to today's usage. The convention changed when interest focused more on the algebra of operators. The quaternion had by this time fallen into disuse and did not succumb to the change in the conventions. This historical oddity has persisted in many works up to the present. The need to abandon the older convention becomes apparent when ..."

What Shuster says is correct but easily misinterpreted. Shuster, as most attitude determination specialists do, uses passive rotations and refers to them simply as rotations. And, as also most attitude determination specialists do, he assumes successive rotations are always referenced to the rotating space (with the only exception of deriving kinematics laws for both interpretations, as also done later in this paper). Therefore the "historical oddity" of composing matrices and quaternions in the "opposite" order is not an oddity at all. They were composed in the most natural order given their assumptions: 1. active rotations and 2. space-referenced successive rotations.

Let's assume we are interested in the attitude of a rigid body using quaternions and that successive rotations are always referenced to the rotating frame. If we lean towards an active interpretation of rotations, we would probably use Hamilton's product in which quaternions multiply in the same order as rotation matrices

$$\bar{\mathbf{q}}_3 = \bar{\mathbf{q}}_1 \circledast \bar{\mathbf{q}}_{2r} \tag{36}$$

Alternatively, we can use Shuster's product in which quaternions multiply in the same order as transformation matrices

$$\bar{\mathbf{q}}_3 = \bar{\mathbf{q}}_{2r} \otimes \bar{\mathbf{q}}_1 \tag{37}$$

either way, the rotation matrix and the transformation matrix are given by

$$\mathbf{R} = \mathbf{I}_{3\times 3} + 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2 \tag{38}$$

$$\mathbf{T} = \mathbf{I}_{3\times 3} - 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2 \tag{39}$$

What is interesting is that the definition of quaternion itself does not change going from the active interpretation to the passive one, while the definition of the matrix does. While there are two definition of matrices (rotation matrix and transformation matrix) most people subscribe to a single definition of quaternion

$$\bar{\mathbf{q}} = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

The designers of the Space Shuttle flight software took a different approach, and they defined the quaternion differently to emphasize it was a parameterization of a transformation, not of a rotation; the so-called left quaternion. [3]

LEFT QUATERNION

Starting from the Euler axis and angle that parameterize the rotation from a reference frame to a body-fixed frame, it should be clear by now that there are two alternative matrix representations of the attitude of a rigid body: rotation matrices:

$$\mathbf{R} = \mathbf{I}_{3\times3} + \sin\theta\,[\mathbf{n}\times]\; + (1-\cos\theta)\,[\mathbf{n}\times]^2$$

and transformation matrices, aka passive rotations

$$\mathbf{T} = \mathbf{I}_{3\times3} - \sin\theta\,[\mathbf{n}\times]\; + (1-\cos\theta)\,[\mathbf{n}\times]^2$$

Under the assumption that in aerospace applications successive rotations are always referenced to rotating space, we have that transformations matrices multiply in the "natural" order and rotation matrices multiply in the "opposite" order. Most aerospace specialists lean towards the passive interpretation which multiplies matrices in an intuitive order. Most aerospace specialists also favor a lower-dimensional attitude representation than the 9 of the 3×3 matrix, often choosing the four-dimensional quaternion. Defining the quaternion as in Eq. (30) has the disadvantage that they multiply in a non-intuitive order when using the original Hamilton multiplication. Most aerospace specialists adhere to one of three conventions, whose arguments are described below.

Hamilton's Convention. There is really not a right order and a wrong order to multiply matrices or quaternions. The fact that we write a functional composition as $\mathcal{R}_2 \circ \mathcal{R}_1$ does not mean we need to have the matrix multiplication or the quaternion multiplication with the two arguments in that same order. The fact that we have rotating-space successive rotations creates a minus sign in the Euler axis composition rule of Eq. (18)

$$\mathbf{n}_3 = \frac{\sin(\theta_1/2)\cos(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_1 + \frac{\sin(\theta_2/2)\cos(\theta_1/2)}{\sin(\theta_3/2)}\mathbf{n}_{2r} - \frac{\sin(\theta_1/2)\sin(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_{2r}\times\mathbf{n}_1$$

but nobody ever writes

$$\mathbf{n}_3 = \frac{\sin(\theta_1/2)\cos(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_1 + \frac{\sin(\theta_2/2)\cos(\theta_1/2)}{\sin(\theta_3/2)}\mathbf{n}_{2r} + \frac{\sin(\theta_1/2)\sin(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_{2r}*\mathbf{n}_1$$

with a new definition of cross product $\mathbf{n}_1 * \mathbf{n}_{2r} = \mathbf{n}_{2r}\times\mathbf{n}_1$, just to have a positive sign and the arguments in the "right" order.[‡] Similarly, we should not change the definition of quaternion multiplication (which was the first introduction of the vector cross product as the vector part of the product of two quaternions with zero scalar part) from that of Hamilton. Hamilton's quaternion algebra follows the well established right-hand rule. From Hamilton's famous equations $\mathbf{i}^2 = \mathbf{j}^2 = \mathbf{k}^2 = -1$, $\mathbf{ijk} = -1$ it follows immediately that $\mathbf{ij} = \mathbf{k}$, which is true for a right-hand triad. Changing the definition of the quaternion multiplication while still using right-hand triads makes little sense.

[‡] Actually, since he always used successive rotations referenced to rotating-space, Shuster noticed that all cross products were preceeded by a minus sign. Therefore, Shuster did not use the skew symmetric cross product matrix $[\mathbf{w}\times]$ but he rather defined its opposite $[[\mathbf{w}]] = -[\mathbf{w}\times]$, effectively creating a left-hand cross product.

Shuster's Convention. Hamilton's multiplication is a good representation of the composition of rotations (assuming fixed-space successive rotations) not of the composition of transformations (assuming rotating-space successive rotations). As a field, we shifted from representing attitude with rotation matrices (active rotations in Eq. (3)) to transformation matrices (passive rotations in Eq. (11)) so that matrices multiply in the "right" order. Similarly, we should change the definition of multiplication from Eq. (34) to Eq. (35) such that quaternions multiply in the same order as attitude matrices. In defining a new product we are not concerned about the group and algebraic properties of quaternions and their operator; rather we are interested in the composition rule of successive rotations referenced to rotating-space.

Space Shuttle Convention The so-called Shuttle convention, or left quaternion, rather than redefining the quaternion multiplication, provides an alternative definition of the quaternion itself. Starting from the Euler axis and angle that parameterize the rotation from a reference frame to a body-fixed frame, we can construct either a rotation matrix or a transformation matrix. Similarly, the usual definition of quaternion is a parameterization of rotations:

$$\bar{\mathbf{q}} = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

while left quaternions ${}^L\bar{\mathbf{q}}$ are parameterizations of coordinate transformations:

$${}^L\bar{\mathbf{q}} = \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix}$$

Under this framework, there is only one definition of quaternion multiplication, the one by Hamilton (like there is only one definition of matrix multiplication) and there is only one formula for a quaternion to operate on a vector: with the same formula Hamilton's quaternions will physically rotate a vector and a left quaternions will transform its coordinates. There is also only one formula to go from quaternions to matrices, if we start from regular quaternions we get a rotation matrix otherwise we get a transformation matrix. Quaternions represent rotations and they multiply in the same order as rotation matrices (this statement assumes the historical development where successive rotations are referenced to fixed-space). Left quaternions represent transformations and they multiply in the same order as transformation matrices. Using a regular quaternion to represent a transformation and then modifying its multiplication is equivalent to expressing a transformation with a rotation matrix and then define a column-by-row product such that they multiply in the natural order. The adoption of the Space Shuttle convention was recently endorsed by Ref. [15].

SUMMARY OF QUATERNION PARAMETERIZATIONS OF ATTITUDE

In summarizing the equations of the various interpretations of the quaternion representation of attitude, we will make the same two assumptions as before:

1. We start from the Euler axis $\mathbf{n}$ and angle θ which are always the parameterization of the active rotation that takes the inertial frame into the body frame

2. Successive rotations are referenced to the rotating space

Hamilton's convention is an active interpretation of the quaternion defined as

$$\bar{\mathbf{q}}_{i\to b} = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \tag{40}$$

the composition rule of a first rotation $\bar{\mathbf{q}}_{i\to b}$ followed by a second one $\bar{\mathbf{q}}_{b\to c}$ referenced to the rotating-space is the Hamilton multiplication with the quaternions appearing in the "opposite" order

$$\bar{\mathbf{q}}_{i\to c} = \bar{\mathbf{q}}_{i\to b} \circledast \bar{\mathbf{q}}_{b\to c} \tag{41}$$

The vector coordinates can be transformed using pure quaternions (quaternions with zero scalar part) as

$$\begin{bmatrix} \mathbf{v}^b \\ 0 \end{bmatrix} = \bar{\mathbf{q}}^*_{i\to b} \circledast \begin{bmatrix} \mathbf{v}^i \\ 0 \end{bmatrix} \circledast \bar{\mathbf{q}}_{i\to b} \tag{42}$$

where superscript $*$ represents the quaternion conjugate. The action of physically rotate vectors is not usually needed, but is given by

$$\begin{bmatrix} \mathbf{v}' \\ 0 \end{bmatrix} = \bar{\mathbf{q}} \circledast \begin{bmatrix} \mathbf{v} \\ 0 \end{bmatrix} \circledast \bar{\mathbf{q}}^* \tag{43}$$

If needed, the rotation and transformation matrices are calculated as

$$\mathbf{R}_{i\to b} = \mathbf{I}_{3\times 3} + 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2 \tag{44}$$

$$\mathbf{T}_i^b = \mathbf{I}_{3\times 3} - 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2 \tag{45}$$

Shuster's convention is an overall passive interpretation but it keeps the quaternion definition as an active one

$$\bar{\mathbf{q}}_i^b = \begin{bmatrix} \sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \tag{46}$$

the composition rule of a first transformation $\bar{\mathbf{q}}_i^b$ followed by a second one $\bar{\mathbf{q}}_b^c$ referenced to the rotating-space is the modified multiplication with the quaternions appearing in the "correct" order

$$\bar{\mathbf{q}}_i^c = \bar{\mathbf{q}}_b^c \otimes \bar{\mathbf{q}}_i^b \tag{47}$$

The vector coordinates can be transformed using pure quaternions

$$\begin{bmatrix} \mathbf{v}^b \\ 0 \end{bmatrix} = \bar{\mathbf{q}}_i^b \otimes \begin{bmatrix} \mathbf{v}^i \\ 0 \end{bmatrix} \otimes (\bar{\mathbf{q}}_i^b)^* \tag{48}$$

If needed, the rotation and transformation matrices are calculated as above

$$\mathbf{R}_{i\to b} = \mathbf{I}_{3\times 3} + 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2$$

$$\mathbf{T}_i^b = \mathbf{I}_{3\times 3} - 2q[\mathbf{q}\times] + 2[\mathbf{q}\times]^2$$

The Space Shuttle convention is a passive interpretation that also defines the quaternion as passive

$$^L\bar{\mathbf{q}}_i^b = \begin{bmatrix} -\sin\left(\frac{\theta}{2}\right)\mathbf{n} \\ \cos\left(\frac{\theta}{2}\right) \end{bmatrix} \tag{49}$$

the composition rule of a first transformation ${}^L\bar{\mathbf{q}}_i^b$ followed by a second one ${}^L\bar{\mathbf{q}}_b^c$ referenced to the rotating-space is the Hamilton multiplication with the quaternions appearing in the "correct" order

$$ {}^L\bar{\mathbf{q}}_i^c = {}^L\bar{\mathbf{q}}_b^c \circledast {}^L\bar{\mathbf{q}}_i^b \tag{50} $$

The vector coordinates can be transformed using pure quaternions

$$ \begin{bmatrix} \mathbf{v}^b \\ 0 \end{bmatrix} = {}^L\bar{\mathbf{q}}_i^b \circledast \begin{bmatrix} \mathbf{v}^i \\ 0 \end{bmatrix} \circledast ({}^L\bar{\mathbf{q}}_i^b)^* \tag{51} $$

The transformation matrix is obtained from the left quaternion using a plus sign in front of the second term

$$ \mathbf{T}_i^b = \mathbf{I}_{3\times3} + 2\,{}^Lq\,[{}^L\mathbf{q}\times] + 2[{}^L\mathbf{q}\times]^2 \tag{52} $$

ANGULAR VELOCITY AND KINEMATICS

The angular velocity is closely related to infinitesimal rotations and the discovery of its vectorial representation follows the discovery that infinitesimal rotations can be added together like vectors [16]. Suppose we have a rigid object with a body-fixed frame b attached to it and at an arbitrary time t we know its orientation with respect to an inertial reference frame i. After some time Δt, the object rotated to a new orientation with respect to the inertial frame and the rotation from time t to $t + \Delta t$ is parameterized by the Euler axis $\mathbf{n}(t, \Delta t)$ and angle $\Delta\theta(t, \Delta t)$. The angular velocity of the object is defined as [17]

$$ \boldsymbol{\omega}(t) = \boldsymbol{\omega}_{b/i}(t) \triangleq \lim_{\Delta t \to 0} \frac{\Delta\theta(t, \Delta t)}{\Delta t}\,\mathbf{n}(t, \Delta t) = \frac{d\theta(t)}{dt}\mathbf{n}(t) \tag{53} $$

the subscript b/i indicates the angular velocity of b is with respect to the reference inertial frame. The angular velocity is an infinitesimal rotation over an infinitesimal time step. Since infinitesimal rotations form a vector space, so does the angular velocity. Eq. (53) is coordinate-free and it can be expressed in any coordinate system of our choice. The attitude kinematic equations depend on whether the time history of the Euler axis (and hence of the angular velocity) is known and expressed in the rotating body frame or in the fixed inertial frame.

Let's first assume the Euler axis is referenced to fixed space and its time history is known in the inertial frame. Therefore rotation matrices compose in the "natural" order and we have that

$$ \mathbf{R}(t + \Delta t) = \Delta\mathbf{R}_f(t, \Delta t)\,\mathbf{R}(t) \tag{54} $$

In terms of Euler axis and angle, and dropping the arguments for ease of notation, we have

$$ \Delta\mathbf{R}_f = \mathbf{I}_{3\times3} + \sin\Delta\theta\,[\mathbf{n}_f\times] + (1 - \cos\Delta\theta)[\mathbf{n}_f\times]^2 \tag{55} $$

The subscript f denotes that the delta-rotation is referenced to fixed space. The Euler angle becomes infinitesimally small and we have that

$$ \Delta\mathbf{R} \to \mathbf{I}_{3\times3} + \Delta\theta\,[\mathbf{n}_f\times] \tag{56} $$

as $\Delta t \to 0$. The derivative of the rotation matrix is therefore given by

$$ \begin{aligned} \dot{\mathbf{R}}(t) &\triangleq \lim_{\Delta t \to 0} \frac{\mathbf{R}(t + \Delta t) - \mathbf{R}(t)}{\Delta t} = \lim_{\Delta t \to 0} \frac{\Delta\mathbf{R}_f - \mathbf{I}_{3\times3}}{\Delta t}\,\mathbf{R}(t) = \lim_{\Delta t \to 0} \frac{\Delta\theta\,[\mathbf{n}_f\times]}{\Delta t}\,\mathbf{R}(t) \\ &= [\boldsymbol{\omega}_f(t)\times]\,\mathbf{R}(t) \end{aligned} \tag{57} $$

this formula is coordinate-independent and valid when the angular velocity $\boldsymbol{\omega}_f(t)$ is referenced to the fixed space. For a coordinate-dependent formulation we have

$$\dot{\mathbf{R}}^i_{i\to b}(t) = [\boldsymbol{\omega}^i(t)\times]\,\mathbf{R}^i_{i\to b}(t)$$

We can take the transpose of this equation and derive how the transformation matrix evolves with respect to an inertial angular velocity

$$\dot{\mathbf{T}}^b_i(t) = \dot{\mathbf{R}}_{i\to b}(t)^{\mathrm{T}} = \left([\boldsymbol{\omega}^i(t)\times]\,\mathbf{R}_{i\to b}(t)\right)^{\mathrm{T}} = -\mathbf{T}^b_i(t)\;[\boldsymbol{\omega}^i(t)\times] \tag{58}$$

Similar steps can be taken when the Euler axis is referenced to the rotating-space, for example its time history is known in the body-fixed frame. The derivation can be done either for transformation matrices starting from $\mathbf{T}^b_i(t+\Delta t) = \Delta\mathbf{T}(t,\Delta t)\;\mathbf{T}^b_i(t)$ or for rotation matrices starting from

$$\mathbf{R}_{i\to b}(t+\Delta t) = \mathbf{R}_{i\to b}(t)\;\Delta\mathbf{R}_r(t,\Delta t) \tag{59}$$

where the subscript r indicates that the delta rotation is referenced to the rotating space. to obtain

$$\begin{aligned}\dot{\mathbf{R}}_{i\to b}(t) &\triangleq \lim_{\Delta t\to 0}\frac{\mathbf{R}_{i\to b}(t+\Delta t)-\mathbf{R}_{i\to b}(t)}{\Delta t} = \lim_{\Delta t\to 0}\mathbf{R}_{i\to b}(t)\frac{\Delta\mathbf{R}_r-\mathbf{I}_{3\times 3}}{\Delta t}\\ &= \mathbf{R}_{i\to b}(t)\lim_{\Delta t\to 0}\frac{\Delta\theta\,[\mathbf{n}_r\times]}{\Delta t} = \mathbf{R}_{i\to b}(t)\;[\boldsymbol{\omega}_r(t)\times]\end{aligned} \tag{60}$$

For a coordinate-dependent formulation we have

$$\dot{\mathbf{R}}^b_{i\to b}(t) = \mathbf{R}^b_{i\to b}(t)\,[\boldsymbol{\omega}^b(t)\times]$$

Taking the transpose we obtain how the transformation matrix evolves with respect to a rotating-space angular velocity:

$$\dot{\mathbf{T}}^b_i(t) = \dot{\mathbf{R}}^b_{i\to b}(t)^{\mathrm{T}} = \Big(\mathbf{R}^b_{i\to b}(t)\;[\boldsymbol{\omega}^b(t)\times]\Big)^{\mathrm{T}} = -[\boldsymbol{\omega}^b(t)\times]\;\mathbf{T}^b_i(t) \tag{61}$$

The same approach can be also used for the quaternion. Assuming $\mathbf{n}(t,\Delta t)$ is referenced to the fixed-space, we have that

$$\bar{\mathbf{q}}_{i\to b}(t+\Delta t) = \Delta\bar{\mathbf{q}}_f(t)\circledast\bar{\mathbf{q}}_{i\to b}(t). \tag{62}$$

In terms of Euler axis ($\mathbf{n}$) and angle ($\Delta\theta$) we have that

$$\Delta\bar{\mathbf{q}}_f \to \begin{bmatrix}\frac{\Delta\theta}{2}\mathbf{n}\\ 1\end{bmatrix},\quad \text{as } \Delta\theta\to 0. \tag{63}$$

Substituting Eq. (63) into Eq. (62) we obtain that

$$\bar{\mathbf{q}}_{i\to b}(t+\Delta t) = \left(\begin{bmatrix}\mathbf{0}\\ 1\end{bmatrix}+\begin{bmatrix}\frac{\Delta\theta}{2}\mathbf{n}\\ 0\end{bmatrix}\right)\circledast\bar{\mathbf{q}}_{i\to b}(t) = \bar{\mathbf{q}}_{i\to b}(t)+\begin{bmatrix}\frac{\Delta\theta}{2}\mathbf{n}\\ 0\end{bmatrix}\circledast\bar{\mathbf{q}}_{i\to b}(t). \tag{64}$$

The derivative of the quaternion is defined as

$$\begin{aligned}\dot{\bar{\mathbf{q}}}_{i\to b}(t) &= \lim_{\Delta t\to 0}\frac{\bar{\mathbf{q}}_{i\to b}(t+\Delta t)-\bar{\mathbf{q}}_{i\to b}(t)}{\Delta t} = \lim_{\Delta t\to 0}\frac{1}{2}\begin{bmatrix}\frac{\Delta\theta}{\Delta t}\mathbf{n}\\ 0\end{bmatrix}\circledast\bar{\mathbf{q}}_{i\to b}(t)\\ &= \frac{1}{2}\begin{bmatrix}\boldsymbol{\omega}^i(t)\\ 0\end{bmatrix}\circledast\bar{\mathbf{q}}_{i\to b}(t).\end{aligned} \tag{65}$$

repeating the same steps for a rotating-space angular velocity we obtain

$$\dot{\bar{\mathbf{q}}}_i^b(t) = \frac{1}{2}\bar{\mathbf{q}}_i^b(t) \circledast \begin{bmatrix} \boldsymbol{\omega}^b(t) \\ 0 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \boldsymbol{\omega}^b(t) \\ 0 \end{bmatrix} \otimes \bar{\mathbf{q}}_i^b(t)$$

Finally, the left quaternion evolution contains a minus sign exactly like the DCM evolution

$$ {}^L\dot{\bar{\mathbf{q}}}_i^b(t) = -\frac{1}{2}\begin{bmatrix} \boldsymbol{\omega}^b(t) \\ 0 \end{bmatrix} \circledast {}^L\bar{\mathbf{q}}_i^b(t) = -\frac{1}{2}\, {}^L\bar{\mathbf{q}}_i^b(t) \circledast \begin{bmatrix} \boldsymbol{\omega}^i(t) \\ 0 \end{bmatrix} \tag{66}$$

CONCLUSIONS

The historical study of rotations takes an external-viewer approach in which everything is known with respect to a fixed (inertial) observer. This approach has an intuitive geometric construction of rotation sequences that directly represents the rotation of the rigid body. As the focus shifts to sequences of rotations referenced to the rotating space, a passive interpretation of rotations becomes, perhaps, more intuitive. The passive interpretation is an internal-viewer approach in which the observer is rotating together with the body, and sees the surroundings artificially rotate opposite to the motion of the body. This interpretation is particularly convenient, for example, when studying attitude dynamics, because the evolution of the angular velocity is more easily calculated attaching it to the rigid body than to the inertial frame. Many researchers refer to both approaches simply as rotations without specifying an active or passive interpretation. A complete treatise of the two approaches and how they affect the formulas used in the study of attitude has so far eluded the literature. This paper provides a single reference where the mathematics of the two approaches are fully developed, together with a complete description of the obscure left quaternion convention used in the NASA Space Shuttle program.

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APPENDIX

In the derivation of the composition rule of two rotations below, it is very important to notice that we assume that the second axis $\mathbf{n}_2$ is fixed *a priori* and independent of the first rotation. The composition rule is different when the second axis is fixed with respect to the vector, i.e. rotates with it during the first rotation. This proof follows that of Paul [8].

We have two successive rotations $\mathcal{R}_1$ followed by $\mathcal{R}_2$

$$\mathbf{v}'' = \mathcal{R}_2(\mathcal{R}_1(\mathbf{v})) = \mathcal{R}_2(\mathbf{v}') = \mathcal{R}_3(\mathbf{v})$$

The Euler axis $\mathbf{n}_1$ of the first rotation remains unchanged across it

$$\mathcal{R}_3(\mathbf{n}_1) = \mathcal{R}_2(\mathcal{R}_1(\mathbf{n}_1)) = \mathcal{R}_2(\mathbf{n}_1) \tag{67}$$

Let’s denote with an asterisk the inverse rotation

$$\mathbf{v}' = \mathcal{R}(\mathbf{v}) \Rightarrow \mathbf{v} = \mathcal{R}^*(\mathbf{v}') \tag{68}$$

The Euler axis $\mathbf{n}_2$ of the second rotation remains unchanged across it.

$$\mathcal{R}_3^*(\mathbf{n}_2) = \mathcal{R}_1^*(\mathcal{R}_2^*(\mathbf{n}_2)) = \mathcal{R}_1^*(\mathbf{n}_2) \tag{69}$$

Using Euler’s formula and Eq. (67), the displacement $\mathcal{R}_3(\mathbf{n}_1) - \mathbf{n}_1$ can be expressed in two equivalent ways

$$\begin{aligned}\mathcal{R}_3(\mathbf{n}_1) - \mathbf{n}_1 &= \mathcal{R}_2(\mathbf{n}_1) - \mathbf{n}_1 \\ &= \sin\theta_2\,[\mathbf{n}_2\times]\mathbf{n}_1 + (1-\cos\theta_2)\,[\mathbf{n}_2\times]^2\mathbf{n}_1 \qquad (70) \\ &= \sin\theta_3\,[\mathbf{n}_3\times]\mathbf{n}_1 + (1-\cos\theta_3)\,[\mathbf{n}_3\times]^2\mathbf{n}_1. \qquad (71)\end{aligned}$$

Similarly we find that

$$\begin{aligned}\mathcal{R}_3^*(\mathbf{n}_2) - \mathbf{n}_2 &= \mathcal{R}_1^*(\mathbf{n}_2) - \mathbf{n}_2 \\ &= -\sin\theta_1\,[\mathbf{n}_1\times]\mathbf{n}_2 + (1-\cos\theta_1)\,[\mathbf{n}_1\times]^2\mathbf{n}_2 \qquad (72) \\ &= -\sin\theta_3\,[\mathbf{n}_3\times]\mathbf{n}_2 + (1-\cos\theta_3)\,[\mathbf{n}_3\times]^2\mathbf{n}_2. \qquad (73)\end{aligned}$$

In deriving the composition rule we will assume that neither θ_1 nor θ_2 are multiples of 180 degrees. The cases in which either or both the angles are multiples of 180 degrees can be easily verified to obey the same composition rule we will now derive.

The displacement due to a rotation is always perpendicular to the rotation axis, therefore both $\mathcal{R}_3(\mathbf{n}_1) - \mathbf{n}_1$ and $\mathcal{R}_3^*(\mathbf{n}_2) - \mathbf{n}_2$ are perpendicular to $\mathbf{n}_3$, hence $\mathbf{n}_3$ must be parallel to the cross product of the two, we will denote the vector obtained from the cross product as $\mathbf{u}$.

$$
\begin{aligned}
\mathbf{u} =& \Big(\mathcal{R}_3(\mathbf{n}_1) - \mathbf{n}_1\Big) \times \Big(\mathcal{R}_3^*(\mathbf{n}_2) - \mathbf{n}_2\Big) \\
=& \sin\theta_2(1-\cos\theta_1)\Big([\mathbf{n}_2\times]\mathbf{n}_1\Big) \times \Big([\mathbf{n}_1\times]^2\mathbf{n}_2\Big) - \sin\theta_1(1-\cos\theta_2)\Big([\mathbf{n}_2\times]^2\mathbf{n}_1\Big) \times \Big([\mathbf{n}_1\times]\mathbf{n}_2\Big) + \\
&+ (1-\cos\theta_2)(1-\cos\theta_1)\Big([\mathbf{n}_2\times]^2\mathbf{n}_1\Big) \times \Big([\mathbf{n}_1\times]^2\mathbf{n}_2\Big) \\
=& \Big\{ -\sin\theta_2(1-\cos\theta_1)\mathbf{n}_1 - \sin\theta_1(1-\cos\theta_2)\mathbf{n}_2 + (1-\cos\theta_2)(1-\cos\theta_1)(\mathbf{n}_1\times\mathbf{n}_2)\Big\} \\
&\cdot \Big(1 - (\mathbf{n}_1\cdot\mathbf{n}_2)^2\Big) \qquad (74)
\end{aligned}
$$

where the following identities are used

$$
\begin{aligned}
\Big([\mathbf{n}_2\times]\mathbf{n}_1\Big) \times \Big([\mathbf{n}_1\times]^2\mathbf{n}_2\Big) &= (\mathbf{n}_1\times\mathbf{n}_2) \times \Big((\mathbf{n}_1\times\mathbf{n}_2)\times\mathbf{n}_1\Big) = -\Big(1-(\mathbf{n}_1\cdot\mathbf{n}_2)^2\Big)\mathbf{n}_1 \\
\Big([\mathbf{n}_2\times]^2\mathbf{n}_1\Big) \times \Big([\mathbf{n}_1\times]^2\mathbf{n}_2\Big) &= -\Big((\mathbf{n}_1\times\mathbf{n}_2)\times\mathbf{n}_2\Big) \times \Big((\mathbf{n}_1\times\mathbf{n}_2)\times\mathbf{n}_1\Big) \qquad (75) \\
&= \Big((\mathbf{n}_1\times\mathbf{n}_2)\cdot(\mathbf{n}_1\times\mathbf{n}_2)\Big)(\mathbf{n}_1\times\mathbf{n}_2) \\
&= \Big(1-(\mathbf{n}_1\cdot\mathbf{n}_2)^2\Big)(\mathbf{n}_1\times\mathbf{n}_2) \qquad (76) \\
\sin\theta &= 2\sin\frac{\theta}{2}\cos\frac{\theta}{2} \qquad (77) \\
\cos\theta &= \cos^2\frac{\theta}{2} - \sin^2\frac{\theta}{2} \qquad (78) \\
1 &= \cos^2\frac{\theta}{2} + \sin^2\frac{\theta}{2}, \qquad (79)
\end{aligned}
$$

A newly defined vector $\mathbf{w}$ parallel to $\mathbf{u}$ and $\mathbf{n}_3$ is defined as

$$
\begin{aligned}
\mathbf{w} &= S_1S_2\Big(1-(\mathbf{n}_1\cdot\mathbf{n}_2)^2\Big)^{-1}(1-\cos\theta_2)^{-1}(1-\cos\theta_1)^{-1}\,\mathbf{u} \\
&= C_2S_1\mathbf{n}_1 + C_1S_2\mathbf{n}_2 - S_1S_2(\mathbf{n}_1\times\mathbf{n}_2) \qquad (80)
\end{aligned}
$$

where the symbols $C_i = \cos(\theta_i/2)$ and $S_i = \sin(\theta_i/2)$ are introduced to ease notation. The Euler axis of the composed rotation is therefore given by

$$
\mathbf{n}_3 = \pm\frac{\mathbf{w}}{\|\mathbf{w}\|}, \qquad (81)
$$

the sign ambiguity is solved choosing the appropriate angle. The norm of $\mathbf{w}$ can be obtained from

$$
\|\mathbf{w}\|^2 = C_2^2S_1^2 + C_1^2S_2^2 + 2C_2S_1C_1S_2(\mathbf{n}_1\cdot\mathbf{n}_2) + \Big(1-(\mathbf{n}_1\cdot\mathbf{n}_2)^2\Big),
$$

To obtain the Euler angle we use

$$
\text{Eq. (71)}\cdot\mathbf{n}_2 - \text{Eq. (73)}\cdot\mathbf{n}_1 = \text{Eq. (70)}\cdot\mathbf{n}_2 - \text{Eq. (72)}\cdot\mathbf{n}_1 \qquad (82)
$$

to obtain that

$$1-\cos\theta_3 = \frac{1-(\mathbf{n}_1-\mathbf{n}_2)^2}{(\mathbf{n}_1\cdot\mathbf{n}_3)^2-(\mathbf{n}_2\cdot\mathbf{n}_3)^2}(\cos\theta_2-\cos\theta_1). \tag{83}$$

Substituting for $\mathbf{n}_3$, after some algebra we finally obtain that

$$\frac{1-\cos\theta_3}{2} = \sin^2\frac{\theta_3}{2} = \|\mathbf{w}\|^2. \tag{84}$$

Noticing that $\cos\frac{\theta_3}{2} = \pm\sqrt{1-\sin^2(\theta_3/2)}$ we finally obtain the composition rule for Euler axis and angle

$$\cos\frac{\theta_3}{2} = \cos\frac{\theta_1}{2}\cos\frac{\theta_2}{2} - \sin\frac{\theta_1}{2}\sin\frac{\theta_2}{2}\,\mathbf{n}_{2f}\cdot\mathbf{n}_1 \tag{85}$$

$$\mathbf{n}_3 = \frac{\sin(\theta_1/2)\cos(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_1 + \frac{\sin(\theta_2/2)\cos(\theta_1/2)}{\sin(\theta_3/2)}\mathbf{n}_{2f} + \frac{\sin(\theta_1/2)\sin(\theta_2/2)}{\sin(\theta_3/2)}\mathbf{n}_{2f}\times\mathbf{n}_1. \tag{86}$$

Eqs. (85) and (86) provide the composition of successive rotations as first introduced by Rodrigues [9]. The subscript f is added to $\mathbf{n}_{2f}$ as a reminder that the axis of the second rotation is referenced to the fixed space. In this derivation we assumed $\mathbf{n}_{2f}$ fixed and known *a priori*. Nowhere in the derivation we made the second Euler axis rotate with the first rotation and hence be a function of $\mathbf{n}_1$ and θ_1.