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Eliciting truthful reports with partial signals in repeated games ☆

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ABSTRACT

We consider a repeated game where a player self-reports her usage of a service and is charged a payment accordingly by a center. The center observes a partial signal, representing part of the player's true consumption, which is generated from a publicly known distribution. The player can report any value that does not contradict the signal and the center issues a payment based on the reported information. Such problems find application in net metering billing in the electricity market, where a customer's actual consumption of the electricity network is masked and complete verification is impractical. When the underlying true value is relatively constant, we propose a penalty mechanism that elicits truthful self-reports. Namely, besides charging the player the reported value, the mechanism charges a penalty proportional to her inconsistent reports. We show how fear of uncertainty in the future incentivizes the player to be truthful today. For Bernoulli distributions, we give the complete analysis and optimal strategies given any penalty. Since complete truthfulness is not possible for continuous distributions, we give approximate truthful results by a reduction from Bernoulli distributions. We also extend our mechanism to a multi-player cost-sharing setting and give equilibrium results.

1. Introduction

Consider the following repeated game where a center owns resources and one or more strategic players pay the center to consume the resources. In every round, a player self-reports their usage, which will then be used to determine their payment to the center. However, it is not always possible for the center to verify the submitted information from the players. Instead, only part of the actual consumption is revealed to the center based on some publicly known distribution. A player can report any value that is at least the revealed amount. Without any external interference, a player will naturally report exactly the revealed amount (potentially lower than the true consumption) to minimize their payment. The center then needs to determine a payment mechanism such that each player is incentivized to report their true value.

The electricity market is facing precisely the described problem. As the number of electricity *prosumers* increases each year, new rate structures are designed to properly calculate the electricity bill for this special type of consumer while ensuring that every

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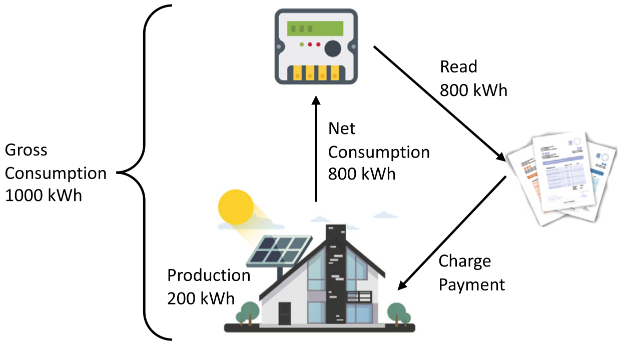

Fig. 1.1. Net metering for electricity prosumers.

customer is still paying their fair share of the network costs. Prosumers are those who not only consume energy but also produce electricity via distributed energy resources such as rooftop solar panels. Among different rate structures, *net metering* is a popular billing mechanism that is currently adopted in more than 40 states in the US [28]. Net metering charges prosumers a payment proportional to their net consumption, i.e., gross consumption minus the production [35], demonstrated in Fig. 1.1. The payment includes the electricity usage as well as grid costs that are incurred by using the electricity network.

The controversy in net metering lies in that prosumers fail to pay their share of the grid costs when they do not have local storage equipment [12]. In the United States, only 4% of the solar panel owners also own a battery to store the produced solar energy [25]. For those who do not own battery storage, the generated power has to be transmitted back to the grid. Accordingly, the daily consumption of power by these prosumers also needs to come from the grid instead of directly from the solar panels. In this way, most prosumers have under-paid their share of the network costs and become “free-riders” of the electricity grid. The grid is often subject to costly line upgrades and net metering unevenly shifts such costs to traditional consumers, who usually come from lower-income households [17]. Indeed, previous research works have suggested that prosumers should pay a part of the grid costs proportionally to their gross consumption, not net consumption [12,21]. However, the gross consumption is hidden from the utility companies since only net consumption can be observed from the meter.[1] Meanwhile, there is no incentive for prosumers to voluntarily report their true consumption as it will only increase their electricity bills.

Fortunately, the production from solar panels usually follows some pattern while the gross consumption of electricity for a typical household stays relatively constant, which is especially true for industrial sites, the major consumers for utilities [31,39]. Thus, the observed usage can be assumed to follow some natural distribution and the center is able to detect dishonesty when a player’s report differs from their reporting history. With this idea, we propose a simple penalty mechanism, the *flux mechanism*, that elicits truthful reports from players in a repeated game setting when only partial verification is possible. Particularly, a player is charged their reported value as well as a penalty due to inconsistency in consecutive reports in each round. The main goal is to ensure that every player reports their true values and no penalty payment is collected. We show that the combining effect of (i) the penalty rate and (ii) the length of the game is sufficient for inducing truthful behavior from the player for the entire game. As the horizon of the game increases, the minimum penalty rate for truth-telling as an optimal strategy decreases. In other words, it is the fear of uncertainty in the future that incentivizes the player to be truthful today.

Besides net metering, our mechanism can be used to address dishonest reporting in other public goods or services. For example, some individuals or enterprises misrepresent their income or deductions to lower their taxes. While tax fraud is a federal crime, the chance of being audited is small compared to the percentage of taxpayers who are not complying with the tax laws [37]. Another application of our mechanism is insurance fraud when an insurer exaggerates claims or provides false information to insurance companies to receive larger payouts. Such behavior sometimes results in a premium increase for every insurer in the same area, including the honest ones. To avoid the erosion of such services and ensure equity among different users, our mechanism presents a simple and automated way of identifying dishonesty in a self-reporting system and incentivizes truthfulness via a single parameter.

1.1. Our contribution

We address the problem of eliciting truthful reports when the center is able to observe a part of the player’s private value based on some publicly known distribution. The strategic player reports some value that is at least the publicly revealed value and is charged a payment accordingly. We propose a truth-eliciting mechanism, *flux mechanism*, that utilizes the player’s fear of uncertainties to achieve truthfulness. In each round, the player is charged a “regular payment” proportional to the consumption they report. Starting from the second round, the player is charged an additional “penalty payment”, which is r times the (absolute) difference between the reports in the current and the previous round, where the penalty rate r is set by the center before the game starts.

[1] We note that there exist net meters such that both the gross consumption and the solar generation are recorded. We focus on the net metering programs where the meter only provides the net usage.

Table 1
Optimal strategy given penalty rate r under $Ber(p)$ distributions.

Bernoulli Prob.	Penalty Rate	Optimal Strategy
$p \geq 0.5$	$r \leq \frac{1}{2p}$	lying-till-end
	$\frac{1}{2p} < r \leq 1$	lying-till-busted + lying last round
	$1 < r < \frac{1-(1-p)^T}{p-p(1-p)^{T-1}}$	lying-till-busted
	$r \geq \frac{1-(1-p)^T}{p-p(1-p)^{T-1}}$	honest-till-end
$p < 0.5$	$r \leq 1$	lying-till-end
	$h(t-1) < r \leq h(t)$	lying-till-end first t rounds + lying-till-busted for rest
	$h(T-1) < r < \frac{1-(1-p)^T}{p-p(1-p)^{T-1}}$	lying-till-busted
	$r \geq \frac{1-(1-p)^T}{p-p(1-p)^{T-1}}$	honest-till-end

Intuitively, a player can save their regular payment by under-reporting their consumption, but they will then face the uncertainty of paying penalties in future rounds due to inconsistent reports. Under most settings, if r is set to be infinitely high, the players will be completely truthful to avoid any penalty payment. However, a severe punishment rule is undesirable and discourages players from participating. Therefore, we want to understand the following question.

What is the minimum penalty rate such that the player is willing to report their true value?

We observe that no finite penalty can achieve complete truthfulness for arbitrary distributions as a player's true consumption may never be revealed exactly. We can, however, obtain approximate truthfulness for a general distribution by analyzing complete truthfulness for a corresponding Bernoulli distribution. For $Ber(p)$, the partial signal equals the true consumption with probability p and 0 with probability $1-p$ for $p \in (0,1)$. We give results for Bernoulli distributions in Main Results 1 and 2. For arbitrary distributions, we redefine p as the probability of having a partial signal that is at least α times the true consumption, for $\alpha \in [0,1]$, to obtain α-truthfulness (Main Result 3).

Main Result 1. (Theorem 3.1) *For a T-round game with Bernoulli distribution $Ber(p)$, the player is completely truthful if and only if the penalty rate is at least*

$$\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}.$$

Main Result 1 gives the minimum penalty rate that guarantees complete truthfulness for $Ber(p)$ distributions. We also want to understand how players would behave if the penalty rate is not as high, which describes the situation when the center is willing to sacrifice some degree of truthfulness by lowering the penalty rate. Given any penalty rate, we show that a player's optimal strategies can be described as one or a combination of three basic strategies, *lying-till-end, lying-till-busted* and *honest-till-end*. Specifically, with a low penalty rate, the player is always untruthful to save regular payment, i.e., lying-till-end is optimal. As the penalty rate increases, the player's optimal strategy *gradually* moves to lying-till-busted, which is to be untruthful until the partial signal is revealed as the true consumption for the first time and then stays truthful for the rest of the game. When the penalty rate is sufficiently high, the player would avoid lying completely and reports the truth, i.e., she is honest-till-end.

Main Result 2. (Theorems 3.1 and 3.2) *For a T-round game with Bernoulli distribution $Ber(p)$, given any penalty rate r, the player's optimal strategy is summarized in Table 1, where*

$$h(t) = \frac{1-(1-p)^t}{2p-p(1-p)^{t-1}}, \text{ for } 1 \leq t \leq T.$$

For arbitrary distributions, including uniform distributions, we explain in Section 4 that it is impossible to obtain complete truthfulness without setting the penalty to infinity. Main Result 3 gives a reduction from Bernoulli distributions to general distributions for approximate truthfulness.

Main Result 3. (Theorem 4.1) *Given $\alpha \in [0,1]$ and an arbitrary distribution with CDF F, if a penalty rate r achieves complete truthfulness for $Ber(p)$ where $p = 1 - F(\alpha D)$ and D is the player's true gross consumption, then the same r achieves α-approximate truthfulness for distribution F.*

Finally, we extend our results to multiple players. We note that if the players are charged independently, applying the flux mechanism to each individual elicits truthful reports. A more complicated and realistic setting is the cost-sharing problem where the players split an overhead cost based on their submitted reports. We propose the *multi-player flux mechanism* where the penalty payment is the same as before but the regular payment is now a share of some overhead cost. Again, if the penalty rate is sufficiently high, the players stay truthful, regardless of others' behavior, to avoid any penalty payment, i.e., the truthful report profile forms a *dominant strategy equilibrium*. As the penalty rate decreases, the truthfulness of a player may depend on other players' actions. That is, with a lower penalty rate, a truthful report profile forms a *Nash equilibrium*. For both equilibrium definitions, we are interested in the following question.

What is the minimum penalty rate for the truthful report profile to form a dominant strategy or Nash equilibrium?

We give exact penalty thresholds for both truthful equilibria under Bernoulli distributions and use a reduction to obtain approximate results under arbitrary distributions in Main Result 4.

Main Result 4. (Theorems 5.1, 5.2, 5.3 and 5.4) *For any T-round game with distribution $Ber(p)$, a truthful strategy profile is a dominant strategy equilibrium if and only if*

$$r \geq \frac{C}{nD}\frac{1-(1-p)^{n-1}}{p}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}},$$

and a Nash equilibrium if and only if

$$r \geq \frac{C}{nD}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}.$$

Given $\alpha \in [0,1]$ and any distribution with cumulative distribution function F, let $p = 1 - F(\alpha D)$, where D is the true gross consumption. Then α-approximate truthful profile is a Nash equilibrium if

$$r \geq \frac{1}{\alpha}\frac{C}{nD}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}},$$

and the α-approximate truthful profile is a dominant strategy equilibrium if

$$r \geq \frac{1}{\alpha}\frac{C}{nD}\frac{1-(1-p)^n}{p}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}.$$

1.2. Related works

The economic effect of the net metering policy has been explored for different countries and regions [7,8,41,34,20]. It has been observed that net metering can cause inequality issues for traditional energy consumers [17,20,29,9]. Accordingly, alternative pricing mechanisms and tariff structures have been proposed to fairly compensate the energy production [36,11,4,12,21]. In particular, Gautier *et al.* [12] and Khodabakhsh *et al.* [21] proposed that individuals should be charged based on their true consumption, not net consumption. Our work is a continuation of [21], where a primitive version of the penalty mechanism is first proposed for promoting a fairer electricity rate structure. We formally define the mechanism and provide the corresponding theoretical analysis.

More broadly, fairness for the power grid has become an increasingly popular subject. First, Heylen *et al.* provided various indices to measure fairness and inequality in power system reliability [15]. Fairness is also explored for load shedding plans [16], electric vehicle charging schemes [1], demand response [2], etc. Moret and Pinson showed fairness can be improved with a "community-based electricity market", where prosumers are allowed to share their production on the community level [26]. Our model, on the other hand, addresses the fairness issues by modifying the current electricity structure, which is easier for utility companies to adopt.

Theoretically, our work is related to information elicitation with limited verification ability. Green and Laffont [13] first formalized the mechanism design problem with partial verification where a center may detect part of a false response submitted by a player. One branch of information elicitation without verification is attributed to crowdsourcing, where the center needs to elicit answers from a large crowd on a simple task. Mechanisms proposed for crowdsourcing include peer prediction [27,19,33], Bayesian truth serum [32,44], collective revelation [14], output agreement [40,42,24], the disagreement mechanism [22], etc. Many of the above mechanisms score a player based on the report from another player, either by using the other player's report as the reference (peer prediction) or by evaluating how close the two reports are to each other (output agreement). It is known that common knowledge is the best a mechanism can do for information elicitation without verification unless there are restrictions on the information structure [42]. Such mechanisms are not applicable to the single-player setting in our problem. In the multi-player setting, however, it is difficult to justify charging players based on the discrepancies between their reports and those from their peers, because the consumption can differ for each player. Another related area is probabilistic verification [6], where the center may catch a lying player by a probability specified by her type. In particular, Ball and Kattwinkel considered the trade-off between the benefit of a successful misreporting and the risk of being detected for a strategic player [3]. The probabilistic verification takes place as tests by the center that a player can either try or skip, which is not applicable for our problem.

Finally, the use of penalty in mechanism design has been widely adopted and proven to be effective in different application areas such as supply chain [43,5], performance-based regulations [18], waste recycling [38], etc. It is also suggested that truthful players are willing to punish "free-riders" [10]. In some cases, a penalty may lead to participation issues and the mechanism needs to satisfy the voluntary participation property such that the players prefer the outcome of joining the game instead of staying out [30,23]. In our application, since not participating indicates receiving no electricity, the voluntary participation property is trivially satisfied.

2. Problem statement

In this section, we formally define our problem under the single-player setting and defer the extension to multiple players to Section 5. The player has a gross consumption $D \geq 0$, which is her private information. The game has T rounds where $T > 1$ as otherwise, the flux mechanism becomes invalid. In each round t, the center observes a partial signal, $y_t \leq D$, which is randomly and independently drawn from a distribution F supported on $[0, D]$. We use $r \geq 0$ to denote the penalty rate. In a flux mechanism, a player cares more about the number of rounds left in the future rather than the number of rounds that have passed. Thus we use $t = T, T-1, \cdots, 1$ to denote the current round, where *t means there are t rounds left*, including the current round. For example, the first round is round T, the last round is round 1, and the previous round of round t is round $t+1$. For the round $t \leq T$, the flux mechanism runs as follows.

- The center observes the player's net consumption $y_t \sim F$.
- The player submits their reported gross consumption which is at least the net consumption, $b_t \geq y_t$. The player may not be truthful, i.e., b_t may not equal D.
- When $t < T$, the player's payment consists of regular payment b_t and penalty payment $r \cdot |b_{t+1} - b_t|$. When $t = T$, the player only pays the regular payment.

For $t < T$, we call b_{t+1} the *history* of round t.[2] In each round t, the player wants to pay the lowest expected total payment by reporting b_t without knowing the partial signals for future rounds. We call a mechanism *truthful* if the player reports D for all rounds. When two reports bring the same expected payment, we break ties in favor of truthfulness.

2.1. Discussions on the value of D

The assumption that the gross consumption D is the same for every round is taken from Khodabakhsh *et al.* [21], the work that inspired our paper. We explain here why this is a valid assumption, especially for the prosumer pricing problem, and discuss an easy extension where D_t is drawn from a known range $[\underline{D}, \overline{D}]$.

According to the U.S. Energy Information Administration, electricity usage typically follows a daily pattern, which means within some period (e.g., day, month, or season), the electricity consumption does not vary much [39]. This is especially true for industrial sites, which are the major consumers of utilities [31]. Therefore, we can always discretize the time horizon into sub-intervals such that the electricity consumption within each interval is relatively constant.

To relax this assumption, let the gross consumption for each round come from a known range, i.e., $D_t \in [\underline{D}, \overline{D}]$. The center can estimate the values of $\underline{D}$ and $\overline{D}$ from historical data but does not necessarily know D_t for any t. We show that our results extend straightforwardly. Recall that in the analysis of Bernoulli distributions, we compare the basic two strategies and find the penalty rate that sets the two expected costs equal. We can find an upper bound for the truthful threshold by bounding the expected cost of lying-till-busted and honest-till-end from below and above. Then the resulting penalty rate is simply the original threshold (3.1) times the ratio $\overline{D}/\underline{D}$. A player's optimal strategy remains the same under the updated penalty rate.

For the reduction from Bernoulli distributions to any arbitrary distribution with cdf F, given $\alpha \in [0,1]$, we now define $p = 1 - F(\alpha\overline{D})$ and use the same argument to obtain an upper bound of penalty rate that achieves α-truthfulness. For the multi-player model, we add the multiplicative ratio $\overline{D}/\underline{D}$ in every expression for an upper bound of the desired penalty rate. This relaxation will also help add heterogeneity to the multi-player model.

We note that with this adjustment, the player(s) can be charged a penalty payment although they have been truthful, due to the fluctuation of the gross consumption. In this case, we suggest that after the game ends, the center returns such wrongful penalty payments to the player(s) as such payments are unrelated to the consumption of the public service and are not supposed to be kept by the center. We emphasize that this recommendation is only applicable to penalty payments incurred from the small discrepancies in gross consumptions. Penalties collected from lying to the center will not be returned, as otherwise it will break the incentive to report truthfully.

2.2. The offline setting

We briefly analyze the offline setting of the flux mechanism, which happens when the player can observe the partial signals, y_t's, for every t, before the game starts. In this case, we show finding the optimal strategy reduces to solving a linear program. In the offline setting, the net consumption for each round t, y_t, is known at the beginning of the game. To obtain the optimal reports in the single-player model, we solve the following mathematical program:

[2] The *history* usually refers to the record from the beginning of the game till the current round. In our mechanism, the history before yesterday does not affect the player's action for today. Therefore, the history in round t only needs to be the report for the previous day.

$$\begin{aligned}&\min_{b_1,\dots,b_T} \ \textstyle\sum_{t=1}^{T} b_t + \sum_{t=1}^{T-1} r \cdot |b_{t+1} - b_t| \\ &\text{subject to} \quad b_t \geq y_t \quad \forall t.\end{aligned}$$

The absolute values in the objective function can be linearized with auxiliary variables. The subsequent linear program can be solved in polynomial time via the ellipsoid method. The online setting, however, is much more complicated as the objective would be a convoluted multi-stage minimization problem. We demonstrate in the following sections how we can tackle the online setting by exploiting the properties of the flux mechanism.

3. Bernoulli distributions

We start with the analysis of Bernoulli distribution as we show later a reduction from an arbitrary distribution to a Bernoulli distribution. We prove it is only optimal for a player to report zero or their true consumption in each round. The optimal strategies can then be characterized by three basic strategies (Definition 3.1). The penalty thresholds are computed by comparing the different combinations of the basic strategies.

3.1. Basic strategies

In a Bernoulli distribution setting, in each round t, the partial signal y_t is D with probability p and 0 with probability $1-p$. When the partial signal equals to the private value, i.e., $y_t = D$, we say that the player is *"busted"* in round t. We first define three basic strategies.

Definition 3.1 *(Basic Strategies)*. For Bernoulli distributed net consumption $y_t \sim Ber(p)$, we define the following as the three basic strategies:

- lying-till-end: Report $b_t = 0$ when $y_t = 0$ and $b_t = D$ otherwise;
- lying-till-busted: Report $b_t = 0$ until $y_t = D$ for the first time, then report D for all future rounds;
- honest-till-end: Report $b_t = D$ for all rounds.

We note that a player's optimal strategy for a given penalty rate r can be solved by backward induction. Let $OptCost(t, r, b_{t+1})$ denote the optimal expected cost for a player starting in round t with penalty rate r and report b_{t+1} for the previous round. Then

$$OptCost(t, r, b_{t+1}) = \min_{b_t} ExpCost(t, r, b_{t+1}, b_t),$$

where $ExpCost(t, r, b_{t+1}, b_t)$ is the expected cost for the player starting in round t and reporting b_t (if she is allowed to), with penalty rate r and history b_{t+1}, i.e.,

$$\begin{aligned}&ExpCost(t, r, b_{t+1}, b_t) \\ &= \mathbb{E}_{y_t}[\max\{y_t, b_t\} + r|\max\{y_t, b_t\} - b_{t+1}| + OptCost(t-1, r, \max\{y_t, b_t\})] \\ &= p\big(D + r(D - b_{t+1}) + OptCost(t-1, r, D)\big) + (1-p)\big(b_t + r|b_t - b_{t+1}| + OptCost(t-1, r, b_t)\big).\end{aligned}$$

The first term on the right-side of the equation above refers to the cost when the partial signal is revealed as D and the player has to report D. The second term refers to the cost when the partial signal is 0 and the player chooses to report b_t. Let $OptCost(0, r, b_1) = 0$ for all b_1. When $t = T$, i.e., the first round, there is no history b_{T+1}. Therefore, the player simply wants to minimize the following total cost,

$$\begin{aligned}OptCost(T, r) &= \min_{b_T} ExpCost(T, r, b_T) \\ &= p(D + OptCost(t-1, r, D)) + (1-p)(b_T + OptCost(t-1, r, b_T)).\end{aligned}$$

Solving the recursion will give the characterization of optimal strategies in Table 1, as we demonstrate in Appendix A.1. In what follows, we discuss a surprisingly simpler and more constructive proof by exploiting the properties of the flux mechanism, which may be of independent interest.

3.2. Main theorems

We observe that there are two key elements that influence the decision-making of the player.

(1) The player's history, b_{t+1} for $t < T$. The value of b_{t+1} directly affects the penalty payment in round t. Intuitively, a player is more reluctant to lie if b_{t+1} is high and better off lying if b_{t+1} is small.
(2) The number of rounds left to play, i.e., t. The value of t indirectly influences the probability and the number of times a player will be busted in the remaining rounds.

Via Lemmas 3.1-3.4, we show these are the *only two* elements that determine a rational player's action. The following lemma shows that it is not optimal for a player to report a value strictly between 0 and D. Moreover, if a player is untruthful in the previous round, it is better to remain untruthful. With this lemma, we largely reduce the strategy space we need to consider.

Lemma 3.1. *For any round $t \le T$, given $y_t = 0$, the optimal report in round t is $b_t \in \{0, D\}$. Moreover, if $t < T$ and $b_{t+1} = y_t = 0$, then the optimal report is $b_t = 0$.*

Proof. To see the first sentence, we can observe that the cost function is a linear function of today's report b_t and thus either 0 or D achieves the optimality. To see the second sentence, we consider the last round t in the optimal strategy such that when $(b_{t+1}, y_t) = (0, 0)$ but $b_t > 0$. It is obvious if t is the last round, and thus we assume $t > 1$. By reporting b_t in round t, the expected total cost afterward is

$$\begin{aligned}
&rb_t + b_t + ExpCost(t-1, r, b_t, b_{t-1})\\
&= \mathbb{E}_{y_{t-1}}[(r+1)b_t + r|\max\{y_{t-1}, b_{t-1}\} - b_{t-1}| + \max\{y_{t-1}, b_{t-1}\}\\
&\qquad + OptCost(t-2, r, \max\{y_{t-1}, b_{t-1}\})]\\
&> \mathbb{E}_{y_{t-1}}[r\max\{y_{t-1}, b_{t-1}\} + \max\{y_{t-1}, b_{t-1}\} + Opt(t-2, r, \max\{y_{t-1}, b_{t-1}\})]\\
&= ExpCost(t-1, r, 0, b_{t-1})
\end{aligned}$$

where the inequality is because for any $x \ge 0$ and $b_t > 0$,

$$(r+1)b_t + r|x - b_t| \ge (r+1)b_t + r(x - b_t) > rx.$$

The last term is exactly the expected total cost by reporting 0 in round t but adopting the same strategy with the optimal one afterward, which is a contradiction with $b_t > 0$ being optimal. Thus we complete the proof of the lemma. □

Next, we prove that in each round, the optimal strategy is determined by a penalty threshold such that a player will be truthful if and only if the penalty rate r is above the threshold. We call them *critical thresholds*.

Lemma 3.2 *(Critical Thresholds). For $t = T$, there is a threshold penalty rate $r_T^{(\emptyset)} \ge 0$ such that reporting D is optimal if and only if the penalty rate is at least $r_T^{(\emptyset)}$; For $t < T$, there is a threshold penalty rate $r_t^{(b_{t+1})} \ge 0$ such that reporting D is optimal for a player in round t with history b_{t+1} if and only if the penalty rate is at least $r_t^{(b_{t+1})}$.*

Proof. Note that by Lemma 3.1, b_{t+1} can only be 0 or D. Moreover, $r_t^{(\emptyset)} = \infty$ for any $t < T$. Therefore, we only need to show the existence of $r_T^{(\emptyset)}$ and $r_t^{(D)}$ for $t < T$. It suffices to show the following claim: For any round $T \ge t \ge 1$ with $y_t = 0$, if the optimal strategy is $b_t = 0$ given penalty rate r, then $b_t = 0$ is also optimal for any $r' \le r$; if the optimal strategy is $b_t = D$ given penalty rate r, then $b_t = D$ is also optimal for any $r' \ge r$.

To prove the claim, we use induction on t. When $t = 1$, it is easy to see that $r_1^{(D)}$ exists and is equal to 1. Consider $t > 1$ rounds left and the optimal strategy is to report 0 given penalty r and $b_{t+1} = D$ (or no history if $t = T$). If we increase penalty r to $r' > r$, by induction, the optimal strategy for future rounds either remains the same or switches to D from 0. Given history D and report D, the payment for this round is independent of the penalty rate r. Therefore, reporting D is still optimal. A similar argument can be made for reporting 0. □

Lemmas 3.1 and 3.2 together imply that the optimal strategy can only be one or a combination of the basic strategies. In particular, by Lemma 3.1, $r_t^{(0)} = \infty$ for any t. Moreover, since b_{t+1} can only be 0 or D, by Lemma 3.2, we only need to determine the values of $r_T^{(\emptyset)}$ and $r_t^{(D)}$ for $t < T$ to complete the picture of optimal strategies. In the following two lemmas, we give some properties of these thresholds.

Lemma 3.3. *$r_t^{(\emptyset)} \ge r_t^{(D)}$ for $t \in \{1, \dots, T\}$.*

Proof. Given the same t rounds left, it is straightforward to see that players with a truthful history are more incentivized to lie compared to when she has no history. This is because she needs to pay an additional payment of rD whenever she has a truthful history. Mathematically, let $r \ge r_t^{(\emptyset)}$. Then we have

$$\begin{aligned}
&ExpCost(t, r, D) \le ExpCost(t, r, 0)\\
\implies &D + OptCost(t-1, r, D) \le p(D + OptCost(t-1, r, D)) + (1-p)\cdot OptCost(t-1, r, 0)\\
\implies &D + OptCost(t-1, r, D) \le OptCost(t-1, r, 0)
\end{aligned}$$

$$\implies D + OptCost(t-1, r, D) \le rD + OptCost(t-1, r, 0)$$
$$\implies ExpCost(t, r, D, D) \le ExpCost(t, r, D, 0)$$

The above inequalities show that when $r \ge r_t^{(\emptyset)}$, the player prefers truth-telling to lying when she has a truthful history, which implies $r_t^{(D)} \le r_t^{(\emptyset)}$. □

Given the same t rounds left, Lemma 3.3 says a player is more inclined to lie without a history than with a truthful history. This is straightforward as lying with a truthful history results in an additional penalty payment.

Lemma 3.4. *Given $r_t^{(\emptyset)} \ge \frac{1}{p}$, $r_t^{(\emptyset)}$ decreases as t increases.*

Proof. An equivalent statement of Lemma 3.4 is that given a penalty $r \ge \frac{1}{p}$, if a player is truthful when there are T rounds left, then she is also truthful when there are $T+1$ rounds left. Let us prove the alternate statement.

Assume $r \ge \frac{1}{p}$ and that the player is truthful when there are T rounds left. Let $OPT(*, T)$ denote the optimal cost for a player if she reports $*$ in the current round and there are T rounds left. Since the player is truthful when there are T rounds left, we have

$$D + OPT(D, T-1) \le OPT(0, T-1).$$

Now assume there are $T+1$ rounds left and the player is free to lie in the first round. By Lemma 3.1, there are the following two cases.

- The player reports D in the first round

If the player also reports D in the second round, she pays $2D + OPT(D, T-1)$. Otherwise she pays $D + rD + OPT(0, T-1) \ge 2D + rD + OPT(D, T-1)$, which is dominated by reporting D for both rounds.

- The player reports 0 in the first round

Then the player's total expected payment is

$$\begin{aligned}&p(D + rD + OPT(D, T-1)) + (1-p)OPT(0, T-1)\\ &\ge D + prD + OPT(D, T-1) \ge 2D + OPT(D, T-1).\end{aligned}$$

Therefore, the optimal strategy is to report D in the first two rounds and the rest of the game is exactly the same as when there are T rounds left. □

Lemmas 3.3 and 3.4 together tell us the player is least incentivized to be truthful on the first round and $r_T^{(\emptyset)}$ is the penalty threshold that ensures truthfulness for the game. We give this important threshold in Theorem 3.1.

Theorem 3.1. *The minimum penalty for truthful reporting in a game of T rounds with $Ber(p)$ distribution is*

$$r_T^{(\emptyset)} = \frac{1-(1-p)^T}{p - p(1-p)^{T-1}}. \tag{3.1}$$

Proof. By the definitions of the thresholds, if the penalty $r \ge r_T^{(\emptyset)}$ and $r \ge r_t^{(1)}$ for any $t \le T$, then the player will be truthful. By Lemma 3.3 and 3.4, $r_T^{(\emptyset)} \ge r_t^{(\emptyset)} \ge r_t^{(1)}$, for any $t \le T$. Therefore, it is only necessary to compute $r_T^{(\emptyset)}$. By Lemma 3.1 and 3.3, it is sufficient to compare lying-till-busted and honest-till-end in the first segment:

$$\begin{aligned}\mathbb{E}[\text{honest}] &= D \cdot \mathbb{E}[\#\text{ days before busted}]\\ &= D + D\left\{\sum_{i=0}^{T-2} i(1-p)^i p + (T-1)(1-p)^{T-1}\right\}\\ &= D + D \cdot \tfrac{1-p}{p}(1-(1-p)^{T-1});\\ \mathbb{E}[\text{lying}] &= rD \cdot \Pr(\text{busted}) = rD(1-(1-p)^{T-1}).\end{aligned}$$

The optimal threshold can be obtained via setting these two expected costs equal,

$$r_T^{(\emptyset)} = \frac{D + D\frac{1-p}{p}(1-(1-p^{T-1}))}{D(1-(1-p)^{T-1})} = \frac{1-(1-p)^T}{p-p(1-p)^{T-1}}. \quad \square$$

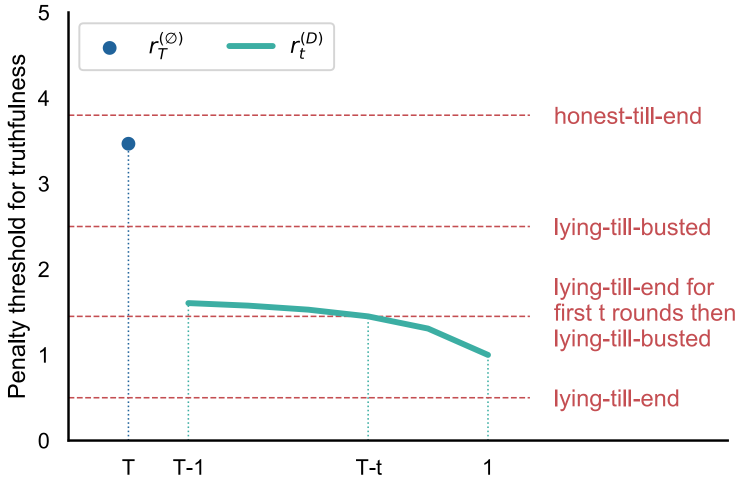

Fig. 3.1. Critical thresholds for each round under $Ber(0.3)$ distribution with examples of optimal strategies.

We see $r_T^{(\emptyset)} \to 1/p$ as $T \to \infty$ and $r_T^{(\emptyset)}$ decreases as T increases. This implies the increasing length of the game incentivizes the player to speak the truth today, even when they do not have to. To understand Theorem 3.1, we observe that it is sufficient to compare lying-till-busted and honest-till-end since $r_T^{(\emptyset)}$ ensures the player to stay truthful after being busted. Before the player is busted for the first time, it is not optimal to oscillate between lying and truth-telling, as it is strictly dominated by lying completely. Therefore, the only viable strategies are lying-till-busted and honest-till-end, and the desired threshold sets the expected cost of these two strategies equal.

With a more involved argument, we get the exact values for the truthful threshold given a truthful history, i.e., the $r_t^{(D)}$'s. The values of $r_T^{(\emptyset)}$ and $r_t^{(D)}$ characterize the optimal strategies for a player and are an alternative representation of Table 1.

Theorem 3.2. *For $p \le \frac{1}{2}$, $r_t^{(D)} = \frac{1-(1-p)^t}{2p-p(1-p)^{t-1}}$. For $p > \frac{1}{2}$, $r_t^{(D)} = 1$ for $t = 1$ and $r_t^{(D)} = \frac{1}{2p}$ for $t \ge 2$.*

Proof. We first show the proof for $p \le \frac{1}{2}$ by induction on t. Let $h(t) = \frac{1-(1-p)^t}{2p-p(1-p)^{t-1}}$. Assume there are t rounds left. Note that $h(t)$ increases in t, which means that if $r \ge h(t)$, then $r \ge h(t')$ for $t' \le t$, i.e., the player stays truthful for the rest of the t rounds. Similar to the argument in Theorem 3.1, we compare the expected payments of the two strategies, namely lying-till-busted ("lying") and being honest, within a segment. Note that the segment now starts with being busted, because the player has a truthful history.

$$\mathbb{E}[\text{honest}] = D \cdot \mathbb{E}[\#\text{ days before busted}] = D + D \cdot \tfrac{1-p}{p}(1-(1-p)^{t-1});$$
$$\mathbb{E}[\text{lying}] = rD + rD \cdot \Pr(\text{busted}) = rD(2-(1-p)^{t-1}).$$

The penalty that results in truthfulness sets these two payments equal, i.e. $r = \frac{1-(1-p)^t}{2p-p(1-p)^{t-1}} = h(t)$.

The proof for $p \ge \frac{1}{2}$ is slightly different. First note that for $t = 1$, it is not hard to see the threshold $r_1^{(D)} = 1$ by comparing the cost of being honest (i.e., D) and the cost of lying (i.e., rD). For $t > 1$ rounds left, we apply the same argument above, with the consideration that the player will switch to lying in the very last round if she is allowed to. Therefore, we have

$$\begin{aligned}\mathbb{E}[\text{honest}] &= D \cdot \mathbb{E}[\#\text{ days before busted}] - (1-r)D \cdot \Pr(\text{not busted in the last day}) \\ &= D + D \cdot \tfrac{1-p}{p}(1-(1-p)^{t-1}) - (1-r)D(1-p)^{t-1};\end{aligned}$$
$$\mathbb{E}[\text{lying}] = rD + rD \cdot \Pr(\text{busted}) = rD(2-(1-p)^{t-1}).$$

The penalty that sets the above two expected costs equal is $\frac{1}{2p}$. $\square$

The optimal strategy is visualized in Figs. 3.1 and 3.2 for $p = 0.3$ and $p = 0.7$, respectively. The x-axis is the number of rounds left (t), and the y-axis is the penalty threshold for truthfulness. We give examples of penalties via the red dashed lines. For the first round, the player refers to the blue dot representing $r_T^{(\emptyset)}$ and is truthful if and only if the penalty is above the blue dot. Afterward, given t rounds left and history D, the player looks at the green curve representing $r_t^{(D)}$ and is only truthful if the penalty is above the curve. If the history is 0, she remains untruthful and reports 0. Figs. 3.1 and 3.2 visualize the optimal strategies given in Table 1. Both green curves are closely related to $\frac{1}{2p}$. An intuition is that in any round $t < T$, a player pays D if she is truthful and roughly

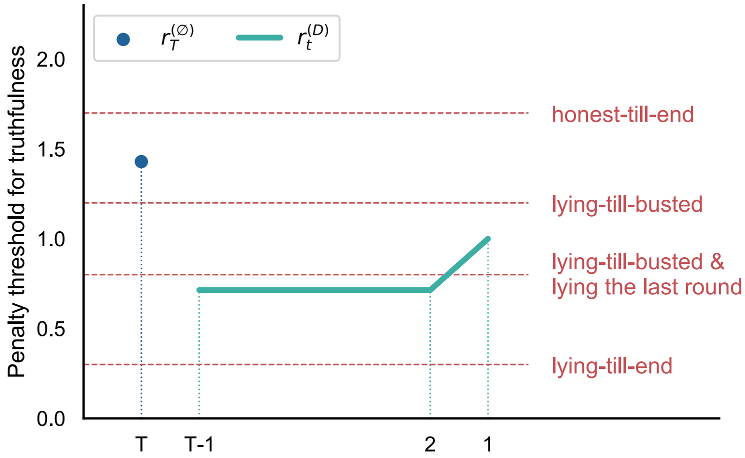

Fig. 3.2. Critical thresholds for each round under $Ber(0.7)$ distribution with examples of optimal strategies.

$2prD$ if she lies, where the penalty payment rD comes from the previous and the next round, each with probability p. The penalty that sets these two costs equal is $\frac{1}{2p}$. The actual $r_t^{(D)}$ thresholds vary upon values of t and p.

4. A reduction for arbitrary distributions

As discussed in the introduction, only the infinite penalty rate will guarantee complete truthfulness under arbitrary distributions. We use the uniform distribution as an example and let the net consumption y_t be drawn from a uniform distribution with support $[0, D]$. As a property of continuous distributions, the probability that y_t attains a specific value in $[0, D]$ is zero because there are an infinite number of values that y_t can assume. This implies that there is no chance the player would ever be forced to report their true gross consumption D. The only way to enforce truthful reporting is via an infinite penalty rate r. To see this, we consider a player who reported $b_t < D$ for some round t. There is a non-zero probability that they will receive a net consumption $y_{t'} > b_t$ at some future round t'. Moreover, they will be forced to report at least $y_{t'}$ and pay an infinite penalty. A superior strategy is to be truthful and report D for all rounds. Since y_t can never exceed D, there is zero probability that this strategy would ever yield an infinite payment.

Although there is no hope in achieving complete truthfulness, we can still work on obtaining approximate results. The trick is to redefine being busted as having a partial signal that is less than α times the true consumption, for $\alpha \in [0, 1]$. Then any arbitrary distribution is reduced to $Ber(p)$ where p is the probability that the partial signal is at least αD. We address that the value of α is chosen by the center based on their risk tolerance. Intuitively, there is a trade-off between α and the value of the penalty threshold. A more stringent center would choose a higher value of α which leads to a reporting profile closer to complete truthfulness but a higher penalty threshold, which can be less desirable. The parameter α provides the center with the flexibility to balance truthfulness and the penalty rate.

For approximate truthfulness, we define being α-truthful as reporting at least αD. We reuse the arguments of comparing basic strategies from Section 3 to determine an upper bound for the penalty rate that guarantees α-truthfulness. We introduce the notion of approximate truthfulness in Definition 4.1 and give the reduction in Theorem 4.1. We demonstrate the reduction with uniform distributions in Example 4.1.

Definition 4.1 *(α-truthfulness).* A reporting $\boldsymbol{b}$ is α-truthful when $b_t \geq \alpha D$ for all $t = 1, \ldots, T$.

Theorem 4.1. *Given $\alpha \in [0, 1]$ and an arbitrary distribution with CDF F, if a penalty rate r achieves complete truthfulness for $Ber(p)$ where $p = 1 - F(\alpha D)$, then the same r achieves α-approximate truthfulness for distribution F.*[3]

Proof. Recall that being "busted" means the player has a D realization. For general distributions, given $\alpha \in [0, 1]$, we redefine being busted as having a realization of at least αD. Then the probability of being busted is $1 - F(\alpha D)$. The proof is essentially the same as that of Theorem 3.1 for $p = 1 - F(\alpha D)$. We analyze the segment between the first day and the day when the player is busted. Assume the minimum report from the player during the segment is βD, $\beta < \alpha$. We compare the savings the player gets from using this strategy versus reporting αD and the corresponding additional penalty that she needs to pay.

[3] The reduction depends on the players' gross consumption, which is private information. In reality, the center can estimate the value of D using historical data. Moreover, if the center has some information on the upper bounds of D, we are still able to set a penalty rate (which may not be minimum) to obtain truthfulness.

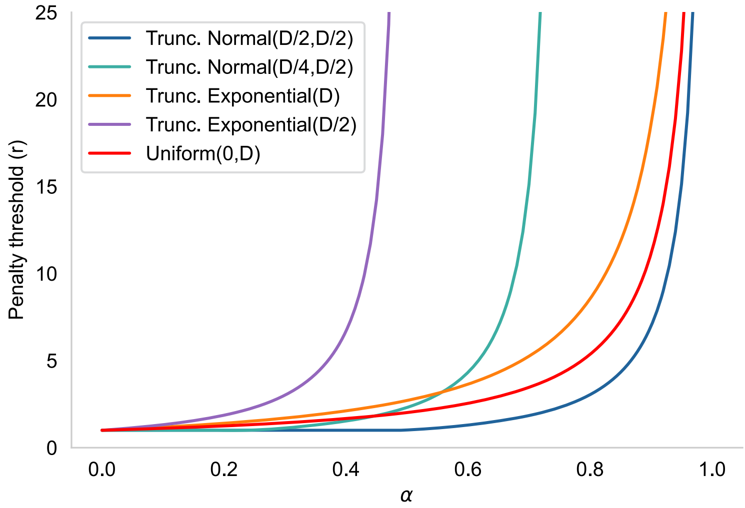

Fig. 4.1. Penalty thresholds for α-approximate truthfulness, assuming $D = 1$.

$$\mathbb{E}[\text{savings}] \le (\alpha - \beta)D \cdot \mathbb{E}[\#\text{ days before busted}]$$

$$\mathbb{E}[\text{penalty}] \ge rD(\alpha - \beta) \cdot \Pr(\text{busted})$$

The player will report αD every round in the segment when the expected penalty exceeds the expected savings. The $(\alpha - \beta)$ term is canceled and the rest calculation is the same as the $Ber(p)$ case where $p = 1 - F(\alpha D)$. □

Example 4.1. Assume partial signals follow a uniform distribution $U(0, D)$. Let r be the truthful threshold of $Ber(p)$ where $p = 1 - \alpha$, i.e. $r = \frac{1-\alpha^T}{(1-\alpha)(1-\alpha^{T-1})}$. Then using r ensures α-truthfulness for $U(0, D)$ by Theorem 4.1. For uniform distributions, it is impossible to obtain complete truthfulness unless $r = \infty$, which can be verified by setting $\alpha = 1$. We plot the penalty thresholds that guarantee approximate truthfulness for $D = 1$ and different α's in Fig. 4.1 for uniform distributions as well as other common distributions like truncated normal and truncated exponential. We see that for some distributions, it is not possible to attain α-truthfulness when α exceeds a certain value.

5. Extension: a cost sharing model

If we consider the case when the players are charged independently, we can simply apply the flux mechanism to each individual player and elicit truthful reports. However, in a more general setting, the players split an overhead cost based on their individual usage of the service. Therefore, we extend the problem to a cost-sharing model for homogeneous players. We formally state the problem in Section 5.1 and give the exact equilibrium results in Section 5.2 for Bernoulli-distributed gross consumption. The proofs for the main theorems are presented in Section 5.3. In Section 5.4, we extend our results for arbitrary distributions.

5.1. Problem statement

Let N be the set of players with $n = |N| \ge 1$. Each player $i \in N$ has a private value $x^i \ge 0$, and we assume all players are symmetric, i.e., $x^i = D$ for all $i \in N$ (see Appendix 2.1 for a relaxation). All players in N split an overhead cost C, which is at least the total gross consumption, i.e., $C \ge nD$. The game has T rounds in total. Given the penalty rate r, we analyze the following multi-player flux mechanism.

- The center observes a partial signal representing player i's net consumption $y_t^i \sim F$ for each player $i \in N$;
- Each player i submits their reported gross consumption that is at least their net consumption, $b_t^i \ge y_t^i$;
- If $t < T$, player i's pays regular payment $C \cdot \frac{b_t^i}{\sum_j b_t^j}$ and penalty payment $r \cdot |b_{t+1}^i - b_t^i|$. If $t = T$, the players only pay regular payments.

We call b_{t+1}^i the *history* for player i in round t and $\boldsymbol{b}_{t+1}$ the *group history*. If everyone lies in a round, the overhead cost is split evenly among all players. A mechanism is *truthful* if every player reports D for every round. We are interested in computing the minimum penalty rates such that truthful reports form a Nash equilibrium (NE) or a dominant strategy equilibrium (DSE). Informally, a strategy profile is an NE if no player wants to unilaterally deviate, and it is a DSE if no player wants to deviate no matter what the other players do. We show that approximate results for any arbitrary distribution can be deducted from an exact analysis for a Bernoulli distribution.

Let $\boldsymbol{\sigma}^i = \{\sigma_1^i, \sigma_2^i, \dots, \sigma_T^i\} \in \boldsymbol{\Sigma}^i$ be a strategy profile for player i for a game with T rounds where $\boldsymbol{\Sigma}^i$ is the space of all strategy profiles for player i. The strategy profile $\boldsymbol{\sigma}^i$ satisfies the following:

- For $t < T$, the strategy profile takes in the history report b_{t+1}^i as well as the realization of the current round y_t^i, and returns the report for the current round, i.e., $b_t^i = \sigma_t^i(b_{t+1}, y_t^i)$;
- For $t = T$, since there is no history, the strategy profile takes in just the realization and returns the report $\sigma_T^i(y_T^i)$.

Fix an arbitrary player i and the other players' strategy $\boldsymbol{\sigma}^{-i}$. Let $\boldsymbol{b}_t^{-i} = \boldsymbol{\sigma}_t^{-i}(\boldsymbol{b}_{t+1}, \boldsymbol{y}_t^{-i})$ denote the reported gross consumption by players $j \neq i$ with group history $\boldsymbol{b}_{t+1}$ and realizations $\boldsymbol{y}_t^{-i}$ in round t. A strategy $\boldsymbol{\sigma}_i$ is called $\boldsymbol{\sigma}^{-i}$'s *best response* if it is the solution of the following recursion, i.e.,

$$\sigma_t^i(b_{t+1}^i, y_t^i) = \arg\min_{b_t^i} Cost(t, r, \boldsymbol{\sigma}^{-i}, \boldsymbol{b}_{t+1}, b_t^i),$$

where the expected cost can be expanded as

$$Cost(t, r, \boldsymbol{\sigma}^{-i}, \boldsymbol{b}_{t+1}, b_t^i) = \frac{C \cdot b_t^i}{\sum_j b_t^j} + r \cdot | b_t^i - b_{t+1}^i | + \mathbb{E}_{\boldsymbol{y}_{t-1}}\left[Cost\left(t-1, r, \boldsymbol{\sigma}^{-i}, \boldsymbol{b}_t, \sigma_{t-1}^i(b_t^i, y_{t-1}^i)\right)\right].$$

The payoff of player i is defined as the expected cost from the first round of the game, i.e.,

$$\mathbb{E}_{\boldsymbol{y}_T}\left[Cost(T, r, \boldsymbol{\sigma}^{-i}, \sigma_T^i(y_T^i))\right].$$

Given a strategy profile $\boldsymbol{\sigma}$, if $\boldsymbol{\sigma}^i$ is a best response to $\boldsymbol{\sigma}^{-i}$ for every player i, then $\boldsymbol{\sigma}$ is called a Nash equilibrium. Mathematically, $\boldsymbol{\sigma}$ is an NE if

$$\mathbb{E}_{\boldsymbol{y}_T}\left[Cost(T, r, \boldsymbol{\sigma}^{-i}, \sigma_T^i(y_T^i))\right] \geq \mathbb{E}_{\boldsymbol{y}_T}\left[Cost(T, r, \boldsymbol{\sigma}^{-i}, \hat{\sigma}_T^i(y_T^i))\right], \text{ for any } \hat{\boldsymbol{\sigma}}^i \in \boldsymbol{\Sigma}^i.$$

If $\boldsymbol{\sigma}_i$ is a best response to any $\boldsymbol{\sigma}'^{-i}$ (not necessarily $\boldsymbol{\sigma}^{-i}$) for any player i, $\boldsymbol{\sigma}_i$ is then called a dominant strategy equilibrium. Mathematically, $\boldsymbol{\sigma}$ is a DSE if

$$\mathbb{E}_{\boldsymbol{y}_T}\left[Cost(T, r, \boldsymbol{\sigma}^{-i}, \sigma_T^i(y_T^i))\right] \geq \mathbb{E}_{\boldsymbol{y}_T}\left[Cost(T, r, \boldsymbol{\sigma}^{-i}, \hat{\sigma}_T^i(y_T^i))\right], \text{ for any } \hat{\boldsymbol{\sigma}}^i \in \boldsymbol{\Sigma}^i, \boldsymbol{\sigma}^{-i} \in \boldsymbol{\Sigma}^{-i}.$$

5.2. Exact results

Similar to the single-player setting, we avoid solving the recursion by exploiting the properties of the mechanism. Again, we start our analysis with F being a Bernoulli distribution and provide a reduction for approximate truthfulness when F is an arbitrary distribution. In the single-player model with Bernoulli-distributed F, we have shown that it is only optimal for a player to report 0 or her actual consumption D. We claim it is the same case for multiple players. Moreover, if a player lied yesterday and also has an observed consumption of 0 today, they will report 0 regardless of other players' actions.

Lemma 5.1. *For Bernoulli-distributed F, reporting anything strictly between 0 and D is sub-optimal in a multi-player flux mechanism. Moreover, if $b_{t+1}^i = y_t^i = 0$, it is optimal to report $b_t^i = 0$.*

Proof. We can use a similar argument in the proof for Lemma 3.1 to prove that if a player lied yesterday, it is better off lying today. We consider the last round t in the optimal strategy such that when $(b_{t+1}^i, y_t^i) = (0, 0)$ but $b_t^i > 0$. It is trivially true if t is the last round, and thus we assume $t > 1$. By reporting b_t^i in round t, the expected total cost afterward is

$$\begin{aligned}
&rb_t^i + \frac{C \cdot b_t^i}{\sum_j b_t^j} + \mathbb{E}_{y_{t-1}}\left[Cost(t-1, r, \boldsymbol{\sigma}^{-i}, \boldsymbol{b}_t, \sigma_{t-1}^i(b_t^i, y_{t-1}^i))\right] \\
&= \left(r + \frac{C}{\sum_j b_t^j}\right) b_t^i + \mathbb{E}_{y_{t-1}}\left\{ r \cdot \left|\sigma_{t-1}^i(b_t^i, y_{t-1}^i) - b_t^i\right| + \frac{C \cdot \sigma_{t-1}^i(b_t^i, y_{t-1}^i)}{\sum_{j \neq i} \sigma_t^j(\boldsymbol{b}_t, \boldsymbol{y}_{t-1}^{-j}) + \sigma_{t-1}^i(b_t^i, y_{t-1}^i)} \right. \\
&\qquad \left. + \mathbb{E}_{y_{t-2}}\left[Cost\left(t-2, r, \boldsymbol{\sigma}^{-i}, \boldsymbol{\sigma}(\boldsymbol{b}_t, \boldsymbol{y}_{t-1}), \sigma_{t-2}^i(\sigma_{t-1}^i(b_t^i, y_{t-1}^i), t_{t-2}^i)\right)\right] \right\} \\
&> \mathbb{E}_{y_{t-1}}\left\{ r \cdot \sigma_{t-1}^i(b_t^i, y_{t-1}^i) + \frac{C \cdot \sigma_{t-1}^i(b_t^i, y_{t-1}^i)}{\sum_{j \neq i} \sigma_t^j(\boldsymbol{b}_t, \boldsymbol{y}_{t-1}^{-j}) + \sigma_{t-1}^i(b_t^i, y_{t-1}^i)} \right.
\end{aligned}$$

Table 2
Expected payment in every round for each player in the multi-player model with $n = 2$.

		Player 2	
		Honest	Lying
Player 1	Honest	$(C/2, C/2)$	$(C, 0)$
	Lying	$(0, C)$	$(C/2, C/2)$

$$+\mathbb{E}_{y_{t-2}}\left[Cost\left(t-2, r, \boldsymbol{\sigma}^{-i}, \boldsymbol{\sigma}(\boldsymbol{b}_t, \boldsymbol{y}_{t-1}), \sigma^i_{t-2}(\sigma^i_{t-1}(b^i_t, y^i_{t-1}), t^i_{t-2})\right)\right]\Bigg\}$$

$$= \mathbb{E}_{y_{t-1}}\left[Cost(t-1, r, \boldsymbol{\sigma}^{-i}, (\boldsymbol{b}^{-i}_t, 0), \sigma^i_{t-1}(b^i_t, y^i_{t-1}))\right],$$

which is the expected total cost by reporting 0 in round t but adopting the same strategy with the optimal one afterward. This contradicts that $b^i_t > 0$ is optimal.

To see that partial reporting is optimal, rewrite the payment for the current round as

$$C \cdot \left(1 - \frac{\sum_{j\neq i} b^i_t}{\sum_{j\neq i} b^j_t + b^i_t}\right) + r \mid b^i_{t+1} - b^i_t \mid,$$

whose second derivative is negative with respect to b^i_t. This means that the payment function is concave in b^i_t and will take minimum at either of the endpoints 0 and D. □

Starting from this point, we assume that every player reports either 0 or D. When $n = 2$, we show that the multi-player model reduces to the single-player model with a multiplicative factor of $\frac{C}{2D}$. The reason for the reduction is that the savings of switching to lying from being truthful for a player is always $\frac{C}{2}$, regardless of what the other player does.

Lemma 5.2. *When $n = 2$, the multi-player model reduces to a single-player model. The truthful penalty threshold is $\frac{C}{2D}$ times (3.1).*

Proof. In the single-player model, if a player switches to lying from being honest, she saves D for regular payment and then pays a penalty of rD if she has a truthful history. Now in the two-player model, since players are symmetric, we fix the action of player 2 and see what happens with player 1 (see Table 2).

No matter if player 2 is honest or lying, for player 1, switching to lying would save $C/2$ and may cost a penalty payment of rD. By applying the same argument seen in Section 3.1 with the new expected savings and penalties, we get the same penalty threshold, except with a $C/2D$ multiplicative factor. □

For general n, we show it is sufficient to analyze the maximum difference between lying and truth-telling for player i in round t given group history $\boldsymbol{b}_{t+1}$. In a DSE, a player achieves the biggest gain from lying if all players were lying in the previous round. We then use $\boldsymbol{b}_{t+1} = \boldsymbol{0}$ to compare lying and truth-telling for a player.

Theorem 5.1. *For the $Ber(p)$ distribution, a truthful strategy profile forms a dominant strategy equilibrium if and only if*

$$r \geq \frac{C}{nD}\frac{1-(1-p)^{n-1}}{p}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}. \tag{5.1}$$

Proof. See Section 5.3. □

If we slowly lower the penalty from (5.1), we will hit a threshold such that truth-telling is an NE. The difference between the truthful NE and the DSE is that now we can assume that every player $j \neq i$ is truthful in the first round and show that player i would not deviate unilaterally. However, we shall not assume that player $j \neq i$ remains truthful for the rest of the game. This is because if player i lies in the first round, player j can observe the report of i in the second round and deviate from truthful behavior. We first show that if $r \geq \frac{C}{nD}\frac{1}{p}$, players with truthful history stay truthful. Then we can safely assume player $j \neq i$ remains truthful throughout the game. In this way, truthful NE is reduced to the case where there is one strategic player and $n-1$ truthful players. It is not hard to see the threshold is precisely $\frac{C}{nD}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}$.

Theorem 5.2. *For the $Ber(p)$ distribution, a truthful strategy profile forms a Nash equilibrium if and only if*

$$r \geq \frac{C}{nD}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}. \tag{5.2}$$

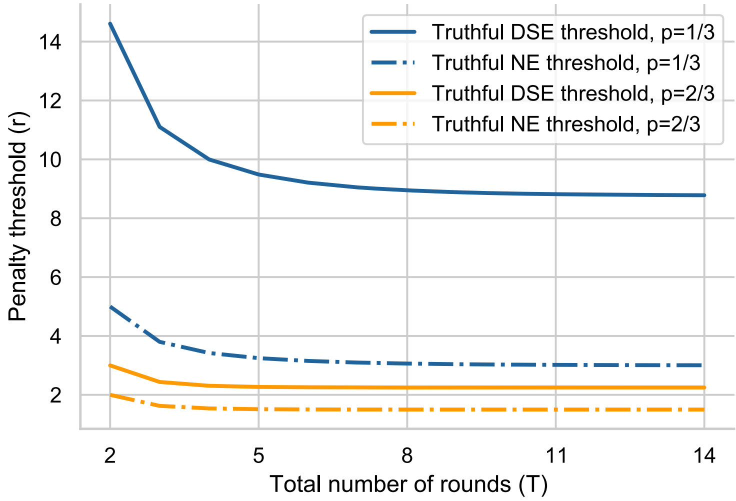

Fig. 5.1. Exact penalty thresholds for truthful DSE and NE, given a total number of rounds T for $Ber(p)$ distributions. We assume $n = 20$, $D = 1$ and $C = n \cdot D = 20$.

Proof. See Section 5.3. □

We visualize $Ber(p)$ penalty thresholds in Fig. 5.1 for different T's and p's. The x-axis is the total number of rounds for a game and the y-axis is the penalty rate that guarantees the specified equilibrium. The blue and orange lines are penalty thresholds for $p = \frac{1}{3}$ and $\frac{2}{3}$, respectively. The solid and dashed lines are thresholds for truthful DSE and NE, respectively. All four thresholds in Fig. 5.1 decrease as T increases, suggesting that the increasing length of the game promotes truthful equilibria. From expressions (5.1) and (5.2), we see that the DSE and NE thresholds tend to be the same as p approaches 1.

5.3. Proofs for the exact results

For general n strategic players, we develop an alternative way to compute the penalty thresholds for NE and DSE. Interestingly, we only need to make use of the following important definition, $\Delta EC_t^i(\boldsymbol{b}_{t+1})$, to derive a universal framework for equilibrium proofs.

Definition 5.1. Let $EC_t^i(\boldsymbol{b}_{t+1})$ denote the expected cost for player i with when there are t rounds left and the group history is $\boldsymbol{b}_{t+1}$. Define

$$\Delta EC_t^i(\boldsymbol{b}_{t+1}^{-i}) \triangleq EC_t^i(b_{t+1}^i = D, \boldsymbol{b}_{t+1}^{-i}) - EC_t^i(b_{t+1}^i = 0, \boldsymbol{b}_{t+1}^{-i})$$

as the difference in the expected payments by reporting D versus 0 for player i, given t rounds left and the reports of other players, $\boldsymbol{b}_{t+1}^{-i}$.

To simplify the notation, we remove the superscript i in the definition and write $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$. By Lemma 5.1, $\boldsymbol{b}_{t+1}^{-i}$ is a string of size $n-1$ consisting of 0's and D's. We first present a technique to obtain upper bounds of $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ given $\boldsymbol{b}_{t+1}^{-i}$.

Lemma 5.3. *Some upper bounds of $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$:*

(i) $\Delta EC_t(\boldsymbol{0})$

$$\leq \frac{C}{n}\frac{1-(1-p)^{n-1}}{p}\sum_{i=1}^{t}(1-p)^i - prD\sum_{i=0}^{t-1}(1-p)^i$$

(ii) $\Delta EC_t(b_{t+1}^j = 0, \boldsymbol{b}_{t+1}^{-i,j} = \boldsymbol{D})$

$$\leq \frac{C}{n-1}\sum_{i=1}^{t}(1-p)^i - prD\sum_{i=0}^{t-1}(1-p)^i$$

Proof. We prove (i) where $\boldsymbol{b}_{t+1}^{-i} = \boldsymbol{0}$ and the proof for (ii) is similar. Let $M = \frac{C}{n}\frac{1-(1-p)^{n-1}}{p}$. We prove this by induction.
Base case. $t = 1$. With probability p, having a D or 0 history pays the same regular payment and the 0 history needs to pay a penalty. With probability $1-p$, only the D history pays the regular payment.

$$\Delta EC_1 = EC_1(D) - EC_1(0) = (1-p)\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}\frac{C}{k+1} - (1-p)^n\frac{C}{n} - prD$$
$$= (1-p)\frac{C}{n}\frac{1-(1-p)^{n-1}}{p} - prD$$
$$= (1-p)M - prD.$$

Note that k in the second equality represents the number of players being busted in $N \setminus \{i\}$.
Induction step. Assume Lemma 5.3 is true for ΔEC_t. Consider $t+1$ rounds left.

$$\begin{aligned}\Delta EC_{t+1} &= EC_{t+1}(D) - EC_{t+1}(0)\\ &= (1-p)\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}\left\{\frac{C}{k+1} + EC_t(D)\right\}\\ &\quad - (1-p)\sum_{k=0}^{n-1}\binom{n-1}{k}p^k(1-p)^{n-1-k}\cdot EC_t(0)\\ &\quad - prD - (1-p)^n\frac{C}{n}\\ &\le (1-p)\left\{M + \Delta EC_t\right\} - prD\\ &\le M\sum_{i=1}^{t+1}(1-p)^i - prD\sum_{i=0}^{t}(1-p)^i. \quad \square\end{aligned}$$

An important property of $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ is that it is monotone increasing as the number of 0's in $\boldsymbol{b}_{t+1}^{-i}$ increases. One way to understand this property is that a player $j \neq i$ with a zero history is more likely to lie in the next rounds, which in turn increases the expected regular payment if player i is truthful. We prove this property mathematically in Lemma 5.4.

Lemma 5.4. *If $\hat{\boldsymbol{b}}_{t+1}^{-i}$ contains more zeros than $\boldsymbol{b}_{t+1}^{-i}$, then*

$$\Delta EC_t(\boldsymbol{b}_{t+1}^{-i}) \le \Delta EC_t(\hat{\boldsymbol{b}}_{t+1}^{-i}).$$

Proof. First note that the only non-trivial case is when the penalty is just high enough such that players with truthful history stay truthful and players with 0 history lie whenever realization is 0. Since every player is symmetric, players with the same history will act the same. If the penalty is too low, $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ does not depend on $\boldsymbol{b}_{t+1}^{-i}$ and $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i}) - \Delta EC_t(\hat{\boldsymbol{b}}_{t+1}^{-i}) = 0$. Same when the penalty is too high then players will be truthful regardless of history. Now we can assume players with truthful history stay truthful regardless of the realization and players with zero history lie whenever possible. We prove by induction on t.
Base case. $t = 1$. Let $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ contain k zero's (and $n-k-1$ D's). Then we have

$$\Delta EC_1(\boldsymbol{b}_{t+1}^{-i}) = p(-rD) + (1-p)\sum_{i=0}^{k}\binom{k}{i}p^i(1-p)^{k-i}\frac{C}{n-k+i} - \mathbb{1}_{\{k=n-1\}}\cdot\frac{C}{n}(1-p)^n$$
$$= p(-rD) + (1-p)\sum_{j=0}^{n-1}\alpha(j,k)H(j) \quad \text{letting } j = k-i$$

where

$$\alpha(j,k) = \begin{cases}\binom{k}{k-j}p^{k-j}(1-p)^j & \text{for } 0 \le j \le k\\ 0 & \text{for } k < j \le n-1\end{cases}$$

and

$$H(j) = \begin{cases}\frac{1}{n-j}\cdot C & \text{for } 0 \le j < n-1\\ \frac{n-1}{n}\cdot C & \text{for } j = n-1\end{cases}$$

Note that $\sum_{j=0}^{n-1}\alpha(j,k) = 1$ and $\alpha(j,k)$'s depend on k. On the other hand, $H(j)$'s do not depend on k and is an increasing sequence in j. Now consider $\hat{\boldsymbol{b}}_{t+1}^{-i}$ that contains $\hat{k}$ zeros, and $k < \hat{k}$. Then we have

$$\Delta EC_1(\hat{\boldsymbol{b}}_{t+1}^{-i}) - \Delta EC_1(\boldsymbol{b}_{t+1}^{-i}) = (1-p)\sum_{j=0}^{n-1}\left\{\alpha(j,\hat{k}) - \alpha(j,k)\right\}H(j)$$

$$
\begin{aligned}
&= (1-p)\left\{\sum_{j=k+1}^{\hat{k}} \alpha(j,\hat{k})H(j) - \sum_{j=0}^{k}(\alpha(j,k)-\alpha(j,\hat{k}))H(j)\right\} \\
&\geq (1-p)\left\{\sum_{j=k+1}^{\hat{k}} \alpha(j,\hat{k})H(k) - \sum_{j=0}^{k}(\alpha(j,k)-\alpha(j,\hat{k}))H(k)\right\} \\
&= (1-p)H(k)\left\{\sum_{j=0}^{\hat{k}} \alpha(j,\hat{k}) - \sum_{j=0}^{k}\alpha(j,k)\right\} \\
&= 0
\end{aligned}
$$

Induction step. Assume the lemma is true for t. We prove for $t+1$ rounds left. Assume again $\boldsymbol{b}_{t+1}^{-i}$ contains k zero's.

$$
\begin{aligned}
\Delta EC_{t+1}(\boldsymbol{b}_{t+1}^{-i}) &= EC_{t+1}(D,\boldsymbol{b}_{t+1}^{-i}) - EC_{t+1}(0,\boldsymbol{b}_{t+1}^{-i}) \\
&= (1-p)\left\{\sum_{i=0}^{k}\binom{k}{i}p^i(1-p)^{k-i}\left(\frac{C}{n-k+i} + \Delta EC_t(k-i \text{ lying})\right)\right\} \\
&\quad - prD - \mathbb{1}_{\{k=n-1\}}(1-p)^n\left\{\frac{C}{n} - EC_t(0,0)\right\} \\
&= -prD + (1-p)\sum_{j=0}^{n-1}\alpha(j,k)H(j)
\end{aligned}
$$

where $\alpha(j,k)$'s are the same as earlier, and $H(j)$'s are now

$$
H(j) = \begin{cases} \frac{1}{n-j}\cdot C + \Delta EC_t(j \text{ lying}) & 0 \leq j < n-1 \\ \frac{n-1}{n}\cdot C + \Delta EC_t(n-1 \text{ lying}) & j = n-1 \end{cases}
$$

By induction, $\Delta EC_t(j \text{ lying})$ increases in j. Thus, $H(j)$'s is again an increasing sequence in j. We re-use the argument in the base case and prove $\Delta EC_{t+1}(\hat{\boldsymbol{b}}_{t+1}^{-i}) \geq \Delta EC_{t+1}(\boldsymbol{b}_{t+1}^{-i})$ for $\hat{\boldsymbol{b}}_{t+1}^{-i}$ with $\hat{k} > k$ zeros. □

With this property, we develop a framework for the equilibrium proofs of both DSE and NE:

1. Determine what $\boldsymbol{b}_{t+1}^{-i}$ look like based on the type of the equilibrium we are trying to compute;
2. Upper bound $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ with an expression using C, D, t, p and r (see Lemma 5.3);
3. Compare player i's expected payment on the first round when she lies or tells the truth using $\Delta EC_{T-1}(\boldsymbol{b}_T^{-i})$;
4. Find the penalty rate that sets the two expected payments equal, and that is the desired penalty threshold.

Proof for Theorem 5.1. Fix a player i. To show that being truthful is a dominant strategy for player i, we want to look at the situation that maximizes the difference between truth-telling and lying for player i, which is precisely when every other player is lying as much as possible, by Lemma 5.4. Now we assume every other player reports 0 whenever they can. We compare the expected cost of being truthful and lying in the very first round.

$$
\begin{aligned}
&\mathbb{E}[\text{lying}] = (1-p)^{n-1}\frac{C}{n} + EC_{T-1}(0,\boldsymbol{0}); \\
&\mathbb{E}[\text{honest}] = \sum_{k=0}^{n-1}\binom{n-1}{k}p^i(1-p)^{n-1-k}\left\{\frac{C}{k+1} + EC_{T-1}(D,\boldsymbol{0})\right\},
\end{aligned}
$$

where k represents the number of players in $N \setminus \{i\}$ that are busted in round T. We would like to find the penalty rate such that $\mathbb{E}[\text{honest}] - \mathbb{E}[\text{lying}] \leq 0$. By Lemma 5.3, we have

$$
\begin{aligned}
\mathbb{E}[\text{honest}] - \mathbb{E}[\text{lying}] &= \frac{C}{n}\frac{1-(1-p)^n}{p} - (1-p)^{n-1}\frac{C}{n} + \Delta EC_{T-1}(\boldsymbol{0}) \\
&\leq \frac{C}{n}\frac{1-(1-p)^{n-1}}{p}\frac{1-(1-p)^T}{p} - rD(1-(1-p)^{T-1}),
\end{aligned}
$$

which is negative when $r \geq \frac{C}{nD}\frac{1-(1-p)^{n-1}}{p}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}$. Since we are analyzing the case that maximizes the differences in lying and truth-telling, we can say that truthfulness is a Nash equilibrium if and only if the penalty rate is above the given threshold. □

Proof for Theorem 5.2. Based on the discussion, we first assume that every player $j \neq i$ is truthful in the first round and $r \geq \frac{C}{nD}\frac{1}{p}$. We want to prove that some player $j \in N \setminus \{i\}$ does not want to deviate from being truthful in the next round. Then it follows that the threshold for truthful NE is equivalent to the case with a single sophisticated player and $n-1$ truthful players. Since the threshold (5.2) is exact in the model with one sophisticated and $n-1$ truthful players, this threshold is the exact threshold for truthful Nash equilibrium.

Fix some player $j \neq i$. Assume there are $t+1$ rounds left. Again, we compare the expected payments of lying and being honest for player j.

$$\mathbb{E}[\text{honest}] = p\left\{\frac{C}{n} + EC_t(D, \boldsymbol{D})\right\} + (1-p)\left\{\frac{C}{n-1} + EC_t(D, (0, \boldsymbol{D}))\right\}$$
$$\mathbb{E}[\text{lying}] = rD + p \cdot EC_t(0, \boldsymbol{D}) + (1-p) \cdot EC_t(D, (0, \boldsymbol{D}))$$

By Lemma 5.4 and Lemma 5.3, we have

$$\mathbb{E}[\text{honest}] - \mathbb{E}[\text{lying}] \leq \frac{C}{n-1} - rD + \Delta EC_t(0, \boldsymbol{D}) \leq 0,$$

for $r \geq \frac{C}{nD}\frac{1}{p}$. Thus, player j will not deviate from being truthful, even when player i is lying in the previous round. $\square$

5.4. Approximate results

Similar to the single-player model, we extend the results for Bernoulli distributions to approximate results for general distributions. Given $\alpha \in [0,1]$, we redefine being busted as having an observed consumption of at least αD. For the dominant strategy equilibrium, we find the threshold such that being α-truthful is a dominant strategy in Theorem 5.3.

Theorem 5.3. *Given $\alpha \in [0,1]$ and some general distribution F, let $p = 1 - F(\alpha D)$. The α-truthful strategy profile forms a dominant strategy equilibrium if*

$$r \geq \frac{1}{\alpha}\frac{C}{nD}\frac{1-(1-p)^n}{p}\frac{1-(1-p)^T}{p - p(1-p)^{T-1}}. \tag{5.3}$$

Proof. Let $p = 1 - F(\alpha D)$. Assume, for contradiction, that the player adopts some strategy that has a minimum reporting of βD, $0 \leq \beta \leq \alpha$. We compare the expected costs of this strategy and the strategy of being α-truthful. We re-define $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ as follows:

$$\Delta EC_t(\boldsymbol{b}_{t+1}^{-i}) \triangleq EC_t(\alpha D, \boldsymbol{b}_{t+1}^{-i}) - EC_t(\beta D, \boldsymbol{b}_{t+1}^{-i}).$$

Similar to the proof in Theorem 5.1, we want to upper bound $\Delta EC_t(\boldsymbol{0})$. Here we show the computation of $\Delta EC_t(\boldsymbol{0})$ for $t = 1$ and using the recursion argument in the proof of Lemma 5.3, we can show that

$$\Delta EC_t(\beta) = \frac{C}{n}\frac{1-(1-p)^n}{p}\sum_{i=1}^{t}(1-p)^i - \alpha prD\sum_{i=0}^{t-1}(1-p)^i. \tag{5.4}$$

After that, we use the same argument in the proof of Theorem 5.1 to obtain the threshold for the first day and Theorem 5.3 follows. Now we prove the statement for $t = 1$. If the net consumption for the last day exceeds αD (which happens with probability p), then the difference between the penalty payments is $(\alpha - \beta)rD$. Otherwise, the player can save some regular payment by reporting some $\beta' D$ where $\beta \leq \beta' \leq \alpha$. Let X denote the number of players being busted beside the target player. Then $X \sim Bin(n-1, p)$ and $P(X = k) = \binom{n-1}{k}p^k(1-p)^{n-1-k}$. Therefore,

$$\begin{aligned}
\Delta EC_1(\beta) &= EC_t(\alpha D) - EC_t(\beta D) \\
&\leq \max_{\beta \leq \beta' \leq \alpha}(1-p)\sum_{k=0}^{n-1}P(X=k)\left(\frac{C\alpha}{k\alpha + \alpha} - \frac{C\beta'}{k\alpha + \beta'}\right) - prD(\alpha - \beta) \\
&\leq \frac{\alpha - \beta}{\alpha}(1-p)M - (\alpha - \beta)prD,
\end{aligned}$$

and

$$\max_{0 \leq \beta \leq \alpha} \Delta EC_1(\beta) = (1-p)M - \alpha prD,$$

given $r > \frac{(1-p)M}{\alpha pD}$, which is satisfied because actual threshold for r in (5.3) is higher. Using the recursion argument in Lemma 5.3, we can obtain the expression (5.4). $\square$

For Nash equilibrium, we first define the approximate truthful NE in Definition 5.2, which is a natural extension of the complete truthful NE. We then give a penalty threshold in Theorem 5.4 such that an approximately truthful profile will form an NE.

Definition 5.2 *(α-truthful Nash equilibrium).* Given $\alpha \in [0,1]$, a reporting profile $\boldsymbol{b} \in [0,D]^{n\times T}$ is an α-truthful Nash equilibrium if $\boldsymbol{b}_t^i \geq \alpha D$ for all i,t and no player wants to deviate from being α-truthful in any round.

Theorem 5.4. *Given $\alpha \in [0,1]$ and some general distribution F, let $p = 1 - F(\alpha D)$. The α-truthful strategy profile forms a Nash equilibrium if*

$$r \geq \frac{1}{\alpha}\frac{C}{nD}\frac{1-(1-p)^T}{p-p(1-p)^{T-1}}. \tag{5.5}$$

Proof. Similar to the proof of Theorem 5.2, we only need to show that players who had a α-truthful history would stay truthful. We redefine $\Delta EC_t(\boldsymbol{b}_{t+1}^{-i})$ as in the proof of Theorem 5.3 and use a similar argument in Theorem 5.2 to show that $\Delta EC_t(\beta D, \tilde{\boldsymbol{\alpha}}\boldsymbol{D}) \leq 0$ for $0 \leq \beta \leq \alpha$ and $\tilde{\boldsymbol{\alpha}} \geq \boldsymbol{\alpha}$. Then we can safely assume that players $j \neq i$ stay α-truthful in the entire game. Now we compare player i's expected savings and penalties by reporting some βD from being α-truthful.

$$\mathbb{E}[\text{savings}] \leq \left\{ \frac{C\cdot\alpha D}{(n-1)\alpha D + \alpha D} - \frac{C\cdot\beta D}{(n-1)\alpha D + \beta D} \right\} \cdot \mathbb{E}[\#\text{ days before busted}]$$

$$\mathbb{E}[\text{lying}] \geq rD(\alpha-\beta)\cdot \Pr(\text{busted}).$$

Expected penalties exceed expected savings when $r = \frac{1}{\alpha}\frac{C}{nD}\frac{1-(1-p)^{t+1}}{p-p(1-p)^t}$. □

We see that both the penalty thresholds, (5.3) and (5.5) are close to $\frac{1}{\alpha}$ times their counterparts of Bernoulli thresholds, (5.1) and (5.2), for $p = 1 - F(\alpha D)$. Recall that in the single-player model, α-truthfulness can be obtained by directly using the Bernoulli threshold with $p = 1 - F(\alpha D)$. In the multi-player model, however, we have to multiply the Bernoulli threshold with a factor of $\frac{1}{\alpha}$, which suggests it is more difficult to get every player to speak the truth under the cost-sharing setting. We note that both penalty rates (5.3) and (5.5) are upper bounds for the actual thresholds. This is because we treat any report greater than αD as αD. We conjecture that the exact thresholds are not far from thresholds (5.3) and (5.5).

6. Conclusions and discussions

We propose a penalty mechanism for eliciting truthful self-reports when only partial signals are revealed in a repeated game. A player faces a trade-off between under-reporting today and paying a penalty in the future due to the uncertainty of partial signals. It is straightforward that if the penalty is infinitely high, the player(s) will be truthful to avoid any potential inconsistency, but a large penalty is not desirable in reality. Instead, we show that it is not necessary to have an infinite penalty to achieve complete truthfulness. In fact, the length of the game naturally reduces the minimum penalty rate that incentivizes truth-telling.

For the theoretical analysis, we aim to understand the behavior of the player(s) and find the minimum penalty rate required to achieve truthfulness. In particular, given any penalty rate, we provide a characterization of the optimal strategies under both single- and multiple-player settings for any distributions. We identify a penalty rate that achieves complete truthfulness for Bernoulli distributions, which can be used in a reduction to obtain approximate truthfulness for arbitrary distributions.

The penalty mechanism we analyze in this work has a preliminary version in [21], in which the authors described the penalty mechanism as a possible electricity rate structure to ensure fairness for different electricity consumers. Our work provides a robust and theoretical analysis of the proposed penalty mechanism as well as guidelines for setting up the penalty rate that guarantees truthfulness. The penalty we analyze particularly focuses on the absolute difference between the two consecutive reports. It may very well be possible to consider other penalty definitions. For example, the penalty can be based on the difference between the current report and the maximum report in the entire history. Using the maximum report in history instead of the report from the round before may increase the chance of "busting" a lying report because it is likely that the electricity user has revealed their true gross consumption at some point in the history of the game. However, we believe in this case the analysis for the penalty mechanism will be simplified and has already been included in our current work. To see this, we first consider the rounds before the player is forced to report their true consumption D for the first time. To find the penalty rate for truthfulness, we simply need to compare the "honest-till-end" and "lying-till-busted" strategies as we did in Section 3. After the true consumption D has been revealed, in a later round when the player has the option to lie, they can either lie and pay a penalty of rD, or report the truth and pay D. We then only need to make sure the penalty rate r is greater than 1 to ensure truthfulness.

Besides various types of penalty mechanisms, there are many other interesting future directions. One possible future development is to extend our results to asymmetric multi-player settings where players do not have the same gross consumption or the same distribution for partial signals. For heterogeneous players, we may then consider, in addition to truthfulness, the fairness of the mechanism. It is both theoretically interesting and practically important to understand and ensure fairness in multi-player systems. One possible way is to define a fairness ratio (e.g., the value of a player's payment over her gross consumption) for the cost-sharing model. A mechanism is then equitable if the fairness ratio for each player is relatively consistent. It is also worthwhile to derive other truthful and fair mechanisms that do not involve a penalty. For example, it is possible for the center to re-distribute part of the collected wealth to traditional energy consumers to compensate for their over-payment of the grid costs.

Declaration of competing interest

The authors declare that they have no known competing financial interests or personal relationships that could have appeared to influence the work reported in this paper.

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Appendix A. Missing materials

A.1. Recursion approach

In Section 3.1, we briefly mentioned that we can solve for the optimal cost for the Bernoulli distribution via recursion. In the recursion proof, we compute explicitly the expression for $OptCost(t, r, b_{t+1})$ for $t < T$ and $ExpCost(T, r, b_T)$ for the first round. Here, we provide such expressions and optimal strategies can be easily derived from these expressions. We note that we presented the alternative proof in the main article because it showcases the essence of our proposed mechanism. Moreover, the recursion approach would be computationally heavy for continuous distributions whereas the proof in the main body can be extended to any general distributions.

The following is the complete proof via backward induction. For simplicity, we set $D = 1$, which does not affect the results. We break the proof into four cases and together, the four cases paint the picture of the optimal strategy under the Bernoulli distributions for the single player model.

Case 1	$p \le \frac{1}{2}$ and $r \le 1$ OR $p > \frac{1}{2}$ and $r \le \frac{1}{2p}$
Case 2	$p > \frac{1}{2}$ and $\frac{1}{2p} < r \le 1$
Case 3	$p > \frac{1}{2}$ and $r > 1$
Case 4	$p \le \frac{1}{2}$ and $r > 1$

Case 1. $p \le \frac{1}{2}$ *and* $r \le 1$ *OR* $p > \frac{1}{2}$ *and* $r \le \frac{1}{2p}$

Lemma A.1. *For any* $1 \le t < T$*, when* $p \le \frac{1}{2}$ *and* $r \le 1$ *OR when* $p > \frac{1}{2}$ *and* $r \le \frac{1}{2p}$*, given yesterday's arbitrary report* b_{t+1}*,*

$$OptCost(t, r, b_{t+1}) = (1-2p)rb_{t+1} + (t-1)p(1-2p)r + tp(1+r),$$

which is achieved by setting $b_t = 0$*. If* $r < 1$*,* $b_t = 0$ *is the unique optimal report; if* $r = 1$*, the optimal report is any value* $b_t \le b_{t+1}$*.*

Proof. We prove the lemma by induction. When $t = 1$,

$$ExpCost(1, r, b_2, b_1) = p(1+r) + (1-p)b_1 - prb_2 + (1-p)r|b_2 - b_1|. \tag{A.1}$$

The coefficient for b_1 is either $(1-p)(1-r)$ (if $b_2 \ge b_1$) or $(1-p)(1+r)$ (if $b_2 < b_1$). Both are non-negative for $r \le 1$. Therefore, by setting $b_1 = 0$, we achieved the optimal cost:

$$OptCost(1, r, b_2) = \min_{b_1} ExpCost(1, r, b_2, b_1) = r(1-2p)p_2 + p(1+r).$$

Assume the lemma is true for round $t - 1 \ge 1$. For round t and given yesterday's report b_{t+1},

$$\begin{aligned}
&ExpCost(t, r, b_{t+1}, b_t) \\
&= p(1+r) + (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t| \\
&\quad + pOptCost(t-1, r, 1) + (1-p)OptCost(t-1, r, b_t) \\
&= p(1+r) + (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t| \\
&\quad + p[(1-2p)r + (t-2)p(1-2p)r + (t-1)p(1+r)] \\
&\quad + (1-p)[(1-2p)rb_t + (t-2)p(1-2p)r + (t-1)p(1+r)] \\
&= tp(1+r) + (t-1)p(1-2p)r + (1-p)[1 + (1-2p)r]b_t + (1-p)r|b_{t+1} - b_t| - prb_{t+1}.
\end{aligned}$$

The coefficient for b_t is as follows

$$\begin{cases} (1-p)(1+(1-2p)r+r) = (1-p)(1+2(1-p)r) & b_t > b_{t+1} \\ (1-p)(1+(1-2p)r-r) = (1-p)(1-2pr) & b_t \le b_{t+1} \end{cases}$$

When $r \le \frac{1}{2p}$, both coefficients are non-negative. Therefore, choosing $b_t = 0$ is optimal and the optimal cost is

$$\begin{aligned} OptCost(t, r, b_{t+1}) &= \min_{pt} ExpCpst(t, r, b_{t+1}, b_t) \\ &= tp(1+r) + (t-1)p(1-2p)r + (1-p)rb_{t+1} - prb_{t+1} \\ &= tp(1+r) + (t-1)p(1-2p)r + (1-2p)rb_{t+1}. \end{aligned}$$

By induction, we proved the lemma. □

Theorem A.1. *If $p \le \frac{1}{2}$ and $r \le 1$, or if $p > \frac{1}{2}$ and $r \le \frac{1}{2p}$, the player's optimal strategy is lying-till-end.*

Proof. Lemma A.1 showed that the theorem is true for every day except the first day. We now show that the theorem is true for the first day.

$$\begin{aligned} ExpCost(T, r, b_T) &= p(1 + OptCost(T-1, r, 1)) + (1-p)(b_T + OptCost(T-1, r, b_T)) \\ &= p[1 + (1-2p)r + (T-2)p(1-2p)r + (T-1)p(1+r)] \\ &\quad + (1-p)[b_T + (1-2p)rb_T + (T-2)p(1-2p)r + (T-1)p(1+r)] \\ &= (T-1)p(1+r) + (T-2)p(1-2p)r + p + (1-p)(1+r-2pr)b_T + (1-2p)pr \end{aligned}$$

The coefficient for b_T is non-negative in both cases. So the optimal choice for the first day is also zero. Along with the Lemma A.1, we've shown the optimal strategy is lying-till-end for $p \le \frac{1}{2}, r \le 1$ and $p > \frac{1}{2}, r \le \frac{1}{2p}$ with optimal cost

$$\begin{aligned} OptCost(T, r) &= \min_{b_T} ExpCost(T, r, b_T) \\ &= (T-1)p(1+r) + (T-2)p(1-2p)r + p + (1-2p)pr. \end{aligned}$$ □

Case 2. $p > \frac{1}{2}$ and $\frac{1}{2p} < r \le 1$ When $p > \frac{1}{2}$ and $\frac{1}{2p} < r \le 1$, as we have seen in Equation (A.1),

$$OptCost(1, r, b_2) = (1-2p)rb_2 + p(1+r),$$

by setting $b_1 = 0$. Next, we consider round $2 \le t < T$.

Lemma A.2. *For any $2 \le t < T$, when $p > \frac{1}{2}$ and $\frac{1}{2p} < r \le 1$, given yesterday's arbitrary report b_{t+1},*

$$OptCost(t, r, b_{t+1}) = \left[2(1-p)^t r + \sum_{l=2}^{t-1} (1-p)^l + (1-p-r) \right] b_{t+1} + const.,$$

which is achieved by setting $b_t = b_{t+1}$.

Proof. We prove the lemma by induction. When $t = 2$,

$$\begin{aligned} &ExpCost(2, r, b_3, b_2) \\ &= \begin{cases} (1-p)[1 + (1-2p)r - r]b_2 + (1-2p)rb_3 + 2p(1+r) + p(1-2p)r, & b_2 \le b_3 \\ (1-p)[1 + (1-2p)r + r]p_2 - rb_3 + 2p(1+r) + p(1-2p)r, & b_2 > b_3 \end{cases} \end{aligned}$$

Since $r > \frac{1}{2p}$, $1 + (1-2p)r - r \le 0$ and $1 + (1-2p)r + r \ge 0$. Thus $ExpCost(2, r, b_3, b_2)$ is a valley function with respect to b_2 and takes minimum by setting $b_2 = b_3$. Therefore, $OptCost$ can be written as

$$OptCost(2, r, b_3) = [2(1-p)^2 r + (1-p-r)]b_3 + 2p(1+r) + p(1-2p)r.$$

Assume up to round $t - 1 \ge 1$, the lemma holds. For round t and yesterday's report b_{t+1},

$$\begin{aligned} &ExpCost(t, r, b_{t+1}, b_t) \\ &= p(1+r) + (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t| \end{aligned}$$

$$
\begin{aligned}
&+ (1-p)\left[2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r)\right] b_t + p(1-2p)r \\
&+ (t-1)p(1+r) + (t-3)p(1-p-r) + \sum_{i=2}^{t-2}\left[2(1-p)^i r + \sum_{l=2}^{i-1}(1-p)^l\right] \\
&+ p\left[2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r)\right] \triangleq M(b_{t+1}, b_t) + const.,
\end{aligned}
$$

where

$$
\begin{aligned}
M(b_{t+1}, b_t) &= (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t| \\
&\quad + (1-p)\left[2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r)\right] b_t.
\end{aligned}
$$

If $b_t \le b_{t+1}$,

$$
\begin{aligned}
M(b_{t+1}, b_t) &= (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t| \\
&\quad + (1-p)\left[2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r)\right] b_t \\
&= (1-p)\left[1 - r + 2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r)\right] b_t + (1-2p)rb_{t+1}.
\end{aligned}
$$

Note that the coefficient of b_t is $(1-p)$ times the following

$$
\begin{aligned}
&1 - r + 2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r) \\
&= 1 + \sum_{l=1}^{t-2}(1-p)^l - 2r[1-(1-p)^{t-1}] = \sum_{l=0}^{t-2}(1-p)^l(1-2pr) \le 0,
\end{aligned}
$$

where the inequality is due to $r \ge \frac{1}{2p}$.
If $b_t > b_{t+1}$,

$$
M(b_{t+1}, b_t) = (1-p)\left[1 + r + 2(1-p)^{t-1}r + \sum_{l=2}^{t-2}(1-p)^l + (1-p-r)\right] b_t - rb_{t+1},
$$

where the coefficient of b_t is positive. Thus the minimum of $M(b_{t+1}, b_t)$ is achieved at $b_t = b_{t+1}$, i.e.,

$$
OptCost(t, r, b_{t+1}) = \left[2(1-p)^t r + \sum_{l=2}^{t-1}(1-p)^l + (1-p-r)\right] b_{t+1} + const.
$$

By induction, we proved the lemma. □

Theorem A.2. *When $p > \frac{1}{2}$ and $\frac{1}{2p} < r \le 1$, the optimal strategy is lying-till-busted for the first $T-1$ rounds and lying in the last round.*

Proof. Let us consider the first day.

$$
\begin{aligned}
ExpCost(T, r, b_T) &= p(1 + OptCost(T-1, r, 1)) + (1-p)(b_T + OptCost(T-1, r, p_T)) \\
&= (1-p)\left[1 + 2(1-p)^t r + \sum_{l=2}^{t-1}(1-p)^l + (1-p-r)\right] b_T + const.
\end{aligned}
$$

The coefficient for b_T is positive when $\frac{1}{2p} < r < 1$, thus $b_T = 0$. □

Case 3. $p > \frac{1}{2}$ and $r > 1$

Lemma A.3. *For $p > \frac{1}{2}$, $r > 1$, and any $1 \le t < T$, given yesterday's arbitrary report b_{t+1},*

$$OptCost(t, r, b_{t+1}) = \left[(1-p-pr)\sum_{i=0}^{t-1}(1-p)^i\right] b_{t+1} + const.,$$

which is achieved by setting $b_t = b_{t+1}$.

Proof. We prove the lemma by induction. When $t = 1$, the expected cost is

$$ExpCost(1, r, b_2, b_1) = p(1+r) + (1-p)b_1 - prb_2 + (1-p)r|b_b - b_1|.$$

The coefficient for b_1 is $(1-p)(1+r)$ for $b_1 \ge b_2$ and is positive. The coefficient is $(1-p)(1-r)$ for $b_1 < b_2$ and is negative. This implies that $ExpCost(1, t, b_2, b_1)$ is a valley function and the minimum is achieved by setting $b_1 = b_2$. Thus the optimal cost for $t = 1$ is

$$OptCost(1, r, b_2) = p(1+r) + (1-p-pr)b_2.$$

Assume up to round $t - 1 \ge 1$, the lemma holds. For round t and yesterday's report b_{t+1},

$$\begin{aligned}
&ExpCost(t, r, b_{t+1}, b_t)\\
&= p(1+r) + (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t| + p(t-1)\\
&\quad + (1-p)\left[b_t(1-p-pr)\sum_{i=0}^{t-2}(1-p)^i + (1+r)p\sum_{i=0}^{t-2}(1-p)^i + t - 1 - \sum_{i=0}^{t-2}(1-p)^i\right]\\
&= M(b_{t+1}, b_t) + const.,
\end{aligned}$$

where

$$\begin{aligned}
M(b_{t+1}, b_t) &= (1-p)b_t - prb_{t+1} + (1-p)r|b_{t+1} - b_t|\\
&\quad + p_t(1-p-pr)(1-p)\sum_{i=0}^{t-2}(1-p)^i.
\end{aligned}$$

When $b_t \ge b_{t+1}$, the coefficient for b_t is as follows

$$\begin{aligned}
(1-p)\left\{1 + (1-p)\sum_{i=0}^{t-2}(1-p)^i - pr\sum_{i=0}^{t-2}(1-p)^i + r\right\} &= (1-p)\left\{\sum_{i=0}^{t-1}(1-p)^i + r\left[1 - p\sum_{i=0}^{t-2}(1-p)^i\right]\right\}\\
&= (1-p)\left\{\sum_{i=0}^{t-1}(1-p)^i + r(1-p)^{t-1}\right\},
\end{aligned}$$

which is always positive. When $b_t < b_{t+1}$, the coefficient is as follows

$$\begin{aligned}
&(1-p)\left\{1 + (1-p)\sum_{i=0}^{t-2}(1-p)^i - pr\sum_{i=0}^{t-2}(1-p)^i + r\right\}\\
&= (1-p)\left\{1 + (1-p)\sum_{i=0}^{t-2}(1-p)^i - r\left[1 + p\sum_{i=0}^{t-2}(1-p)^i\right]\right\},
\end{aligned}$$

which is negative when

$$r > \frac{1 + (1-p)\sum_{i=0}^{t-2}(1-p)^i}{1 + p\sum_{i=0}^{t-2}(1-p)^i} = \frac{\sum_{i=0}^{t-1}(1-p)^i}{2 - (1-p)^{t-1}} = \frac{1 - (1-p)^t}{2p - p(1-p)^{t-1}} \tag{A.2}$$

Note that from Equation (A.2), we see when $p > \frac{1}{2}$, the right-hand-side is smaller than 1. Thus given $r > 1$, the M function is a valley function, and the minimum is achieved by setting $b_t = b_{t+1}$. The optimal cost in round t is then

$$\begin{aligned}
OptCost(t, r, b_{t+1}) &= M(b_{t+1}, b_{t+1}) + const.\\
&= (1-p-pr)\left[1 + (1-p)\sum_{i=0}^{t-2}(1-p)^i\right] b_t + const.\\
&= \left[(1-p-pr)\sum_{i=0}^{t-1}(1-p)^i\right] b_t + const.
\end{aligned}$$

By induction, we proved the lemma. □

Theorem A.3. *When $p > \frac{1}{2}$, if $r \geq \frac{1-(1-p)^T}{p(1-(1-p)^{T-1})}$, honest-till-end is the optimal strategy; if $1 < r < \frac{1-(1-p)^T}{p(1-(1-p)^{T-1})}$, lying-till-busted is optimal.*

Proof. We write out the expected cost on the first round, i.e., $t = T$.

$$\begin{aligned} ExpCost(T, r, b_T) &= p(1 + OptCost(T-1, r, 1)) + (1-p)(b_T + OptCost(T-1, r, b_T)) \\ &= (1-p)\left[1 + (1-p-pr)\sum_{i=0}^{T-2}(1-p)^i\right] b_T + const. \\ &= (1-p)\left[1 + (1-p)\sum_{i=0}^{T-2}(1-p)^i - pr\sum_{i=0}^{T-2}(1-p)^i\right] b_T + const. \\ &= (1-p)\left[\sum_{i=0}^{T-1}(1-p)^i - pr\sum_{i=0}^{T-2}(1-p)^i\right] b_T + const. \end{aligned}$$

The coefficient for b_T is positive when

$$r < \frac{\sum_{i=0}^{T-1}(1-p)^i}{p\sum_{i=0}^{T-2}(1-p)^i} = \frac{\frac{1-(1-p)^T}{p}}{p\frac{1-(1-p)^{T-1}}{p}} = \frac{1-(1-p)^T}{p(1-(1-p)^{T-1})}. \tag{A.3}$$

The optimal strategy for the first day is therefore setting $b_T = 0$ when r is smaller than (A.3) and $b_T = 1$ otherwise. Along with Lemma A.3, we have proved the theorem. □

Case 4. $p \leq \frac{1}{2}$ and $r \geq 1$ For any $2 \leq t \leq T-1$, let

$$h(t) = \frac{\sum_{i=0}^{t-1}(1-p)^i}{1 + p\sum_{i=0}^{t-2}(1-p)^i} = \frac{1-(1-p)^t}{2p - p(1-p)^{t-1}},$$

and

$$A(t) = (1-r)\sum_{i=1}^{t-1}(1-p)^i + (1-p-pr)(1-p)^{t-1} + (1-2p)r\sum_{i=1}^{t-1}(1-p)^{i-1}.$$

Claim A.1. *When $p < \frac{1}{2}$, $1 = h(1) < h(2) < \cdots < h(T-1) < h(T) < \frac{1}{2p}$.*

Proof. The derivative of $h(t)$ with respect to t is strictly positive:

$$\begin{aligned} \frac{d}{dt}h(t) &= \frac{d}{dt}\frac{1-(1-p)^t}{2p - p(1-p)^{t-1}} \\ &= \frac{[2p - p(1-p)^{t-1}][-(1-p)^t \ln(1-p)] - [1-(1-p)^t][-p(1-p)^{t-1}\ln(1-p)]}{[2p - p(1-p)^{t-1}]^2} \\ &= \frac{p(1-p)^{t-1}\ln(1-p)[1-2p(1-p)]}{[2p - p(1-p)^{t-1}]^2} \\ &> \frac{p(1-p)^{t-1}\ln(1-p)[1-2p]}{[2p - p(1-p)^{t-1}]^2} \geq 0 \end{aligned}$$

The edge cases can be checked manually. Thus, $h(t)$ is increasing w.r.t. t. □

Claim A.2. *When $p < \frac{1}{2}$ and $t \geq 2$, $A(t) + (1-r) \geq 0$ if and only if and $r \leq h(t+1)$.*

Proof. We simplify the expression $A(t) + 1 - r$ as follows.

$$\begin{aligned} A(t) + 1 - r &= (1-r)\sum_{i=0}^{t-1}(1-p)^i + (1-p-pr)(1-p)^{t-1} + (1-2p)r\sum_{i=1}^{t-1}(1-p)^{i-1} \\ &= \frac{1-(1-p)^t}{p} + (1-p)^t - r(2-(1-p)^t). \end{aligned}$$

Thus $A(t) + 1 - r \geq 0$ if and only if

$$r \le \frac{\frac{1-(1-p)^t}{p} + (1-p)^t}{2-(1-p)^t} = \frac{1-(1-p)^{t+1}}{2p - p(1-p)^t} = h(t+1). \quad \square$$

Next, we further distinguish the following subcases $h(t-1) < r \le h(t)$ for each $t = 2, \dots, T-1$.

SubCase 4.1. $p \le \frac{1}{2}$ and $h(t-1) < r \le h(t)$ We start with the last day,

$$ExpCost(1, r, b_2, b_1) = \begin{cases} (1-p)(1-r)b_1 + (1-2p)rb_2 + const., & b_2 \ge b_1 \\ (1-p)(1+r)b_1 - rb_2 + const., & b_2 < b_1. \end{cases}$$

Given $r \ge h(t-1) \ge 1$, $OptCost(1, r, b_2) = (1-p-pr)b_2 + const.$, achieved by setting $b_1 = b_2$.

Now we consider the rounds after the first t rounds, i.e., the last $T-t$ rounds.

Lemma A.4. *When $p < \frac{1}{2}$ and $h(t-1) < r \le h(t)$, given yesterday's arbitrary report $b_{t'+1}$,*

$$OptCost(t', r, b_{t'+1}) = A(t')b_{t'+1} + const., \tag{A.4}$$

which is achieved by setting $b_{t'} = b_{t'+1}$ for any $2 \le t' < t$ and by setting $b_{t'} = 0$ for $t \le t' \le T-1$.

Proof. For $t' = 2$,

$$ExpCost(2, r, b_3, b_2) = \begin{cases} (1-p)[1-r+(1-p-pr)]b_2 + (1-2p)rb_3 + const, & b_3 \ge b_2 \\ (1-p)[1+r+(1-p-pr)]b_2 - rb_3 + const, & b_3 < b_2. \end{cases}$$

Given $p < \frac{1}{2}$ and $r > h(t-1) \ge h(2) = \frac{2-p}{1+p}$, $ExpCost(2, r, b_3, b_2)$ is a valley function and takes minimum at $b_2 = b_3$. Thus,

$$OptCost(2, r, b_3) = (2-p)(1-p-pr)b_3 + const. = A(2)b_3 + const.$$

In general, for any $t' \ge 3$,

$$ExpCost(t', r, b_{t'+1}, b_{t'}) = \begin{cases} (1-p)[1-r+A(t'-1)]b_{t'} + (1-2p)rb_{t'+1} + const., & b_{t'+1} \ge b_{t'} \\ (1-p)[1+r+A(t'-1)]b_{t'} - rb_{t'+1} + const., & b_{t'+1} < b_{t'}. \end{cases}$$

By Claim A.2, for $2 \le t' < t$, $1-r+A(t'-1) \le 0$ since $r > h(t-1) > h(t'-1)$. Then $ExpCost(t, r, b_{t'+1}, b_{t'})$ is a valley function and takes minimum at $b_{t'} = b_{t'+1}$. For $t \le t' \le T-1$, $1-r+A(t'-1) \ge 0$ since $r \le h(t) \le h(t')$. Then the coefficient for $b_{t'}$ in both cases is positive and the expected cost takes minimum at $b_{t'} = 0$. $\square$

Finally, we consider the first day,

$$ExpCost(T, r, p_T) = (1-p)[1 + A(T-1)]b_T + const.,$$

where the coefficient for b_T is positive. Thus on the first day, the optimal $b_T = 0$. In conclusion, we have the following theorem.

Theorem A.4. *When $p < \frac{1}{2}$, $2 \le t \le T-1$, and $h(t-1) < r \le h(t)$, the optimal strategy is lying-till-end for the first t rounds, and lying-till-busted for the rest of the game.*

SubCase 4.2. $r \ge h(T-1)$ When $r \ge h(T-1)$, as we have seen in previous subcase,

$$ExpCost(T, r, b_T) = (1-p)[1 + A(T-1)]b_T + const.,$$

where

$$\begin{aligned} 1 + A(T-1) &= 1 + (1-r)\sum_{i=1}^{T-2}(1-p)^i + (1-p-pr)(1-p)^{T-2} + (1-2p)r\sum_{i=1}^{T-2}(1-p)^{i-1} \\ &= 1 + (1-p-pr)\sum_{i=1}^{T-1}(1-p)^{i-1}. \end{aligned}$$

Thus $1 + A(T-1) \ge 0$ if and only if

$$r \le \frac{1-(1-p)^T}{p(1-(1-p)^{T-1})}.$$

In conclusion, we have the following theorem.

Theorem A.5. *When* $p < \frac{1}{2}$, *if* $\frac{1-(1-p)^{T-1}}{2p-p(1-p)^{T-2}} < r \leq \frac{1-(1-p)^{T}}{p(1-(1-p)^{T-1})}$, *the optimal strategy is lying-till-busted; if* $r > \frac{1-(1-p)^{T}}{p(1-(1-p)^{T-1})}$, *the optimal strategy is honest-till-end.*

References

[1] I. Komang Adi Aswantara, Kab Seok Ko, Dan Keun Sung, A centralized ev charging scheme based on user satisfaction fairness and cost, in: 2013 IEEE Innovative Smart Grid Technologies-Asia (ISGT Asia), IEEE, 2013, pp. 1–4.

[2] Zahra Baharlouei, Massoud Hashemi, Hamed Narimani, Hamed Mohsenian-Rad, Achieving optimality and fairness in autonomous demand response: benchmarks and billing mechanisms, IEEE Trans. Smart Grid 4 (2) (2013) 968–975.

[3] Ian Ball, Kattwinkel Deniz, Probabilistic verification in mechanism design, in: Proceedings of the 2019 ACM Conference on Economics and Computation, 2019, pp. 389–390.

[4] Scott Burger, Ian Schneider, Audun Botterud, Ignacio Pérez-Arriaga, Fair, equitable, and efficient tariffs in the presence of distributed energy resources, in: Consumer, Prosumer, Prosumager: How Service Innovations will Disrupt the Utility Business Model, 2019, p. 155.

[5] Cheng-Kang Chen, M. Akmalul'Ulya, Analyses of the reward-penalty mechanism in green closed-loop supply chains with product remanufacturing, Int. J. Prod. Econ. 210 (2019) 211–223.

[6] Ioannis Caragiannis, Edith Elkind, Mario Szegedy, Lan Yu, Mechanism design: from partial to probabilistic verification, in: Proceedings of the 13th ACM Conference on Electronic Commerce, 2012, pp. 266–283.

[7] Naïm R. Darghouth, Galen Barbose, Ryan Wiser, The impact of rate design and net metering on the bill savings from distributed pv for residential customers in California, Energy Policy 39 (9) (2011) 5243–5253.

[8] Rodolfo Dufo-López, José L. Bernal-Agustín, A comparative assessment of net metering and net billing policies. Study cases for Spain, Energy 84 (2015) 684–694.

[9] Cherrelle Eid, Javier Reneses Guillén, Pablo Frías Marín, Rudi Hakvoort, The economic effect of electricity net-metering with solar pv: consequences for network cost recovery, cross subsidies and policy objectives, Energy Policy 75 (2014) 244–254.

[10] Ernst Fehr, Simon Gächter, Cooperation and punishment in public goods experiments, Am. Econ. Rev. 90 (4) (2000) 980–994.

[11] Ephram Glass, Victor Glass, Power to the prosumer: a transformative utility rate reform proposal that is fair and efficient, Electr. J. 34 (9) (2021) 107023.

[12] Axel Gautier, Julien Jacqmin, Jean-Christophe Poudou, The prosumers and the grid, J. Regul. Econ. 53 (1) (2018) 100–126.

[13] Jerry R. Green, Jean-Jacques Laffont, Partially verifiable information and mechanism design, Rev. Econ. Stud. 53 (3) (1986) 447–456.

[14] Sharad Goel, Daniel M. Reeves, David M. Pennock, Collective revelation: a mechanism for self-verified, weighted, and truthful predictions, in: Proceedings of the 10th ACM Conference on Electronic Commerce, 2009, pp. 265–274.

[15] Evelyn Heylen, Marten Ovaere, Stef Proost, Geert Deconinck, Dirk Van Hertem, Fairness and inequality in power system reliability: summarizing indices, Electr. Power Syst. Res. 168 (2019) 313–323.

[16] Evelyn Heylen, Marten Ovaere, Dirk Van Hertem, Geert Deconinck, Fairness of power system load-shedding plans, in: 2018 IEEE International Conference on Systems, Man, and Cybernetics (SMC), IEEE, 2018, pp. 1404–1409.

[17] Quentin Hoarau, Yannick Perez, Network tariff design with prosumers and electromobility: who wins, who loses?, Energy Econ. 83 (2019) 26–39.

[18] M. Jooshaki, A. Abbaspour, M. Fotuhi-Firuzabad, M. Moeini-Aghtaie, A. Ozdemir, A new reward-penalty mechanism for distribution companies based on concept of competition, in: IEEE PES Innovative Smart Grid Technologies, Europe, IEEE, 2014, pp. 1–5.

[19] Radu Jurca, Boi Faltings, Collusion-resistant, incentive-compatible feedback payments, in: Proceedings of the 8th ACM Conference on Electronic Commerce, 2007, pp. 200–209.

[20] Ioannis Koumparou, Georgios C. Christoforidis, Venizelos Efthymiou, Grigoris K. Papagiannis, George E. Georghiou, Configuring residential pv net-metering policies–a focus on the Mediterranean region, Renew. Energy 113 (2017) 795–812.

[21] Ali Khodabakhsh, Jimmy Horn, Evdokia Nikolova, Emmanouil Pountourakis, Prosumer pricing, incentives and fairness, in: Proceedings of the Tenth ACM International Conference on Future Energy Systems, 2019, pp. 116–120.

[22] Yuqing Kong, Grant Schoenebeck, Equilibrium selection in information elicitation without verification via information monotonicity, arXiv preprint arXiv:1603.07751, 2016.

[23] Mohammad Mahdi Khalili, Xueru Zhang, Mingyan Liu, Incentivizing effort in interdependent security games using resource pooling, in: Proceedings of the 14th Workshop on the Economics of Networks, Systems and Computation, 2019, pp. 1–6.

[24] Yang Liu, Yiling Chen, Learning to incentivize: eliciting effort via output agreement, arXiv preprint arXiv:1604.04928, 2016.

[25] Laura Leavitt, Solar batteries: how renewable battery backups work, https://www.cnet.com/home/energy-and-utilities/solar-batteries-how-renewable-battery-backups-work/. (Accessed 13 August 2022), 2021.

[26] Fabio Moret, Pierre Pinson, Energy collectives: a community and fairness based approach to future electricity markets, IEEE Trans. Power Syst. 34 (5) (2018) 3994–4004.

[27] Nolan Miller, Paul Resnick, Richard Zeckhauser, Eliciting informative feedback: the peer-prediction method, Manag. Sci. 51 (9) (2005) 1359–1373.

[28] National Conference of State Legislators. State net metering policies, https://www.ncsl.org/research/energy/net-metering-policy-overview-and-state-legislative-updates.aspx. (Accessed 13 August 2022), 2017.

[29] Ahlmahz I. Negash, Daniel S. Kirschen, Combined optimal retail rate restructuring and value of solar tariff, in: 2015 IEEE Power & Energy Society General Meeting, IEEE, 2015, pp. 1–5.

[30] Parinaz Naghizadeh, Mingyan Liu, Exit equilibrium: towards understanding voluntary participation in security games, in: IEEE INFOCOM 2016-the 35th Annual IEEE International Conference on Computer Communications, IEEE, 2016, pp. 1–9.

[31] Steven Nadel, Rachel Young, Why is electricity use no longer growing?, in: American Council for an Energy-Efficient Economy Washington, 2014.

[32] Dražen Prelec, A bayesian truth serum for subjective data, Science 306 (5695) (2004) 462–466.

[33] Victor Shnayder, Arpit Agarwal, Rafael Frongillo, David C. Parkes, Informed truthfulness in multi-task peer prediction, in: Proceedings of the 2016 ACM Conference on Economics and Computation, 2016, pp. 179–196.

[34] Chelsea Schelly, Edward P. Louie, Joshua M. Pearce, Examining interconnection and net metering policy for distributed generation in the United States, Renew. Energy Focus 22 (2017) 10–19.

[35] Solar Energy Industry Associations. Net metering, https://www.seia.org/initiatives/net-metering. (Accessed 1 September 2021), 2017.

[36] Siddharth Prakash Singh, Alan Scheller-Wolf, That's not fair: tariff structures for electric utilities with rooftop solar, Manuf. Serv. Oper. Manag. 24 (1) (2022) 40–58.

[37] Torgler Benno, What do we know about tax fraud?: an overview of recent developments, Soc. Res. Int. Q. 75 (4) (2008) 1239–1270.

[38] Yanyan Tang, Qi Zhang, Yaoming Li, Hailong Li, Xunzhang Pan, Benjamin Mclellan, The social-economic-environmental impacts of recycling retired ev batteries under reward-penalty mechanism, Appl. Energy 251 (2019) 113313.

[39] United States Energy Information Administration. Hourly electricity consumption varies throughout the day and across seasons, https://www.eia.gov/todayinenergy/detail.php?id=42915. (Accessed 3 November 2021), 2020.

[40] Luis Von Ahn, Laura Dabbish, Designing games with a purpose, Commun. ACM 51 (8) (2008) 58–67.

[41] Daniel Vieira, Rafael Amaral Shayani, Marco Aurelio Goncalves De Oliveira, Net metering in Brazil: regulation, opportunities and challenges, IEEE Lat. Am. Trans. 14 (8) (2016) 3687–3694.
[42] Bo Waggoner, Yiling Chen, Output agreement mechanisms and common knowledge, in: Second AAAI Conference on Human Computation and Crowdsourcing, 2014.
[43] Wenbin Wang, Junfei Ding, Hao Sun, Reward-penalty mechanism for a two-period closed-loop supply chain, J. Clean. Prod. 203 (2018) 898–917.
[44] Jens Witkowski, David C. Parkes, A robust bayesian truth serum for small populations, in: Twenty-Sixth AAAI Conference on Artificial Intelligence, 2012.