We study the single-choice prophet inequality problem where arrivals are i.i.d. random variables.

When the distribution is known, a single-threshold algorithm is \((1 - 1/e)\)-competitive. When the distribution is unknown, past works focus on the setting where uniform samples are given.

We propose a new setting where the algorithm has access to an oracle that answers quantile queries about the distribution. We derive algorithms when only a few queries are allowed.

Main results

1. The \(k\)-threshold algorithms, where the \(k\) thresholds are obtained from carefully chosen queries, have ratios strictly higher than \(1 - 1/e\). The ratio improves gracefully with the number of thresholds.

2. The observe-then-accept algorithm beats \(1 - 1/e\) with just a single query.

**Single-choice Prophet Inequality – IID Model**

- A gambler faces a sequence of \(n\) online i.i.d. random variables from an unknown distribution.
- When a variable reveals its value, the gambler decides irrevocably whether to accept the value.
- We want to maximize the competitive ratio between the expected gain of the gambler and that of the maximum variable (i.e., the prophet value).

**Research Question:** How much knowledge of the distributions is required to achieve a good competitive ratio?

**Preliminaries**

\(\text{ALG}\): the expected gain of the algorithm; \(\text{OPT}\): the expected maximum of all variables.

The competitive ratio is the minimum of \(\text{ALG}/\text{OPT}\) over all possible distributions.

Given a quantile query \(q \in [0, 1]\), the distribution quantile oracle returns the corresponding value \(v(q)\) s.t.

\[
v(q) := \inf_{\theta \in \Delta} \{ F(\theta) \geq q \} = F^{-1}(q).
\]

Zero Queries (equiv. to the Secretary Problem): There exists a \(1/e\)-competitive algorithm that makes zero queries for the prophet inequality problem on unknown i.i.d. distributions.

One Query (equiv. to using a single threshold): There exists a \((1 - 1/e)\)-competitive algorithm for the prophet inequality problem on unknown i.i.d. distributions that makes a single query to the distribution oracle.

**Beating \(1 - 1/e\) with Two or More Queries**

The \(k\)-threshold algorithm is parameterized by \([c_\ell, \rho_\ell]\) for \(k \leq \ell \leq n\).

- The \(\ell\)-th segment contains variables indexed from \(n \cdot \sum_{j < \ell} \rho_j + 1\) to \(n \cdot \sum_{j \leq \ell} \rho_j - 1\).
- The ratio improves gracefully with the number of thresholds.

We set a lower bound of \(\text{ALG}\) as the objective and three upper bounds of \(\text{OPT}\) as constraints:

\[
\text{minimize} \quad \sum_{\ell=1}^{k} e^{-\sum_{j=1}^{\ell-1} n \cdot \rho_j} \cdot (1 - e^{-\gamma \cdot (\theta_{\ell} - \rho_{\ell})})
\]

subject to

- \(0 \leq \theta_1 \leq \cdots \leq \theta_\ell \leq \Delta_\ell \leq \Delta_\ell + 1\), \(\theta_\ell = \Delta_\ell = n \cdot E[(x - \theta_\ell)^+]\) and \(\delta_\ell := \int_{\theta_\ell}^{\delta_\ell} \Pr[\max_i(x_i) \geq \ell] \, dt\).

We find appropriate \([c_\ell, \rho_\ell]\) via enumeration and give the competitive ratios for different \(k\):

<table>
<thead>
<tr>
<th># thresholds</th>
<th>2</th>
<th>3</th>
<th>4</th>
<th>5</th>
</tr>
</thead>
<tbody>
<tr>
<td>ratio</td>
<td>0.6786</td>
<td>0.6883</td>
<td>0.6946</td>
<td>0.7004</td>
</tr>
</tbody>
</table>

**Beating \(1 - 1/e\) with One Query**

The observe-then-accept algorithm is parameterized by \(c \in [0, n]\) and \(\rho \in [0, 1]\).

- For the first \(\rho n\) variables, we use \(c\) as the threshold, where \(c\) is obtained by making a query at \(1-c/n\).
- If the first \(\rho n\) variables are not accepted, we use their maximum variable \(\text{max}_{i \leq \rho n} x_i\) as the threshold for the remaining \((1-\rho)n\) variables.

**Theorem.** The observe-then-accept algorithm performs strictly better than the single-threshold algorithm, achieving a competitive ratio of \(0.6718\) for the prophet inequality problem on unknown i.i.d. distributions.

The analysis is using a factor-revealing LP that is similar to that of the \(k\)-threshold algorithm.