Nonlinear Control under Polytopic Input Constraints with Application to the Attitude Control Problem

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Abstract— This work studies the control problem for non-linear affine systems, subject to polytopic input constraints. A new paradigm is suggested as an alternative to the min-norm approach, to regulate the system while allowing it to operate within or at its saturation limits and minimizing a given pointwise performance index. The control input is obtained through the solution of a quadratic programming optimization problem. The applicability and efficacy of the method are illustrated on the input constrained attitude control problem.

I. INTRODUCTION

In most control applications, the actuation process is typically achieved through electromechanical means, like, for example, servomechanisms, other motors in general or valves. All these devices are inherently subject to saturation limits with regards to the maximum control that they can apply to the dynamical system. The presence of input saturation introduces a nonlinearity in the system, which can cause a performance degradation in the closed loop dynamics or, even worse, destabilize the system. The design of controllers which can handle input constraints is, therefore, an important topic in control theory and applications.

Related Work A simplistic solution to account for input saturation could be to appropriately tune the design parameters of an existing controller (which, itself, is unaware of the saturation limits) so that the largest anticipated reference signals will generate admissible control inputs. This trial and error approach, however, is not very straightforward in the case of nonlinear systems, and most likely will result in reduced performance. For linear systems, [1] provides a systematic way to design controllers by taking the input constraints into consideration. For a chain of integrators, [2] suggests an appropriate bounded control scheme. As far as nonlinear systems are concerned, optimal control solutions can account explicitly for input constraints [3], however, their derivation is usually rather challenging. Receding horizon controllers for nonlinear systems can also take input constraints into consideration explicitly [4]; yet, their online computational requirements may render their implementation unfeasible in real time. As far as the stabilization of nonlinear, control-affine systems is concerned, unit ball and p-norm unit ball input constraints have been considered in [5] and [6] respectively. A backstepping based solution which results in bounded control input and input rates for nonlinear affine systems with a scalar input has been proposed in [7].

For the same class of nonlinear, affine systems, the control design problem based on the concept of Control

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Lyapunov Functions (CLF) is studied in [8], where it is suggested that it can be cast as a minimization problem. This formulation, usually referred in the literature as *min-norm control*, attempts to minimize the norm of the control input at every point in time, while providing performance equal to or better than a specified performance index based on the rate of decrease of the associated CLF. The desired performance requirement is imposed to the system through a constraint in the control norm minimization problem. Ref. [9] further elaborates on the min-norm control paradigm, for the case of polytopic input constraints, by parameterizing the corresponding pointwise optimization problem and casting it as a quadratic programming problem. Input constrained, CLF-based controllers have been further analyzed in [10]. A problem-specific extension has been developed in [11], utilizing the satisficing algorithm. An extension of unconstrained pointwise min-norm CLF control to receding horizon control has been suggested in [12], [13], while [13] also discusses briefly the effect of input constraints on the pointwise min-norm CLF controller. Since the literature on the topic is considerably rich, the references mentioned here do not, by any means, form an inclusive list. However, they are indicative of the different approaches to the problem.

Contributions Using the development in [8], [9] and the min-norm paradigm as a starting point for control of nonlinear affine systems, we identify certain pitfalls of this method. We propose a new solution to the problem of nonlinear control subject to polytopic input constraints. Asymptotic stability is provided in an enlarged subset of the state space where the controller is and remains feasible, the given polytopic input constraints not violated and the performance gap between the CLF decrease rate and a prescribed, state-dependent function is minimized. The minimization assumes the form of a quadratic programming problem and takes place pointwisely in time. Contrary to the standard min-norm controller, the formulation presented in this work allows for the input constraints to become active; the controller ensures asymptotic stability and the satisfaction of the input constraints while trying to minimize the performance gap. The proposed formulation results in a feasibility region that is significantly larger than the one corresponding to the min-norm controller for the same input value set and CLF.

To illustrate the capabilities of the proposed controller in terms of its overall performance and, especially, its handling of the constraints, we solve the attitude control problem and we compare the results with an a posteriori constrained backstepping controller. Based on the results, we identify significant performance advantages.

Paper Structure The problem statement for the input constrained nonlinear control problem is given in Sec. II-

A. A brief review of the min-norm solution is provided in Sec. II-B, followed by a discussion on its main limitations through a motivating example in Sec. II-C. The main result is presented in Sec. III. The application on the attitude problem under input constraints is studied in Sec. IV. Numerical simulations follow in Sec. V. Sec. VI concludes the paper.

Notation   The symbols $\succ$, $\prec$, $\succeq$, $\preceq$ refer to pointwise vector inequalities, in contrast to their scalar counterparts $>$, $<$, $\geq$, $\leq$. Positive definite and semidefinite (symmetric) matrices are denoted by $A > 0$ and $B \geq 0$, respectively. $\mathbb{Z}$ denotes the set of integers. The $n \times n$ identity matrix is denoted by I_n. The symbol $\hat{}$ denotes the 3×3 cross product matrix operator, that is $\hat{a}b = a \times b$ for any $a, b \in \mathbb{R}^3$. Its inverse is denoted with $\vee$, that is $(\hat{a})^\vee = a$, for any $a \in \mathbb{R}^3$. Let $\mathrm{Sk}(A) := \frac{1}{2}\left(A - A^\mathsf{T}\right)$, for any real, square matrix A. The boundary of a set $\mathbb{S}$ is denoted by $\partial\mathbb{S}$. SO(3) denotes the set of 3×3 orthogonal rotation matrices R with $\det(R) = 1$.

II. PRELIMINARIES

A. Problem Statement

We consider nonlinear systems with dynamics which are affine in the control, that is,

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0, \qquad (1)$$

where $x \in \mathbb{R}^n$ is the state vector at time t with initial value x_0, u is the input vector, with $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ for all $t \geq 0$, and $f : \mathbb{R}^n \to \mathbb{R}^n$, $g : \mathbb{R}^n \to \mathbb{R}^{n \times m}$ are locally Lipschitz continuous, vector and matrix valued functions of the state x, respectively, with $f(0) = 0$. The solution of (1) at time t is denoted as $\phi(t; x_0, u(\cdot))$, where $u(\cdot)$ denotes here the time history of the control input in the time interval $[0, t)$. The convex and compact set $\mathbb{U}$, described by $\mathbb{U} := \{u \in \mathbb{R}^m : Au \preceq b\}$, where $A \in \mathbb{R}^{p \times m}$ and $b \in \mathbb{R}^p$ with $b \succeq 0$ (note that $0 \in \mathbb{U}$), contains the values which the input u is allowed to attain.

A C^1 function $V : \mathbb{R}^n \to \mathbb{R}$ is a Control Lyapunov Function (CLF) for (1) if it is positive definite, radially unbounded, and satisfies [8]

$$\inf_{u \in \mathbb{U}} \psi(x, u) < 0, \qquad (2)$$

where $\psi : \mathbb{R}^n \times \mathbb{U} \to \mathbb{R} \cup \{\pm\infty\}$ and $\psi(x, u) := \nabla V(x)\left(f(x) + g(x)u\right)$. Given the convex and compact input value set $\mathbb{U}$, and a CLF such that (2) is true, the control problem for system (1) is reduced, in a pointwise way, to finding a $u^* \in \mathbb{U}$ such that $\psi(x, u^*) < 0$ for all $x \in \mathbb{R}^n \backslash \{0\}$. The criteria with respect to which the choice of u^* takes place are free to be determined by any particular control algorithm, as will be shown next.

B. Review of Pointwise Min-norm Control

The pointwise min-norm control paradigm assumes the presence of a prescribed *minimum* rate of decrease for V, along the closed loop system trajectories [8], that is

$$\inf_{u \in \mathbb{U}} \psi(x, u) < -\alpha(x), \qquad (3)$$

for all $x \in \mathbb{R}^n \backslash \{0\}$, where $\alpha(x) > 0$. A reasonable choice would be $\alpha(x) = \epsilon V(x)$, where $\epsilon > 0$ is a tuning parameter.

Assuming convex polytopic input constraints and following [9], the stabilization problem for (1) reduces to the satisfaction of (3) at every time t. The min-norm paradigm suggests minimizing the norm of u pointwisely (with respect to time), which results in a "locally" minimum effort controller. This can be cast as a quadratic programming (QP) problem. In particular, the QP problem is

$$\min_{u} \quad \mathcal{J}(u) = u^\mathsf{T} u, \qquad (4)$$

$$\text{s.t.} \quad \begin{bmatrix} \nabla V(x)g(x) \\ A \end{bmatrix} u \preceq \begin{bmatrix} -\alpha(x) - \nabla V(x)f(x) \\ b \end{bmatrix}, \quad (5)$$

and it has m decision variables subject to $1 + p$ polytopic constraints. We denote by $\mathbb{X}_f^\alpha$, where $\mathbb{X}_f^\alpha := \{x \in \mathbb{R}^n \backslash \{0\} : \inf_{u \in \mathbb{U}} \psi(x, u) < -\alpha(x)\}$, the subset of $\mathbb{R}^n$ where the QP problem (4), (5) is *feasible*. Additionally, given the feedback control law resulting from the solution of (4), (5) at each $x \in \mathbb{X}_f$, let it be $u_{mn} : \mathbb{R}^n \to \mathbb{U}$, we denote the set of *recursive feasibility* by $\mathbb{X}_{rf}^\alpha$, where $\mathbb{X}_{rf}^\alpha := \{x_0 \in \mathbb{X}_f^\alpha : \phi(t; x_0, u_{mn}(\cdot)) \in \mathbb{X}_f^\alpha, \forall t \geq 0\}$. The min-norm controller possesses certain inverse optimality attributes, on which we do not elaborate in this work (the reader may refer to [14], [8]). We focus, instead, on the effect of input constraints on the stabilization problem.

C. Motivating Example

We consider a pendulum which we intend to stabilize at its upper, unstable equilibrium point. The pendulum's dynamics are given by $\dot{\theta} = \omega$ and $\dot{\omega} = \sin\theta + u$, where $\theta \in \mathbb{R}$ is the angular displacement at time t, measured from the upper vertical position, with $\theta(0) = \theta_0$, $\omega \in \mathbb{R}$ is the angular velocity at time t, with $\omega(0) = \omega_0$, and $u \in \mathbb{U}$ is the input, with $\mathbb{U} = \{u \in \mathbb{R} : -2.2 \leq u \leq 2.5\}$. Our objective is to stabilize the system at $\theta = 0$. A suitable CLF is $V(\theta, \omega; c_1, c_2) = c_1(1 - \cos\theta) + \frac{1}{2}c_2(\sin\theta + \omega)^2$, where $c_1, c_2 > 0$. This CLF takes the periodicity of the problem into account and penalizes the distance from the unstable equilibrium $(\theta, \omega) = (2k\pi, 0)$ for $k \in \mathbb{Z}$. Due to symmetry, the analysis that follows and the resulting solutions are *non-global*, since there is an ambiguity at $(\theta, \omega) = ((2k+1)\pi, 0)$ where $\nabla V((2k+1)\pi, 0) = 0$. Moreover, V is unbounded only along the $\pm\omega$ axis. This is an issue inherent to rotational systems, however, it is not limiting for the purposes of our analysis. We refer to [15] for global solutions to a similar problem.

The desired pointwise performance is parameterized as $\alpha(\theta, \omega) = \epsilon V(\theta, \omega)$. We choose $c_1 = 1.5$, $c_2 = 1$. Fig. 1 illustrates the $\mathbb{X}_f^\alpha$ regions in the (θ, ω) plane where the min-norm controller is feasible for $u \in \mathbb{U}$, that is, where the pointwise QP problem (4), (5) has a solution for different values of ϵ. Additionally, Fig. 1 illustrates the closed loop system trajectories, under the min-norm controller for $\epsilon = 0.4$. One can observe that $\mathbb{X}_f^\alpha$ is shrinking for increasing ϵ values, as a consequence of the fact that the constraint (5) becomes increasingly harder to satisfy with respect to the same $\mathbb{U}$. The input history which stabilizes the system for $(\theta_0, \omega_0) = \left(\frac{\pi}{2}, 0.49\right)$ and $\epsilon = 0.4$ is illustrated in Fig. 3. It is clear that the input does not saturate, as it moves away from the boundary $\partial\mathbb{U}$. To improve the rate of convergence, the necessary increase of ϵ will render the controller *unfeasible* instead of using any available control authority. Similarly, changing the initial condition to $(\theta_0, \omega_0) = \left(\frac{\pi}{2}, 0.5\right)$ results to loss of feasibility, unless ϵ is lowered.
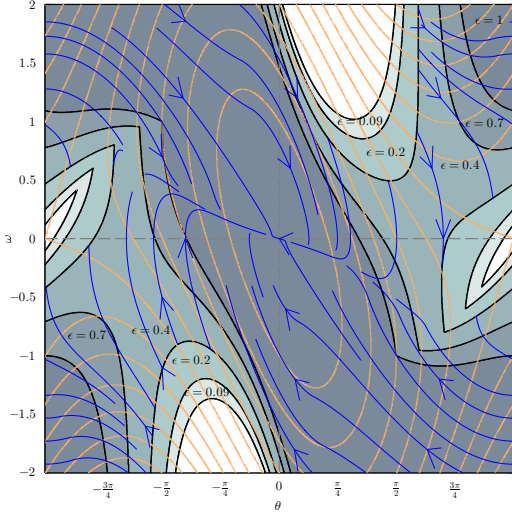
Fig. 1: Regions of feasibility $\mathbb{X}_f^\alpha$ (shaded) for the min-norm controller, for various ϵ and $u \in \mathbb{U}$. The white regions indicate unfeasibility. The closed loop trajectories for $\epsilon = 0.4$ are indicated with blue lines and arrows. The solid orange lines indicate the level sets of V. Apart from the shrinking of $\mathbb{X}_f^\alpha$ for increasing ϵ, it can also be seen that the min-norm controller cannot "escape" the regions close to the attractive equilibrium $(\pm\pi, 0)$ under the given problem formulation (that is, the chosen CLF and its weights) and $\mathbb{U}$.

The common cause for both the shrinking of $\mathbb{X}_f^\alpha$ and the reduced performance of the min-norm controller is that this controller *cannot* effectively tolerate the presence of input saturation. The same argument is also true for a wider class of dynamical systems, beyond the pendulum example. When the satisfaction of the minimum desired performance constraint (5) requires additional control authority than it is available, the QP problem (4), (5) becomes unfeasible and the min-norm controller fails. If one wishes to enlarge the $\mathbb{X}_f^\alpha$ region, ϵ has to decrease, reducing the overall closed loop performance and resulting in slower convergence. It is true that for points $x \notin \mathbb{X}_f^a$, there could be a different choice of ϵ, or even a completely different $\alpha(x)$, which can yield stabilizing inputs taking values in $\mathbb{U}$. In any case, though, two questions arise. First, is it possible to decouple the performance of the input constrained controller from its feasibility? Additionally, since control inputs on the saturation limits $\partial\mathbb{U}$ are allowed, could operating on the $\partial\mathbb{U}$ limits offer any performance or other benefits? This is apparently not always possible with the min-norm controller. Our alternative framework attempts to improve upon these issues, by decoupling feasibility and performance and by providing a formulation which can tolerate control inputs getting saturated.

The reader should be warned that any pointwise control scheme can exhibit a particular limitation under input constraints. Specifically, as one can observe in Fig. 1, $\mathbb{X}_f^\alpha$ is not invariant under the min-norm controller for $\epsilon = 0.4$, since there are trajectories pointing towards the unfeasible regions. This issue is not related exclusively to the choice of ϵ, the particular problem in question or the min-norm algorithm. It is a limitation of the pointwise concept employed in the solution and in the formulation of the associated QP problem. In particular, a pointwise scheme is inherently lacking some form of prediction of the state evolution or some sense of the proximity of the current state to $\partial\mathbb{X}_f^\alpha$. In Fig. 1, though,

one can identify an invariant set $\mathbb{X}_{rf}^\alpha \subsetneq \mathbb{X}_f^\alpha$ which provides recursive feasibility.

III. PROPOSED CONTROLLER

Enforcing a minimum required rate of decrease for V as a hard constraint for the system, following the min-norm paradigm, can be rather strict for an input constrained system. Such a controller may not, actually, tolerate constraints which inevitably cause a performance degradation, that is, slower convergence, to the system. We propose an optimization based solution, which can effectively handle polytopic input constraints and minimize an objective function pointwisely in time. We assume that the *desired* pointwise performance along the closed loop trajectories is $\dot{V}(x) = -\beta(x)$, where $\beta(x) > 0$ for all $x \in \mathbb{R}^n \setminus \{0\}$. We define the *performance gap* for the system, when the latter attains the state x, as follows: $\mathcal{H}(u; x) := (\psi(x, u) + \beta(x))^2$. Expanding $\mathcal{H}$ yields

$$\mathcal{H}(u; x) = (\nabla V(x)f(x))^2 + (\nabla V(x)g(x)u)^2 + \beta^2(x) \\ + 2\nabla V(x)f(x)\beta(x) + 2\nabla V(x)g(x)u\beta(x) \\ + 2\nabla V(x)f(x)\nabla V(x)g(x)u.$$

If we drop the terms which do not contain u and, thus, would play no role in a pointwise minimization of the performance gap, we can define $\mathcal{J}(u; x) := L(x)u + u^\mathsf{T}Q(x)u$, where

$$L(x) := 2\nabla V(x)\left[f(x)\nabla V(x) + \beta(x)\right]g(x), \qquad (6)$$

$$Q(x) := g^\mathsf{T}(x)\left(\nabla V(x)\right)^\mathsf{T}\nabla V(x)g(x) + \mu I_m. \qquad (7)$$

The term μI_m, where μ is a small positive number, is added in order to guarantee that $Q(x) > 0$.

Definition 1: Let $u_f : \mathbb{R}^n \to \mathbb{U}$ be a feedback control law for (1). We define the sets of *feasibility* and *recursive feasibility* as $\mathbb{X}_f := \{x \in \mathbb{R}^n \setminus \{0\} : \inf_{u \in \mathbb{U}} \psi(x, u) < 0\}$ and $\mathbb{X}_{rf}(u_f(\cdot)) := \{x_0 \in \mathbb{X}_f : \phi(t; x_0, u_f(\cdot)) \in \mathbb{X}_f, \forall t \geq 0\}$, respectively.

Definition 2: Let V be a CLF for the system (1) and let the sets $\mathbb{X}_f$ and $\mathbb{X}_{rf}$ be defined as in Def. 1. Then, the feedback control law $u_f : \mathbb{R}^n \to \mathbb{U}$ is said to be asymptotically stabilizing over $\mathbb{X}_f$, if for all $x_0 \in \mathbb{X}_{rf}(u_f(\cdot))$ the function V decreases monotonically along the trajectories $\phi(t; x_0, u_f(\cdot))$ of the system for all $t \geq 0$.

Proposition 1: Suppose that V is a CLF for (1), $\mathbb{U} := \{u \in \mathbb{R}^m : Au \preceq b\}$ is a compact set, and let $\mathbb{X}_f$, $\mathbb{X}_{rf}$, which are defined as in Def. 1, be non-empty. The feedback control law $u_f : \mathbb{R}^n \to \mathbb{U}$, with $u_f(x) = u^*$ being the solution of the linearly constrained QP problem

$$\min_{u \in \mathbb{U}} \quad \mathcal{J}(u; x) = L(x)u + u^\mathsf{T}Q(x)u, \qquad (8)$$

$$\text{s.t.} \quad \begin{bmatrix} \nabla V(x)g(x) \\ A \end{bmatrix} u \preceq \begin{bmatrix} -\nabla V(x)f(x) - \gamma V(x) \\ b \end{bmatrix}, \quad (9)$$

where $x = \phi(t; x_0, u_f(\cdot))$, $L(x)$ and $Q(x)$ are given by (6), (7), and $0 < \gamma \ll 1$, is asymptotically stabilizing over $\mathbb{X}_f$, in the sense of Def. 2, for all $x_0 \in \mathbb{X}_{rf}(u_f(\cdot))$. In addition, the mapping $t \mapsto u_f(x(t))$ is continuous.

Proof: First we show that the QP problem is feasible and admits a unique solution u^*. To this aim, we note that (9) determines a set in $\mathbb{R}^m$ that is necessarily convex and closed as the intersection of a finite number of closed half-spaces in $\mathbb{R}^m$. In addition, by hypothesis, the set $\mathbb{U}$ is

compact; consequently, the domain of the QP problem, call it $\mathcal{D}$, is compact as the intersection of a closed set and a compact set. Moreover, $\mathcal{D}$ is convex as the intersection of convex sets. Furthermore, $\mathcal{D}$ is non-empty for all $x \in \mathbb{X}_f$, where $\mathbb{X}_f$ is non-empty by hypothesis. Consequently, the QP problem is *feasible* for all $x \in \mathbb{X}_f$. Additionally, since $Q(x)$ is a (symmetric) positive definite matrix, $Q(x) > 0$, the QP problem admits a *unique solution* (Prop. 3.1.1, [16]); we denote this solution by u^*. Next, we show that the feedback control law $u_f : \mathbb{R}^n \to \mathbb{U}$, where $u_f(x) = u^*$, is asymptotically stabilizing over $\mathbb{X}_f$ in the sense of Def. 2. From (9), it follows readily that $u_f(x) \in \mathbb{U}$ for all $x \in \mathbb{R}^n$, and that V decreases monotonically along the system trajectories, provided that $x = \phi(t; x_0, u_f(\cdot)) \in \mathbb{X}_f$ for all $t \geq 0$, which is always true, since by hypothesis, $x_0 \in \mathbb{X}_{rf}(u_f(\cdot))$. Finally, we show that the mapping $t \mapsto u_f(x(t))$ is continuous. As shown in [17], the mapping $x \mapsto u^*(x)$ is continuous. In addition, the mapping $t \mapsto x(t)$, where $x(t) = \phi(t; x_0, u_f(\cdot))$, is (at least) continuous as the solution of the differential equation (1), whose right hand side is a locally Lipschitz continuous function. (Note that if $x(t)$ is well defined at some time $t = \tau$, then the mapping $t \mapsto x(t)$ is continuous and well defined for all $t \in (\tau - \epsilon, \tau + \epsilon)$, for some $\epsilon > 0$). Consequently, we can deduce that the mapping $t \mapsto u_f(x(t))$ is continuous for all $t \geq 0$, as the composition of continuous functions. ∎

Remark 1: The term $\gamma V(x)$ is included in (9) in order to maintain compatibility with the strict inequality in the CLF property (2). It has no effect on the performance of the closed loop system and the feasibility regions, since one can choose a rather small γ value, even $\gamma \approx 0$. The reason is that the regulation action of the proposed controller results from the effort to minimize the pointwise performance objective $\mathcal{J}$. In contrast, the regulation action of the min-norm controller results from the presence of $\alpha(x)$ in (5). The term $\alpha(x)$, though, has a significant, potentially adverse, effect on performance and feasibility. Let $\alpha(x) = \epsilon V(x)$. In the example in Section II-C, we have shown that large values of ϵ, which would yield fast convergence, can cause unfeasibility, depending on the particular input saturation limits. Smaller ϵ values tend to result in increasingly slower convergence.

Remark 2: The feasible set of the proposed controller, $\mathbb{X}_f$, is a superset of the feasible set of the min-norm controller, $\mathbb{X}_f^\alpha$, that is, $\mathbb{X}_f^\alpha \subseteq \mathbb{X}_f$ for the same f, g, V, $\mathbb{U}$ and for any function $\alpha(x)$ with $\alpha(x) > 0$ for $x \neq 0$. This follows trivially from the definitions of $\mathbb{X}_f^\alpha$ and $\mathbb{X}_f$.

Remark 3: The proposed QP problem minimizes the performance gap $\mathcal{J}$, whereas the associated constraints are related to asymptotic stability, in the sense of Def. 2, and the input value set $\mathbb{U}$. In this way, the algorithm respects the input constraints and ensures stability, while allowing for the inevitable performance degradation when the input constraints become active.

Remark 4: In general, $\mathbb{X}_f$ is not invariant under the proposed controller, similarly to $\mathbb{X}_f^\alpha$ for the min-norm controller. It is possible that some trajectories may drive the system towards $x \in \mathbb{X}_f$. The pointwise QP problem (8), (9), is non-predictive in nature and thus it cannot proactively steer the system away from the unfeasible region. Nevertheless, the

fact that the $\mathbb{X}_f$ set is "maximized", following Remark 2, has been observed to cause a corresponding enlargement to the $\mathbb{X}_{rf}$ set as well, providing some additional assurance for recursive feasibility compared to the min-norm controller (for the same system, CLF, and $\mathbb{U}$). Incorporating information about $\partial \mathbb{X}_f$ in the formulation of the proposed controller in order to further enlarge $\mathbb{X}_{rf}$ is a topic for future work.

IV. AN APPLICATION OF THE PROPOSED CONTROLLER TO THE ATTITUDE CONTROL PROBLEM

If we represent the attitude of a rigid body at time t using the rotation matrix $R \in \mathrm{SO}(3)$, then its rotational dynamics and kinematics with three degrees of freedom are given by

$$J\dot{\omega} = -\hat{\omega}J\omega + \tau, \quad \dot{R} = R\hat{\omega}, \quad (10)$$

where $\omega \in \mathbb{R}^3$ is the angular velocity around the 1-2-3 body fixed axes at time t with initial value ω_0, R_0 is the initial attitude, $J \in \mathbb{R}^{3\times3}$, $J > 0$, is the inertia tensor, and τ is the torque input [18]. We assume that $\tau(t) \in \mathbb{T}$ for all $t \geq 0$, where $\mathbb{T}$ is a convex, compact polytope given by $\mathbb{T} = \{\tau \in \mathbb{R}^3 : A\tau \preceq b\}$ with $b \succeq 0$. Our objective is to design a controller capable of tracking a trajectory $(R_d(t), \omega_d(t))$, while respecting the torque input constraints. First, we formulate the attitude error dynamics in control affine form, following the standard backstepping algorithm [19]. Subsequently, the problem is solved using the controller proposed in this work. Moreover, we also utilize the backstepping algorithm to complete the control design and obtain an unconstrained backstepping-based static feedback controller, as proposed in [20] for a quaternion attitude parametrization and in [21] for the combined position and attitude control problem using a rotation matrix attitude representation. In the backstepping design, we do not account for the input constraints; however, the resulting control inputs are saturated a posteriori, as they are applied to the system. We compare the two solutions in order to demonstrate the performance and stability improvements provided by our approach.

A. Formulation of Attitude Error Dynamics

Following [21], [22], we define an attitude error metric as $e_\theta(R, R_d) := \frac{1}{2}\mathrm{Tr}(I_3 - RR_d^\mathsf{T})$. Its time derivative is

$$\dot{e}_\theta = \left(\mathrm{Sk}\left(R_d^\mathsf{T}R\right)^\vee\right)^\mathsf{T}(\omega - \omega_d). \quad (11)$$

We define the Lyapunov function $V_1(e_\theta; k_\theta) := k_\theta e_\theta$, where $k_\theta > 0$ is a tuning parameter. Along the system's trajectories, we have $\dot{V}_1 = k_\theta \left(\mathrm{Sk}\left(R_d^\mathsf{T}R\right)^\vee\right)^\mathsf{T}(\omega - \omega_d)$. According to the standard backstepping algorithm, ω is chosen as $\omega = \omega_d - \mathrm{Sk}\left(R_d^\mathsf{T}R\right)^\vee + e_\omega$. The new error variable $e_\omega(R, R_d, \omega, \omega_d)$ has dynamics given by $\dot{e}_\omega = \dot{\omega} - \dot{\omega}_d + \frac{d}{dt}\left[\mathrm{Sk}\left(R_d^\mathsf{T}R\right)^\vee\right]$. Analytical calculation of the time derivative term and substitution of $\dot{R}$ from (10) yield

$$\dot{e}_\omega = -J^{-1}\hat{\omega}J\omega + J^{-1}\tau - \dot{\omega}_d$$
$$+ \mathrm{Sk}\left(\hat{\omega}_d^\mathsf{T}R_d^\mathsf{T}R + R_d^\mathsf{T}R\hat{\omega}\right)^\vee. \quad (12)$$

We now define an augmented Lyapunov function as $V_2(e_\theta, e_\omega; k_\theta, k_\omega) := V_1(e_\theta; k_\theta) + \frac{1}{2}k_\omega e_\omega^\mathsf{T}e_\omega$, where $k_\omega > 0$

is a second tuning parameter. The time derivative of V_2 along the attitude error trajectories is

$$\dot{V}_2 = -k_\theta \left\| \mathrm{Sk}\left(R_d^\mathsf{T} R\right)^\vee \right\|^2 + k_\theta \, \mathrm{Sk}\left(R_d^\mathsf{T} R\right)^\vee e_\omega$$
$$+ k_\omega e_\omega^\mathsf{T}\left(-J^{-1}\hat{\omega}J\omega + J^{-1}\tau - \dot{\omega}_d \right.$$
$$\left. + \mathrm{Sk}\left(\hat{\omega}_d^\mathsf{T} R_d^\mathsf{T} R + R_d^\mathsf{T} R\hat{\omega}\right)^\vee \right).$$

An appropriate τ will make $\dot{V}_2$ negative semidefinite and thus allow us to conclude that the attitude error dynamics, in the absence of control input constraints, are Lyapunov stable. Using LaSalle's invariance principle, asymptotic stability is deduced.

B. Solution Using the Proposed Controller

Let $x = [e_\theta \; e_\omega^\mathsf{T}]^\mathsf{T}$. By combining (11) and (12), the error dynamics can be written in the control-affine form (1), with

$$f(x) = \begin{bmatrix} \left(\mathrm{Sk}\left(R_d^\mathsf{T} R\right)^\vee\right)^\mathsf{T}(\omega - \omega_d) \\ -J^{-1}\hat{\omega}J\omega - \dot{\omega}_d + \mathrm{Sk}\left(\hat{\omega}_d^\mathsf{T} R_d^\mathsf{T} R + R_d^\mathsf{T} R\hat{\omega}\right)^\vee \end{bmatrix},$$
$$g(x) = \begin{bmatrix} 0 \\ J^{-1} \end{bmatrix}.$$

An asymptotically stabilizing, in the sense of Def. 2, control input $\tau \in \mathbb{T}$ can be found by solving the QP problem (8), (9), given the attitude error dynamics, the CLF V_2 and the input value set $\mathbb{T}$. The constants k_θ, k_ω which appear in V_2, as well as the parameter ϵ in $\beta(e_\theta, e_\omega) = \epsilon V_2(e_\theta, e_\omega)$, can be used to tune the solution.

C. Unconstrained Backstepping-Based Solution

By completing the backstepping design, the stabilizing τ for the unconstrained case is obtained as

$$\tau = J\Big[J^{-1}\hat{\omega}J\omega + \dot{\omega}_d - \mathrm{Sk}\left(\hat{\omega}_d^\mathsf{T} R_d^\mathsf{T} R + R_d^\mathsf{T} R\hat{\omega}\right)^\vee$$
$$- e_\omega - \frac{k_\theta}{k_\omega}\left(\mathrm{Sk}\left(R_d^\mathsf{T} R\right)^\vee\right)^\mathsf{T}\Big]. \tag{13}$$

The ratio k_θ/k_ω is a tuning parameter, affecting the magnitude of the control input. However, as it will be demonstrated in Section V, this controller does not necessarily make $\dot{V} \le 0$ along the system's trajectories for all t, under the given input constraints.

V. SIMULATIONS

Pendulum Stabilization Example We consider again the pendulum problem of Section II-C, with the same dynamics, CLF, and input value set $\mathbb{U}$. The desired pointwise performance is parameterized as $\beta(\theta, \omega) = \epsilon V(\theta, \omega)$. For the min-norm controller, we choose again $\epsilon = 0.4$; higher values, which would yield faster performance, render the min-norm controller unfeasible for $(\theta_0, \omega_0) = \left(\frac{\pi}{2}, 0.49\right)$, as it has been already discussed. There are no such restrictions for the proposed controller, since feasibility $\mathbb{X}_f$ and performance $\beta(\theta, \omega)$ are essentially decoupled; we therefore choose $\epsilon = 1$ to demonstrate the potential for performance improvement.

Fig. 2 illustrates the feasible set $\mathbb{X}_f$ for the proposed controller, which is independent of ϵ, on top of the feasible sets $\mathbb{X}_f^\alpha$ for the min-norm case for different ϵ. The input histories, state trajectories and $\dot{V}$ are illustrated in Fig. 3
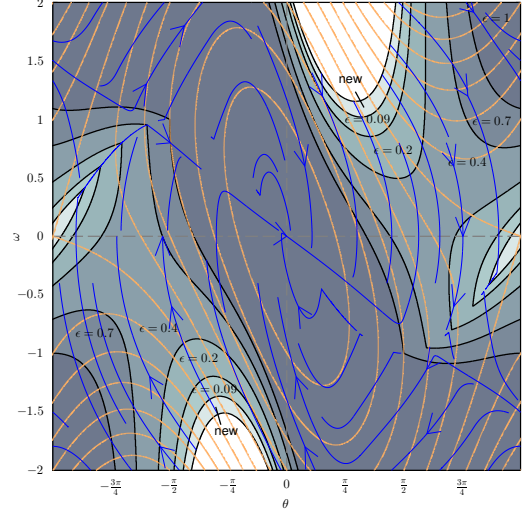
Fig. 2: Regions of feasibility for the proposed controller, $\mathbb{X}_f$, and for the min-norm controller for different ϵ, $\mathbb{X}_f^\alpha \subsetneq \mathbb{X}_f$. The trajectories correspond to the closed loop system under the proposed controller for $\epsilon = 0.7$. The system does not exhibit any problems *near* the attractive equilibria $(\pm\pi, 0)$ for any $\beta(\theta, \omega)$, in contrast to the min-norm solution.
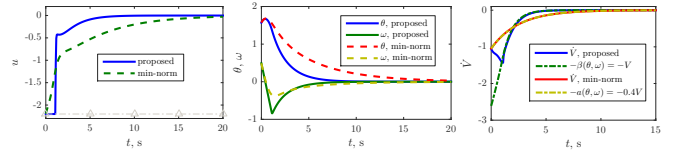
Fig. 3: Input, state and $\dot{V}$ as functions of time for the closed loop system with $(\theta_0, \omega_0) = \left(\frac{\pi}{2}, 0.49\right)$ under the proposed controller and the min-norm controller. The proposed controller is able to operate on $\partial\mathbb{U}$. The min-norm controller is forced to perform equal or better, in terms of $\dot{V}$, than $\alpha(\theta, \omega)$ at any time. The proposed controller is allowed to initially exert significant control input, at the saturation limits, in order to attain the "faster" performance objective $\beta(\theta, \omega)$.

as functions of time. One can observe that the proposed controller exhibits a much shorter settling time. Initially it falls short of its pointwise performance objective, as Fig. 3 shows. However, the proposed formulation utilizes the available control authority to accelerate, and subsequently reach its desired pointwise performance $\beta(\theta, \omega)$.

The results presented here reveal another interesting aspect of both controllers. It can be seen that the difference in terms of control input applied initially, from 0s to 1.5s, is small. Yet, subsequently, the proposed controller uses much less control effort in total, compared to the min-norm controller which, by definition, attempts to minimize the control effort for every t. This is a consequence of both the particular formulation of each controller, as discussed before, and the common pointwise nature of the two solutions. In fact, due to the absence of any kind of predictive element in the formulation, the minimization of the control effort for every t which the min-norm controller attempts does not cause an equivalent minimization of control effort for the whole maneuver. Such performance conclusions depend on many factors, specifically, the system in question, the associated CLF, the input constraints and the initial conditions. Nevertheless, we anticipate that the capability of the proposed controller to operate on the system's saturation limits will yield significantly improved performance using, potentially,
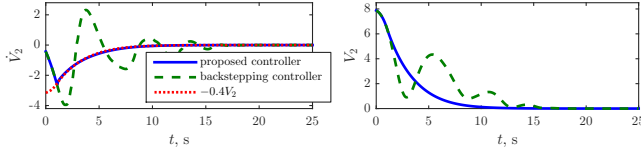
Fig. 4: V_2 and $\dot{V}_2$ along the system's trajectory as a function of time for the two controllers. The dotted red line corresponds to the desired performance objective for the proposed controller. The a posteriori saturated backstepping controller exhibits temporary increases of V_2 along the ensuing trajectory.
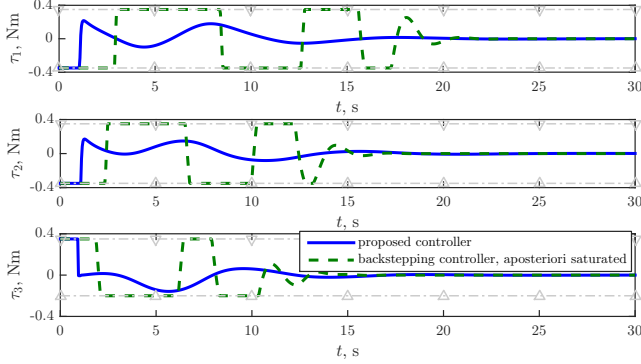

Fig. 5: Torque inputs around the 1,2,3 body axes

less total control effort, for a wide class of dynamical systems, compared to the min-norm controller.

Input Constrained Attitude Control We consider the attitude dynamics derived in Sec. IV, for a rigid body with $J = \mathrm{diag}(1.2, 1.2, 1.1)\,\mathrm{kg\,m}^2$. The control inputs are constrained in the *asymmetric* set $[-0.35, -0.35, -0.2]^\mathsf{T} \preceq \tau \preceq [0.35, 0.35, 0.35]^\mathsf{T}\,\mathrm{N\,m}$. We choose $k_\theta = 5$, $k_\omega = 1$, $\epsilon = 0.4$. At $t = 0$, the system is at rest at the initial attitude corresponding to a rotation of $2\pi/3$ radians around the $[2/3, 2/3, -1/3]$ axis, while the desired state is $(R_d, \omega_d) = (I_3, 0\,\mathrm{rad/s})$. The backstepping control law (13) is a posteriori saturated using the same constraint set $\mathbb{T}$.

Fig. 4 illustrates the fact that the system converges a little faster under the proposed controller. The backstepping controller tends to generate rather aggressive control inputs, as illustrated in Fig. 5. From the same data, it can be observed that the control effort used for the same maneuver is considerably lower for the proposed controller. It can also be seen that the backstepping controller, when subjected to input saturation, renders the system temporarily "unstable" for short intervals around $4\mathrm{s}$, $9\mathrm{s}$, and $14\mathrm{s}$, in the sense that V_2 does not decrease monotonically along the ensuing trajectories. This behavior does not prevent the backstepping controller from ultimately achieving (in this particular example) its regulation goal. Nevertheless, any kind of oscillatory behavior, as exhibited here, is typically undesirable.

Choosing different values for k_θ and k_ω may prevent the generation of control inputs that tend to exceed the system's saturation limits. However, such a conservative design would result in lower performance and, eventually, longer settling times with respect to the proposed controller.

VI. DISCUSSION AND CONCLUSION

A new solution to the input constrained control problem of control-affine nonlinear systems has been presented. The proposed CLF based controller is built around a pointwise optimization problem with a state-dependent performance index. The particular formulation provides asymptotic stability, provided that the closed loop trajectories stay in the feasible state space, while it allows the system to operate at or close to its input saturation limits. In addition, the proposed controller can utilize the available control authority appropriately, in order to provide as much performance is necessary, if it is possible. Numerical simulations on the pendulum and attitude stabilization problems have illustrated advantages of the proposed controller over the min-norm controller and an unconstrained, a posteriori saturated nonlinear controller, in terms of performance, total control effort and stability.

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