Ensemble Kalman Filter with Bayesian Recursive Update

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***Abstract*—Nonlinear measurement models pose a challenge to linear filters. The ensemble Kalman filter (EnKF) is a popular choice despite its tendency to diverge in systems with highly accurate, highly nonlinear measurements. In this work, we present the Bayesian Recursive Update EnKF (BRUEnKF): a novel EnKF that employs the Bayesian Recursive Update Filter (BRUF) measurement update. The BRUF divides the the extended Kalman filter (EKF) update into an integer number of steps, allowing for the recomputation of the measurement Jacobian at regular intervals. We adapt the BRUF update for an ensemble filter, taking advantage of the EnKF's numerical covariance computation at each update step. The BRUEnKF is shown to outperform the EnKF for systems with range measurements.**

***Index Terms*—statistical estimation, ensemble kalman filter, bayesian recursive update**

I. Introduction

The objective of state estimation is to approximate the states of a dynamical system given information from measurements. In *statistical estimation*, the uncertainty of the state estimate is represented by a probability distribution. The state estimate at any time is given by a statistic of the distribution (e.g., the mean). Uncertainty in the state estimate is driven by process noise and measurement noise. Process noise arises primarily due to unmodeled dynamics and uncertainty in control inputs. Measurement noise is caused by imperfect sensors. For linear systems with additive Gaussian noise, the Kalman filter provides the optimal state estimate in the mean-square error sense [1].

For nonlinear systems, the fundamental linearity assumption of the Kalman filter breaks down. In general, the probability distribution is no longer Gaussian. Numerous techniques for extending the Kalman filter to nonlinear applications have been proposed. Perhaps the most common approach is the Extended Kalman Filter (EKF), which simply assumes linearizations of the dynamics and measurement models are valid in a small area around the current state estimate. The Ensemble Kalman Filter [2]–[5] (EnKF) represents the probability distribution of a nonlinear system with an *ensemble* of M state vectors. Each ensemble member is propagated and updated as in the EKF, with the advantage that the covariance matrices required for the Kalman update can be computed numerically at measurement time. Still, artificial inflation of the ensemble and artificially-generated measurement noises are required to maintain statistical consistency [6], [7].

In this work, we replace the usual EKF update in the EnKF with the Bayesian Recursive Update Filter (BRUF) measurement update introduced in [8]. Taking inspiration from particle flow [9], [10], progressive Gaussian filtering [11], [12], and other iterative methods [13]–[15], the BRUF applies the EKF update gradually by breaking it up into N steps. The main benefit of iterative update methods is the recalculation of the measurement Jacobian at each step. In this way, iterative methods are able to "follow" the nonlinearity in the measurement model as the state is herded toward the updated value. Ref. [16] presents an iterative EnKF update inspired by the randomized maximum likelihood method.

The remainder of this paper is organized as follows: Section II describes the BRUF and the EnKF and, notably, includes two proofs regarding the equivalence of the BRUF update to the Kalman update for linear systems; Section III introduces the Bayesian Recursive Update EnKF (BRUEnKF), the filter proposed in this work; Section IV compares the performance of the BRUEnKF and the EnKF for a Lorenz '63 system with a range measurement; and, finally, Section V presents conclusions and future work.

II. Background

Consider nonlinear measurement,

$$\mathbf{y} = h(\mathbf{x}^*) + \boldsymbol{\eta}, \tag{1}$$

where $\mathbf{x}^*$ is the true state, and

$$\boldsymbol{\eta} \sim \mathcal{N}(\mathbf{0}, R) \tag{2}$$

is an additive zero-mean Gaussian noise. A prior estimate of $\mathbf{x}^*$, $\hat{\mathbf{x}}^-$, is available with distribution $p(\mathbf{x})$. The task of the measurement update step is to approximate the Bayesian posterior distribution $p(\mathbf{x}|\mathbf{y})$.

The BRUF solves this problem by breaking up the EKF update into N steps. A new Kalman gain is computed for each step using inflated measurement noise covariance NR. Algorithm 1 summarizes the BRUF update. The computational

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cost of the BRUF is N times the cost of the EKF update, where N is an integer parameter chosen by the designer.

Algorithm 1 The Bayesian Recursive Update

Require: $\hat{\mathbf{x}}^-$, the prior state estimate; P^-, the prior covariance; $\mathbf{y}$, a measurement; R, the measurement covariance; N, the number of steps

$\mathbf{x}_0 \leftarrow \hat{\mathbf{x}}^-$
$P_0 \leftarrow P^-$
for $i = 1 \ldots N$ **do**
  $H_i = \frac{d\mathbf{y}}{d\mathbf{x}}\Big|_{\mathbf{x}=\hat{\mathbf{x}}_{i-1}}$
  $K_i = P_{i-1}H_i^T(H_iP_{i-1}H_i^T + NR)^{-1}$
  $\hat{\mathbf{x}}_i \leftarrow \hat{\mathbf{x}}_{i-1} + K_i(\mathbf{y} - h(\hat{\mathbf{x}}_{i-1}))$
  $P_i \leftarrow (I - K_iH_i)P_{i-1}$
end for
$\hat{\mathbf{x}}^+ \leftarrow \hat{\mathbf{x}}_N$
$P^+ \leftarrow P_N$

By recursively recalculating the measurement Jacobian, the BRUF is able to better approximate the *maximum a posteriori* (MAP) estimate than a single EKF update. However, if the measurement is linear, the BRUF yields the same state estimate, $\hat{\mathbf{x}}^+$, and covariance estimate, P^+, as the EKF update. This is true for any number of steps N. For the sake of completeness, we explicitly prove that the EKF update is equivalent to the BRUF update for linear measurements $h(\mathbf{x}) = H\mathbf{x}$. In other words, for linear measurements, performing N Kalman updates with measurement error covariance NR is equivalent to performing a single Kalman update with measurement error covariance R. The proof is presented in two parts; first, a proof that $P^+_{BRUF} = P_{EKF}$, and second that $\hat{\mathbf{x}}^+_{BRUF} = \hat{\mathbf{x}}^+_{EKF}$.

Theorem II.1 (BRUF Linear Covariance Convergence)**.** *Given P^-, the prior covariance, the iterative BRUF update,*

$$P_i = \left[I - P_{i-1}H^T\left(HP_{i-1}H + NR\right)^{-1}H\right]P_{i-1}, \quad 2 < i < N, \tag{3}$$

with $P_1 = P^-$, and a linear observation H, the update (3) *is exactly equivalent to the Kalman update,*

$$P^+ = P_N = (I - KH)\,P^-. \tag{4}$$

Proof. Consider the matrix inversion lemma [17]:

$$(P^{-1} + H^TR^{-1}H)^{-1} = \left(I - PH^T\left(HPH^T + R\right)^{-1}H\right)P. \tag{5}$$

Using (5), it is possible to rewrite the Kalman covariance update (3) as

$$P_i = \left[P_{i-1}^{-1} + \frac{1}{N}H^TR^{-1}H\right]^{-1}, \quad 2 < i < N. \tag{6}$$

Thus, by simple manipulation,

$$\begin{aligned} P_N &= \left[\left(P^-\right)^{-1} + \frac{1}{N}\sum_{i=1}^{N}H^TR^{-1}H\right]^{-1}, \\ &= \left[\left(P^-\right)^{-1} + H^TR^{-1}H\right]^{-1} = P^+, \end{aligned}$$

as required. □

Theorem II.2 (BRUF Linear State Convergence)**.** *Given P^-, the prior covariance, $\hat{\mathbf{x}}^-$, the prior state estimate, and the iterative BRUF update,*

$$\hat{\mathbf{x}}_i \leftarrow \hat{\mathbf{x}}_{i-1} + K_i(\mathbf{y} - h(\hat{\mathbf{x}}_{i-1})), \quad 1 < i < N, \tag{7}$$

with $P_0 = P^-$ and $\mathbf{x}_0 = \hat{\mathbf{x}}^-$, and a linear observation H, the update (7) *is exactly equivalent to the Kalman update,*

$$\hat{\mathbf{x}}^+ = \mathbf{x}_N = \hat{\mathbf{x}}^- + K(\mathbf{y} - H\hat{\mathbf{x}}^-) \tag{8}$$

where K is the Kalman gain expressed in terms of the updated covariance [18],

$$K = P^+H^TR^{-1}. \tag{9}$$

Proof. We begin by re-expressing (7) in terms of the *information state* [18]:

$$\hat{\mathbf{z}}_i = P_i^{-1}\hat{\mathbf{x}}_i \tag{10}$$

where

$$\hat{\mathbf{z}}_{i+1} \leftarrow \hat{\mathbf{z}}_i + H^T(NR)^{-1}\mathbf{y} \tag{11}$$

is the information state update for BRUF step i.

The ith information state update can be expressed in terms of the prior information state, $\hat{\mathbf{z}}^-$, as

$$\hat{\mathbf{z}}_{i+1} \leftarrow \hat{\mathbf{z}}^- + \frac{i}{N}H^TR^{-1}\mathbf{y}. \tag{12}$$

Thus,

$$\hat{\mathbf{z}}^+ = \hat{\mathbf{z}}_N = \hat{\mathbf{z}}^- + H^TR^{-1}\mathbf{y}. \tag{13}$$

Equation (13) is the usual expression for the information state update.

Next, we can express the updated state in terms of the updated information state as

$$\hat{\mathbf{x}}_N = P_N\hat{\mathbf{z}}_N. \tag{14}$$

From Theorem II.1, we know that

$$P_N = P^+ = \left[\left(P^-\right)^{-1} + H^TR^{-1}H\right]^{-1}. \tag{15}$$

Rearranging (15), it is easy to show that

$$P^- = \left[(P^+)^{-1} - H^TR^{-1}H\right]^{-1}. \tag{16}$$

Finally,

$$\begin{aligned} \hat{\mathbf{x}}_N &= P_N\hat{\mathbf{z}}_N = P^+\hat{\mathbf{z}}_N \\ &= P^+\hat{\mathbf{z}}^- + P^+H^TR^{-1}\mathbf{y} \\ &= P^+(P^-)^{-1}\hat{\mathbf{x}}^- + P^+H^TR^{-1}\mathbf{y} \\ &= P^+\left[(P^+)^{-1} - H^TR^{-1}H\right]\hat{\mathbf{x}}^- + P^+H^TR^{-1}\mathbf{y} \\ &= \hat{\mathbf{x}}^- + P^+H^TR^{-1}(\mathbf{y} - H\hat{\mathbf{x}}^-), \end{aligned} \tag{17}$$

which concludes the proof. □

In this work, we propose integrating the BRUF update into an EnKF. Of note is that there are two different flavours of the ensemble Kalman filter that differ in their computation of the Kalman gain [3]. In what we call the *statistical EnKF*, the Kalman gain is computed in a statistical manner,

$$K = \mathrm{Cov}(\mathbf{X}, h(\mathbf{X})) \left(\mathrm{Cov}(h(\mathbf{X}), h(\mathbf{X})) + R\right)^{-1}, \tag{18}$$

where $\mathbf{X}$ is an ensemble of state vectors $\mathbf{x}_j$, $j = 1 \dots M$, and the computations of the covariances require the measurement perturbations

$$h(\mathbf{X}) - \frac{1}{M}\sum_{j=1}^{M} h(\mathbf{x}_j). \tag{19}$$

The alternative form of the EnKF behaves more similar to the EKF, whereby each ensemble member has its own Kalman gain that utilizes the linearization of the measurement (1) computed at the ensemble state,

$$\begin{aligned} K_j &= \mathrm{Cov}(\mathbf{X}, \mathbf{X}) H_j^T \left(H_j \,\mathrm{Cov}(\mathbf{X}, \mathbf{X}) H_j^T + R\right)^{-1}, \\ H_j &= \left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_j}, \end{aligned} \tag{20}$$

which we call the *linearized EnKF*.

In this work we focus on systems for which the computation of the covariance matrix $\mathrm{Cov}(\mathbf{X}, \mathbf{X})$ is tractable, and the measurement (1) is sufficiently non-linear, thus we make use of the linearized EnKF form (20). The linearized EnKF update is summarized in Algorithm 2.

Algorithm 2 The Linearized EnKF Update

Require: $\mathbf{X}^-$, an ensemble of M samples $\mathbf{x}_j$ from the prior distribution, where $j = 1 \dots M$; α, an inflation factor

$\mathbf{m} = \frac{1}{M}\sum_{j=1}^{M} \mathbf{x}_j$ ▷ Recompute mean
$\mathbf{X} \leftarrow \mathbf{m} + \alpha(\mathbf{X}^- - \mathbf{m})$ ▷ Perform inflation
$P = \frac{1}{M-1}(\mathbf{X}^- - \mathbf{m})(\mathbf{X}^- - \mathbf{m})^T$ ▷ Recompute cov.
for $j = 1 \dots M$ **do**
  $\hat{\mathbf{y}}_j = h(\mathbf{x}_j) + \gamma_j$ ▷ $\gamma_j \sim \mathcal{N}(\mathbf{0}, R)$
  $H_j = \left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_j}$
  $S = H_j P H_j^T + R$
  $K = P H_j^T S^{-1}$
  $\mathbf{x}_j \leftarrow \mathbf{x}_j + K(\mathbf{y} - \hat{\mathbf{y}}_j)$ ▷ Build $\mathbf{X}^+$
end for

The Linearized EnKF update begins with an inflation step. Inflation is required for the EnKF to converge [7]. The inflated samples are each updated according to the usual EKF update using the ensemble covariance. The predicted measurement $\hat{\mathbf{y}}$ is also perturbed with artificially generated measurement noise, γ_j, in order to correct the posterior covariance [4].

III. The Bayesian Recursive Update EnKF

In this work, we apply the Bayesian recursive update in an ensemble Kalman filter. Given ensemble $\mathbf{X}^-$ with members $\mathbf{x}_j$, where $j = 1, \dots M$, and nonlinear measurement (1), we replace the usual EnKF update with the Bayesian recursive update. This approach is summarized in Algorithm 3.

Algorithm 3 The Bayesian Recursive Update Linearized EnKF (BRUEnKF) Update Step

Require: $\mathbf{X}^-$, an ensemble of M samples $\mathbf{x}_j$ from the prior distribution, where $j = 1 \dots M$; N, the number of BRUF steps; α, an inflation factor

$\mathbf{X}_0 \leftarrow \mathbf{X}^-$
for $i = 1 \dots N$ **do**
  $\mathbf{m}_{i-1} = \frac{1}{M}\sum_{j=1}^{M} \mathbf{x}_{i-1,j}$
  $\mathbf{X}_{i-1} \leftarrow \mathbf{m}_{i-1} + \alpha^{1/N}(\mathbf{X}_{i-1} - \mathbf{m}_{i-1})$
  $P_{i-1} = \frac{1}{M-1}(\mathbf{X}_{i-1} - \mathbf{m}_{i-1})(\mathbf{X}_{i-1} - \mathbf{m}_{i-1})^T$
  for $j = 1 \dots M$ **do**
    $\hat{\mathbf{y}}_j = h(\mathbf{x}_{i-1,j}) + \gamma_j$ ▷ $\gamma_j \sim \mathcal{N}(\mathbf{0}, R)$
    $H_j = \left.\frac{\partial h}{\partial \mathbf{x}}\right|_{\mathbf{x}=\mathbf{x}_{i-1,j}}$
    $S = H_j P_{i-1} H_j^T + NR$
    $K = P_{i-1} H_j^T S^{-1}$
    $\mathbf{x}_{i,j} \leftarrow \mathbf{x}_{i-1,j} + K(\mathbf{y} - \hat{\mathbf{y}}_j)$ ▷ Build $\mathbf{X}_i$
  end for
end for
$\mathbf{X}^+ \leftarrow \mathbf{X}_N$

The BRUEnKF applies each successive BRUF update step to the entire ensemble at once. Each BRUF update step begins with an inflation of the ensemble $\mathbf{X}_i$ by inflation factor α. Then the ensemble covariance P_i is computed numerically. The ensemble covariance is used in a Kalman filter update of each ensemble member $\mathbf{x}_{i,j}$. In Algorithm 3, the subscript i indicates the BRUF step number and the subscript j is the index of state $\mathbf{x}_{i,j}$ in ensemble $\mathbf{X}_i$. For each ensemble member, the predicted measurement $\hat{\mathbf{y}}_j$ is perturbed with artificial noise γ_j, where $\gamma_j \sim \mathcal{N}(\mathbf{0}, R)$. The rest of BRUF update step i continues as detailed in Section II. When BRUF update step i has been completed for each ensemble member, the next BRUF update step is performed on ensemble $\mathbf{X}_{i+1}$. The updated ensemble $\mathbf{X}^+$ is the ensemble $\mathbf{X}_N$ built in the final BRUF update step.

The following example illustrates the BRUEnKF update for a Gaussian-distributed prior ensemble $\mathbf{X}^-$ and a nonlinear measurement.

A. Two Dimensional Range Observation Example

We now illustrate how the proposed BRUEnKF differs from the EnKF through a simple two dimensional example.

For our prior, we take a Gaussian distribution,

$$\mathcal{N}\left(\begin{bmatrix}-3.5\\0\end{bmatrix}, \begin{bmatrix}1 & \frac{1}{2}\\ \frac{1}{2} & 1\end{bmatrix}\right) \tag{21}$$

that is off-center from the origin with the two variables correlated. For the non-linear measurement, we take the range from the origin,

$$h(\mathbf{x}) = \sqrt{x_1^2 + x_2^2}, \tag{22}$$

and measure the value of

$$y = 1, \tag{23}$$

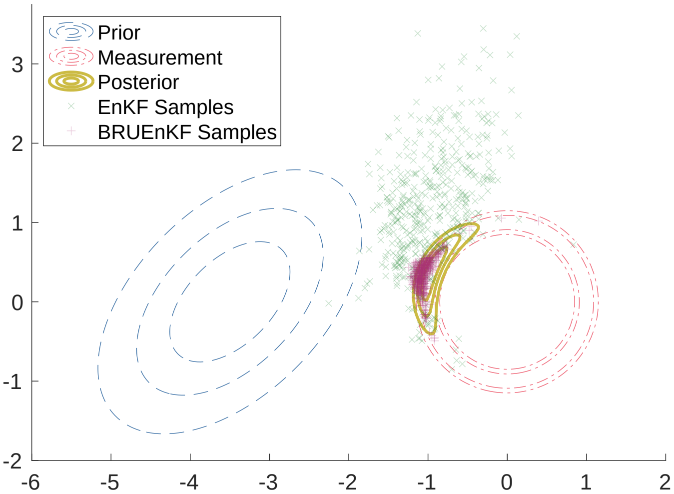

Fig. 1. A plot of the two dimensional range observation example. The blue dashed contour lines represent the prior distribution, the red dash-dotted contour lines represent the measurement distribution, and the bold solid yellow lines represent the posterior distribution. Green $\times$'s represent the posterior ensemble generated by the EnKF, while purple $+$'s represent the posterior ensemble generated by the BRUEnKF.

meaning that the measured value was poorly predicted by our prior knowledge. We additionally take an error covariance of,

$$R = 0.1^2, \tag{24}$$

implying that we have high confidence that our measurement is correct.

Taking $M = 500$ samples from the prior (21), taking a high inflation factor $\alpha = 1.5$, and by using $N = 100$ steps in the BRUEnKF (see Algorithm 3), we can look at the difference between the EnKF and BRUEnKF at approximating the true posterior. We visualize this in Fig. 1.

As can be seen, the true posterior is highly non-Gaussian, has nonlinear curvature, and the region of high likelihood is almost entirely a subset of the observation distribution. The EnKF entirely fails to capture the posterior distribution. Most of the samples that it generates would be considered outliers to the truth, and are thus of little use. The BRUEnKF generates samples that are highly likely in the posterior. It captures the mean of the true posterior well. However, it is slightly over confident, and does not fully represent the higher order moments.

IV. Numerical Simulation

In this example, we compare the performance of the EnKF and the BRUEnKF on a modified version of the numerical experiment in [19]. We use the Lorenz '63 dynamics [20]:

$$\begin{aligned} \dot{x}_1 &= 10(x_2 - x_1), \\ \dot{x}_2 &= x_1(28 - x_3) - x_2, \\ \dot{x}_3 &= x_1 x_2 - \frac{8}{3} x_3. \end{aligned} \tag{25}$$

The states are propagated using (25) in the classical 4th-order Runge-Kutta integrator. The initial states are:

$$\mathbf{x}_0 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}^T. \tag{26}$$

The filter begins tracking the system after 10 time units. The initial uncertainty is

$$P_{10} = I_{3\times 3}. \tag{27}$$

For the nonlinear measurement, we take the range from one of the equilibrium points:

$$y = \sqrt{(x_1 - 6\sqrt{2})^2 + (x_2 - 6\sqrt{2})^2 + (x_3 - 27)^2} + \eta \tag{28}$$

where the measurement error variance is $R = 1/16$. A measurement is recorded every $\Delta t = 0.12$ time units. Fig. 2 shows the trajectory for the first 30 time units of the simulation. The dynamics form a distinct butterfly shape.

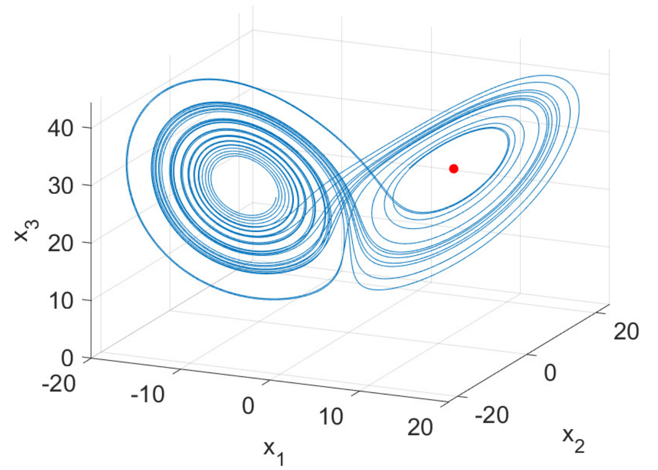

Fig. 2. Lorenz '63 trajectory. An instrument placed at the equilibrium point $\begin{bmatrix} 6\sqrt{2} & 6\sqrt{2} & 27 \end{bmatrix}^T$ measures the range to the target.

Like the example in Section III-A, this is a challenging scenario for linear filters. The relatively long propagation time step allows the states to spread out sufficiently in between measurement updates. The prior covariance is large, but the measurements are highly accurate.

Both the EnKF and the BRUEnKF are given $M = 15$ ensemble members drawn from the initial distribution $\mathcal{N}(\mathbf{x}_{10}, P_{10})$, where $\mathbf{x}_{10}$ is the true state after 10 time units. For both filters, we use the inflation factor $\alpha = 1.01$. For the BRUEnKF, we choose $N = 25$ EKF updates.

Fig. 3 compares the performance of the BRUEnKF and the EnKF for this example. We show results for the second half of the simulation after a 60 time unit spin-up. The covariance bounds shown in the figure are the mean 3σ standard deviation bounds for all 100 filters.

The average behavior of the filters is similar for this problem setting. For the EnKF, certain observations cause the filters to diverge. We compare the performance of the two

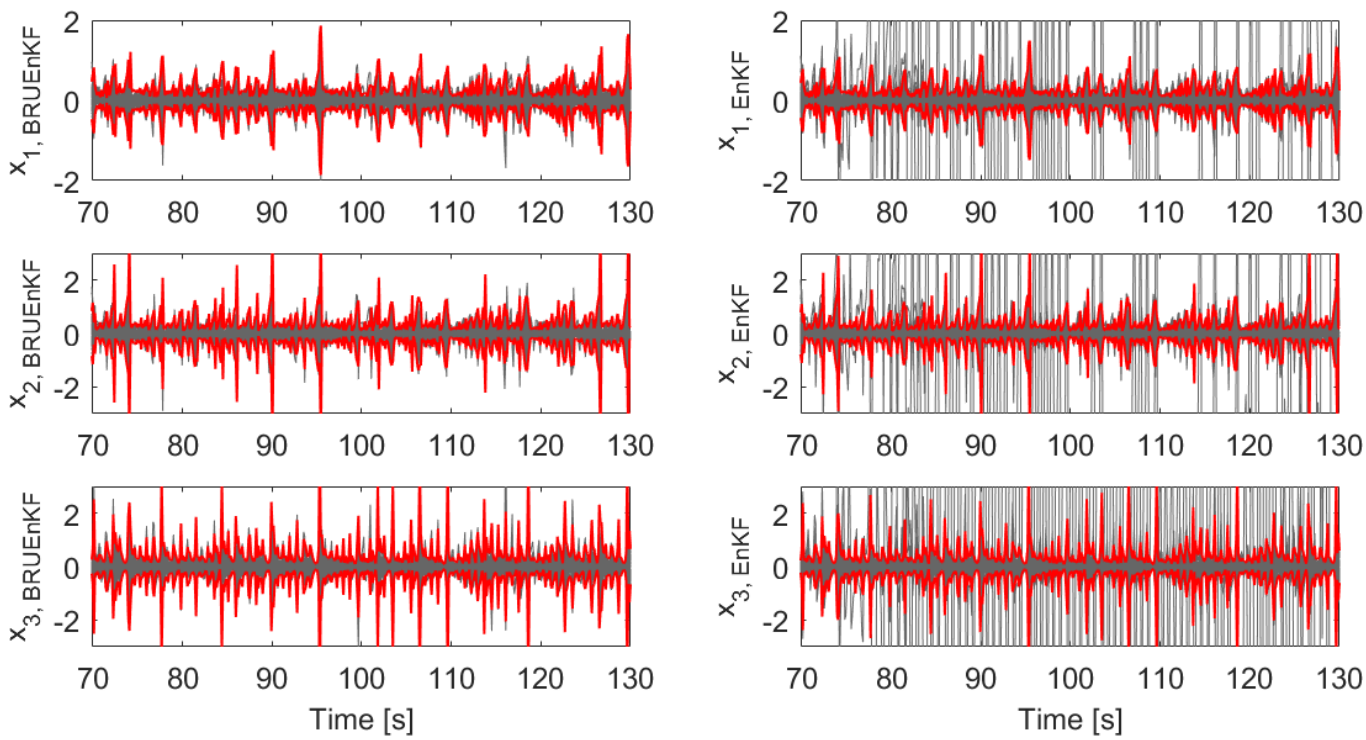

Fig. 3. Comparison of BRUEnKF (left) and EnKF (right) error results for 100 Monte Carlo runs. The plots show the estimation error (gray) and mean filter 3σ bounds (red).

approaches using the mean spatio-temporal root mean square error (RMSE):

$$\text{RMSE} = \frac{1}{n_m}\sum_{m=1}^{n_m}\sqrt{\frac{1}{n_x n_t}\sum_{n=1}^{n_x}\sum_{k=1}^{n_t} e_{n,k}^2} \tag{29}$$

where n_m is the number of Monte Carlo runs, n_x is the dimension of the state, n_t is the number of time steps, and $e_{n,k}$ is the error in the estimate of state n at time step k. The mean spatio-temporal RMSE for the BRUEnKF is 0.2405, while the mean spatio-temporal RMSE for the EnKF is 0.3435. The difference in RMSE values implies that, on average, the performance of the EnKF is worse than the BRUEnKF.

Finally, we would like give more insight into the behavior of the BRUEnKF update. Fig. 4 shows the error and covariance bounds for a portion of the trajectory with three measurement updates. This time, $\Delta t = 0.01$. There are 20 propagation steps in between each measurement update. The sections of the plot highlighted in pink show the changes in the state error and covariance during the BRUEnKF update. In other words, the pink sections of the plot contain the $N = 25$ "pseudotime" steps during which the BRUEnKF update is carried out. The EnKF update is also included for comparison. The changes in the EnKF covariance bounds appear as straight lines through the highlighted portions of the plot, since the EnKF update is applied in a single step.

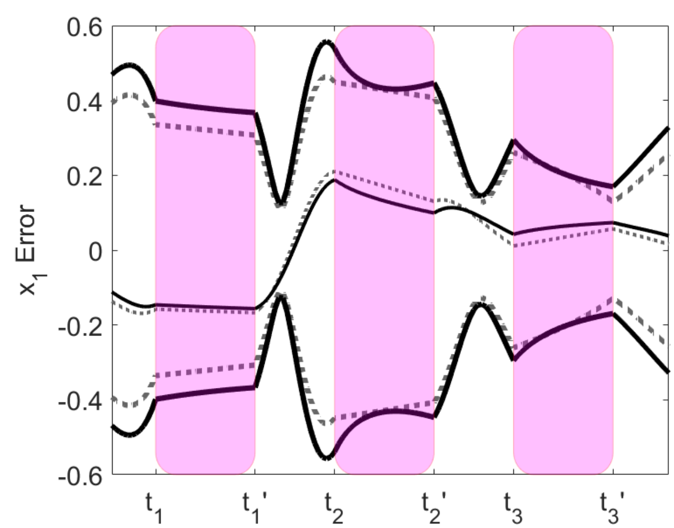

Fig. 4. The BRUEnKF update. The highlighted portions of the plot (pink) the BRUEnKF state error and 3σ covariance bounds (black) during $N = 25$ update steps. The corresponding EnKF state error and covariance values are shown in gray, dashed lines. The intervals $[t_k, t_k']$ denote the BRUEnKF update "pseudotime" steps. The evolution of the error dynamics in real time occurs during intervals $[t_k', t_{k+1}]$.

The second BRUEnKF update in Fig. 4 reveals an important aspect of the BRUF update: the covariance may shrink *and then grow again*, all in the course of a single update. This is possible due to the iterative recomputation of the measurement Jacobian. The BRUEnKF ends the update step with a slightly higher covariance than the EnKF. For observations that cause the BRUEnKF covariance to form this J-curve shape, the EnKF covariance is overconfident. This behavior helps the BRUEnKF maintain consistency during measurement updates where the EnKF cannot. Note that the BRUEnKF covariance still *decreases overall* during the measurement update. The filter's uncertainty does not increase due to the incorporation of information from a nonlinear measurement. It simply decreases *less* than the EnKF uncertainty.

V. Conclusion

In this work, we adapt the BRUF update [8] for use in an ensemble Kalman filter. The BRUEnKF is shown to improve on the EnKF for systems with large prior uncertainty and highly accurate, nonlinear measurements. The BRUEnKF measurement update is able to "follow" the nonlinearity in the measurement by recursively recalculating the measurement Jacobian. We present two examples with range measurements. In the two-dimensional example, the BRUEnKF matches the true Bayesian posterior better than the EnKF for a single measurement update. In the second example, the BRUEnKF successfully tracks the Lorenz '63 dynamics. On average, the EnKF is prone to higher error.

In the future, we will continue to explore new applications for the BRUF update. We will adapt it for other types of filters, including particle filters. Among single-state filters, the BRUF is most closely related to the Recursive Update Filter [14] and the Iterated EKF [13]. Of the three, it is the simplest to compute. It performs exactly N EKF updates for each measurement, making its computational cost N times the computational cost of the EKF. We will also continue to probe the convergence properties of the BRUF. In this work, we show that the BRUF update is equivalent to the Kalman update for linear measurements. In the future, we would like to provide stronger convergence guarantees for nonlinear measurements.

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