Optimal Covariance Control for Discrete-Time Stochastic Linear Systems Subject to Constraints

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***Abstract*— This work deals with an optimal covariance control problem for stochastic discrete-time linear systems subject to mean sum constraints involving quadratic functions of the state and the control input sequences under the assumption of full state information. We show that the stochastic optimal control problem is equivalent to a deterministic nonlinear program, which, under a judicious choice of the decision variable, can be brought to a form in which its performance index is a convex, quadratic function subject to both equality and inequality quadratic constraints. The key challenge here stems from the fact that the equality constraints that result from the terminal constraints on the state covariance may not be necessarily convex. We show, however, that by employing a simple relaxation technique, the nonlinear program is associated with a convex program, which can be addressed by means of robust and efficient algorithms. Despite the fact that the solution to the relaxed convex program will not necessarily give closed-loop trajectories whose endpoints follow exactly the goal Gaussian distribution, a representative sample of such trajectories are expected to have endpoints that will be more concentrated near the origin than if there were drawn from the goal Gaussian distribution. Finally, numerical simulations that illustrate some key ideas of the paper are presented.**

I. Introduction

We consider a finite-horizon stochastic optimal control problem that seeks for the sequence of control inputs that will steer the uncertain state of a discrete-time stochastic linear system, which is initially drawn from a known Gaussian distribution, to a terminal state, which is drawn from another known (goal) Gaussian distribution. Besides the terminal constraints on the state covariance, in the formulation of the stochastic optimal control problem, we also consider mean sum constraints that involve quadratic functions of the state and the control input sequences. This work is a natural extension of our previous work on similar finite-horizon covariance control problems for *continuous-time* stochastic linear systems subject to integral quadratic state constraints [1]. In contrast with the previous reference, in this paper we will primarily focus on the control synthesis problem rather the analysis problem. In particular, we will show that the stochastic optimal control problem can be reduced to a deterministic nonlinear program, which will be subsequently associated, via a simple relaxation technique, with a convex program, which can be addressed by means of efficient and robust algorithms [2], [3].

Literature Review: The covariance control problem was first introduced in the controls community by Hotz and Skelton [4], [5]. This class of problems has been studied in detail in the literature for both continuous-time and discrete-time stochastic linear systems (the reader may refer, for instance, to [6]–[9]). All these references focus on the infinite-horizon problem in which the objective is to steer the covariance of the state of a stochastic linear system to a positive definite matrix that satisfies a relevant Lyapunov (or Stein) algebraic, matrix equation (*steady state* covariance matrix). The finite-horizon covariance control problem for continuous-time stochastic linear systems has been recently addressed in [10], [11]. In these two references, it is shown that when the input and the noise channels of the system are identical, the covariance control problem becomes significantly more amenable to analysis and computation whereas the other case is a much harder problem, whose solvability is in general difficult to be concluded a priori. The finite-horizon covariance control problem for continuous-time stochastic linear systems in the presence of integral quadratic state constraints was addressed in our previous work [1]. A finite-horizon covariance control problem with a soft constraint on the terminal state covariance is addressed in [12].

Main Contribution: In this paper, we address the optimal finite-horizon covariance control problem for stochastic discrete-time linear systems using a stochastic optimal control framework. The problem considered herein can be viewed as an extension of the classic finite-horizon LQG (Linear Quadratic Gaussian) problem for discrete-time stochastic linear systems [13]–[15] under full-state observation, which explicitly accounts for hard terminal constraints on the covariance of the (random) state vector along with mean sum constraints that involve quadratic functions of the state and the control input sequences. We henceforth refer to this class of stochastic optimal control problems as the *constrained* discrete-time LQG covariance control, or more compactly, constrained DTLQGCC problem.

To address the constrained DTLQGCC problem, we restrict our attention to, possibly suboptimal, feedback control policies comprised of linear feedback laws. This choice is motivated by a significant body of work on the synthesis of feedback control laws for stochastic linear systems (see [16] and references therein). First, we show that the stochastic optimal control problem can be associated with a deterministic nonlinear program (NLP). Following [16], we show that by judiciously selecting the decision variable of the optimization problem, both the performance index

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and the mean sum constraint function of the latter become convex quadratic functions of the new decision variable. The key challenge in addressing the resulting NLP stems from the fact that the terminal constraints on the state covariance correspond to a system of quadratic constraints, which are not necessarily convex. To handle these non-convex constraints, we employ a simple relaxation technique that allows us to associate the NLP with a convex program. Despite the fact that the convex program and the original stochastic control problem are not equivalent in the strict sense, that is, the endpoints of the closed-loop trajectories associated with the latter do not follow exactly the goal Gaussian distribution, the results that one obtains by solving the convex program are desirable for practical purposes. This is because the formulation of the convex program favors the generation of closed-loop trajectories whose endpoints are more likely to be concentrated near the origin than a sample of points drawn from the goal Gaussian distribution.

Structure of the paper: The rest of the paper is organized as follows. In Section II, we formulate the covariance control problem as a stochastic optimal control problem and in Section III, we show how to reduce the latter problem to a deterministic nonlinear program, which we subsequently associate with a convex program. Illustrative numerical simulations are presented in Section IV, and finally, Section V concludes the paper with a summary of remarks.

II. Problem Formulation

A. Notation

We denote by $\mathbb{R}^n$ the set of n-dimensional real vectors. We write $\mathbb{N}_{\geq 0}$ and $\mathbb{N}_{>0}$ to denote the set of non-negative integers and strictly positive integers, respectively. Given a probability space $(\Omega, \mathfrak{F}, P)$ and $N \in \mathbb{N}_{>0}$, we denote by $\ell_2^n(\{0,\dots,N\};\Omega,\mathfrak{F},P)$ the Hilbert space of mean square summable random sequences $\{x(t) : \ t \in \{0,\dots,N\} \subset \mathbb{N}_{\geq 0}\}$ on $(\Omega, \mathfrak{F}, P)$, where $x(t)$ is a n-dimensional (random) vector at each $t \in \{0,\dots,N\}$. We write $\mathbb{E}[\cdot]$ to denote the expectation operator. Given a square matrix $\mathbf{A}$, we denote its trace by $\mathrm{trace}(\mathbf{A})$. Furthermore, we denote by $\mathrm{bdiag}(\mathbf{A}_1,\dots,\mathbf{A}_\ell)$ the block diagonal matrix whose diagonal elements are the matrices $\mathbf{A}_i$, $i \in \{1,\dots,\ell\}$. We write $\mathbf{0}$ and $\mathbf{I}$ to denote the zero matrix and the identity matrix, respectively. Finally, we will denote the convex cone of $n\times n$ symmetric positive definite and positive semi-definite matrices by $\mathbb{P}_n$ and $\overline{\mathbb{P}}_n$, respectively.

B. Formulation of the Optimal Covariance Control Problem

We consider a discrete-time stochastic linear system that satisfies the following stochastic difference equation:

$$x(t+1) = \mathbf{A}(t)x(t) + \mathbf{B}(t)u(t) + \mathbf{C}(t)w(t), \qquad (1)$$

with $x(0) = x_0$, for $t \in \{0,\dots,N-1\}$, where $N \in \mathbb{N}_{>0}$ is given, $\{x(t),\ t \in \{0,\dots,N\}\}$ and $\{u(t) :\ t \in \{0,\dots,N-1\}\}$, or simply $\{x(t)\}_{t=0}^{N}$ and $\{u(t)\}_{t=0}^{N-1}$, denote the state and the control input (random) sequences, respectively, on a complete probability space $(\Omega, \mathfrak{F}, P)$. At each stage $t \in \{0,\dots N\}$ and $t \in \{0,\dots N-1\}$, $x(t)$ and $u(t)$ are n-dimensional and m-dimensional vectors, respectively. The control input sequence $\{u(t)\}_{t=0}^{N-1}$ is assumed to belong to $\ell_2^m(\{0,\dots,N-1\};\Omega,\mathfrak{F},P)$ and to have finite k-moments for all $k > 0$. We will henceforth refer to a control input sequence that satisfies these properties as *admissible*. In addition, $\{w(t) \ : \ t \in \{0,\dots,N-1\}\}$, or simply $\{w(t)\}_{t=0}^{N-1}$, is a sequence of independent normal random variables with zero mean and unit covariance, that is,

$$\mathbb{E}\left[w(t)w(\tau)^{\mathrm{T}}\right] = \mathbf{0}, \qquad (2)$$

for all $t, \tau \in \{0,\dots,N-1\}$ with $t \neq \tau$, and

$$\mathbb{E}\left[w(t)\right] = \mathbf{0}, \quad \mathbb{E}\left[w(t)w(t)^{\mathrm{T}}\right] = \mathbf{I}, \qquad (3)$$

for all $t \in \{0,\dots,N-1\}$. At each time t the (random) vector $w(t)$ is p-dimensional. It is assumed that x_0 and $\{w(t)\}_{t=0}^{N-1}$ are mutually independent. Finally, $\{\mathbf{A}(t) :\ t \in \{0,\dots,N-1\}\}$, $\{\mathbf{B}(t) \ : \ t \in \{0,\dots,N-1\}\}$, and $\{\mathbf{C}(t) :\ t \in \{0,\dots,N-1\}\}$ are bounded sequences of matrices of appropriate dimensions.

Now, let us assume that the initial state x_0 is a random vector drawn from the multi-variate normal distribution $\mathcal{N}(0, \mathbf{\Sigma}_0)$, where $\mathbf{\Sigma}_0 \in \mathbb{P}_n$ is the initial state covariance, that is, $\mathbb{E}\left[x_0 x_0^{\mathrm{T}}\right] = \mathbf{\Sigma}_0$. We are interested in steering the state of the stochastic discrete-time linear system described by the difference equation (1) from the initial (random) vector $x_0 \sim \mathcal{N}(0, \mathbf{\Sigma}_0)$ to a terminal (random) vector $x_{\mathsf{f}} = x(N)$ at a given stage $t = N$, where $x_{\mathsf{f}} \sim \mathcal{N}(0, \mathbf{\Sigma}_{\mathsf{f}})$, and where $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{P}_n$ is the terminal state covariance, that is, $\mathbb{E}\left[x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\right] = \mathbf{\Sigma}_{\mathsf{f}}$. We refer to this problem as the covariance control problem. In this work, we are interested in addressing the covariance control problem in the presence of state and input constraints by using a stochastic optimal control framework.

Problem 1: Constrained DTLQGCC Problem: Let $N \in \mathbb{N}_{>0}$, $c > 0$, and $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{P}_n$ be given. In addition, assume that for all $t \in \{0,\dots,N-1\}$ the matrices $\mathbf{Q}(t)$ and $\mathbf{R}(t)$ belong to $\overline{\mathbb{P}}_n$ and $\mathbb{P}_m$, respectively. Let Π denote the class of feedback control policies, $\pi = \{\mu(x;t)\}_{t=0}^{N-1}$, where $\mu(x;t) := -\mathbf{K}(t)x$, $\mathbf{K}(t) \in \mathbb{R}^{m\times n}$, for $t \in \{0,\dots,N-1\}$, such that the sequence of control inputs $\{u(t)\}_{t=0}^{N-1}$ with $u(t) = \mu(x(t);t)$ is admissible. Then, find an optimal feedback control policy $\pi^{\circ} := \{\mu^{\circ}(x;t)\}_{t=0}^{N-1} \in \Pi$ that minimizes the performance index:

$$J(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}(t)u(t)\Big], \qquad (4)$$

subject to the difference equation constraints (1), the mean sum constraint:

$$h\left(\{x(t)\}_{t=0}^{N-1}, \{u(t)\}_{t=0}^{N-1}\right) \leq c,$$

where

$$h\left(\{x(t)\}_{t=0}^{N-1}, \{u(t)\}_{t=0}^{N-1}\right) := \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}_c(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}_c(t)u(t)\Big], \quad (5)$$

where $\mathbf{Q}_c(t)$ and $\mathbf{R}_c(t)$ belong to $\overline{\mathbb{P}}_n$ and $\overline{\mathbb{P}}_m$, respectively, for all $t \in \{0, \ldots, N-1\}$. In addition, the covariance of the (random) state vector $x(t)$ at stage $t = 0$ and at stage $t = N$ the following boundary conditions:

$$\mathbb{E}\left[x_0 x_0^{\mathrm{T}}\right] = \mathbf{\Sigma}_0, \qquad \mathbb{E}\left[x_{\mathsf{f}} x_{\mathsf{f}}^{\mathrm{T}}\right] = \mathbf{\Sigma}_{\mathsf{f}}, \quad (6)$$

where $x_0 = x(0)$ and $x_{\mathsf{f}} = x(N)$.

Note that in the formulation of Problem 1, we explicitly required that the minimizing feedback control policy is comprised of linear feedback control laws at each stage. It is likely that this additional assumption on the structure of the feedback control policy is unnecessary. To see this, let Π' denote the class of feedback control policies which correspond to sequences of feedback control laws $\{\mu(\cdot;t),\ t \in \{0, \ldots, N-1\}\}$, where $\mu(x;t) \in \mathbb{R}^m$ for each $x \in \mathbb{R}^n$ and $t \in \{0, \ldots, N-1\}$, such that the control input sequence $\{u(t)\}_{t=0}^{N-1}$ with $u(t) = \mu(x(t);t)$ is admissible. One can conjecture at this point that the solution to Problem 1, even when the solution space is augmented to the class $\Pi' \supseteq \Pi$, will still be a feedback control policy which corresponds to a sequence of linear feedback control laws.

In the special case, when (5) is removed, Problem 1 reduces to the standard finite-horizon optimal covariance control problem for discrete-time stochastic linear systems. This problem was very recently addressed for the continuous-time case in [10], [11]. In the discrete-time case, when (5) is removed, the optimal control policy, π°, will belong to Π necessarily and is defined as follows: $\pi^{\circ} = \{\mu^{\circ}(x;t)\}_{t=0}^{N-1}$, where, for all $t \in \{0, \ldots, N-1\}$,

$$\mu(x;t) = -\mathbf{K}(t)x, \quad (7)$$

where the gain matrix $\mathbf{K}(t)$ satisfies

$$\mathbf{K}(t) = \left(\mathbf{R}(t) + \mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t+1)\mathbf{B}(t)^{\mathrm{T}}\right)^{-1} \mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t+1)\mathbf{A}(t), \quad (8)$$

and $\mathbf{S}(t)$ denotes the solution to the following discrete-time Riccati (matrix) recursive equation:

$$\mathbf{S}(t) = \mathbf{Q}(t) + \mathbf{A}(t)^{\mathrm{T}}\mathbf{S}(t+1)\mathbf{A}(t) - \mathbf{A}(t)^{\mathrm{T}}\mathbf{S}(t+1)\mathbf{B}(t) \times \left(\mathbf{R}(t) + \mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t+1)\mathbf{B}(t)\right)^{-1}\mathbf{B}(t)^{\mathrm{T}}\mathbf{S}(t+1)\mathbf{A}(t), \quad (9)$$

with boundary condition $\mathbf{S}(N) = \mathbf{S}_{\mathsf{f}}$, where the matrix $\mathbf{S}_{\mathsf{f}}$ belongs to $\overline{\mathbb{P}}_n$. In addition, the matrix $\mathbf{S}_{\mathsf{f}}$ is such that the state covariance $\mathbf{\Sigma}(t)$ of the closed-loop system driven by the optimal feedback control policy π°, which evolves according to the following matrix difference equation:

$$\mathbf{\Sigma}(t+1) = (\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t))\mathbf{\Sigma}(t)(\mathbf{A}(t) - \mathbf{B}(t)\mathbf{K}(t))^{\mathrm{T}} + \mathbf{C}(t)\mathbf{C}(t)^{\mathrm{T}}, \quad (10)$$

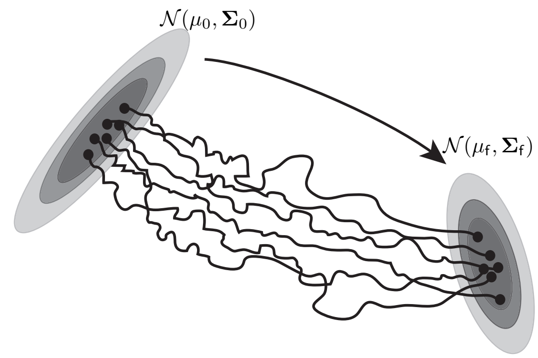


Fig. 1. The problem of steering the initial Gaussian distribution $\mathcal{N}(\mu_0, \mathbf{\Sigma}_0)$ of the state of a stochastic discrete-time linear system to a desired terminal Gaussian distribution $\mathcal{N}(\mu_{\mathsf{f}}, \mathbf{\Sigma}_{\mathsf{f}})$, at a given final stage $t = N$ subject to mean sum constraints involving quadratic functions of the state and the input sequences.

satisfies at stage $t = 0$ and at stage $t = N$ the following boundary conditions:

$$\mathbf{\Sigma}(0) = \mathbf{\Sigma}_0, \qquad \mathbf{\Sigma}(N) = \mathbf{\Sigma}_{\mathsf{f}}. \quad (11)$$

Next, we will address Problem 1 in the general case when the constraint (5) is present based on optimization techniques.

III. Practical Numerical Solution Techniques

In this section, we will first reduce Problem 1 to a deterministic nonlinear program (NLP) which we will subsequently associate with a convex program via simple relaxation techniques. We will show in particular, that Problem 1 is equivalent to a deterministic nonlinear program whose performance index is a convex quadratic function and its constraints are also quadratic functions without being, however, necessarily convex. The existence of non-convex quadratic constraints is due to the constraints on the terminal state covariance. Subsequently, we will employ a simple convex relaxation technique to associate the nonlinear program with a (suboptimal) convex program.

Next, we summarize the key steps for associating Problem 1 with a nonlinear program. By making use of standard results from the theory of discrete-time stochastic linear systems, we can express the solution to difference equation (1) as follows:

$$\boldsymbol{x} = \mathbf{H}\boldsymbol{u} + \mathbf{G}\boldsymbol{w} + \boldsymbol{x}_0, \quad (12)$$

where $\boldsymbol{x} := [x(0)^{\mathrm{T}}, \ldots, x(N)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n}$, $\boldsymbol{u} := [u(0)^{\mathrm{T}}, \ldots, u(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}$, $\boldsymbol{w} := [w(0)^{\mathrm{T}}, \ldots, w(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Np}$, and the matrices $\mathbf{H} \in \mathbb{R}^{(N+1)n \times (Nm)}$, $\mathbf{G} \in \mathbb{R}^{(N+1)n \times (Np)}$ are given by

$$\mathbf{H} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{B}(0) & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{\Phi}(2,1)\mathbf{B}(0) & \mathbf{B}(1) & \ldots & \mathbf{0} \\ \vdots & \vdots & \ldots & \vdots \\ \mathbf{\Phi}(N,1)\mathbf{B}(0) & \mathbf{\Phi}(N,2)\mathbf{B}(1) & \ldots & \mathbf{B}(N-1) \end{bmatrix},$$

$$\mathbf{G} := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{C}(0) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{\Phi}(2,1)\mathbf{C}(0) & \mathbf{C}(1) & \dots & \mathbf{0} \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{\Phi}(N,1)\mathbf{C}(0) & \mathbf{\Phi}(N,2)\mathbf{C}(1) & \dots & \mathbf{C}(N-1) \end{bmatrix},$$

and $\boldsymbol{x}_0 := \mathbf{\Gamma} x_0$, where

$$\mathbf{\Gamma} := \begin{bmatrix} \mathbf{I} & \mathbf{\Phi}(1,0)^{\mathrm{T}} & \dots & \mathbf{\Phi}(N,0)^{\mathrm{T}} \end{bmatrix}^{\mathrm{T}},$$
$$\mathbf{\Phi}(t,\tau) := \mathbf{A}(t-1)\dots\mathbf{A}(\tau), \qquad \mathbf{\Phi}(t,t) = \mathbf{I},$$

for $t \in \{1,\dots,N\}$ and $\tau \in \{0,\dots,t-1\}$. Because $u(t) = -\mathbf{K}(t)x(t)$ for all $t \in \{0,\dots,N-1\}$, we have that

$$\boldsymbol{u} = -\mathrm{bdiag}(\mathbf{K}(0),\mathbf{K}(1),\dots,\mathbf{K}(N-1)) \times [x(0)^{\mathrm{T}},\dots,x(N-1)^{\mathrm{T}}]^{\mathrm{T}},$$

or equivalently,

$$\boldsymbol{u} = \mathbf{F}\boldsymbol{x}, \quad \mathbf{F} := \begin{bmatrix} \mathrm{bdiag}(-\mathbf{K}(0),\dots,-\mathbf{K}(N-1)), & \mathbf{0} \end{bmatrix}. \tag{13}$$

It follows readily that

$$\boldsymbol{x} = \mathbf{P}_{\boldsymbol{x}}\boldsymbol{w} + \boldsymbol{\chi}, \quad \boldsymbol{u} = \mathbf{P}_{\boldsymbol{u}}\boldsymbol{w} + \boldsymbol{\nu}, \tag{14}$$

where

$$\mathbf{P}_{\boldsymbol{x}} := (\mathbf{I}-\mathbf{HF})^{-1}\mathbf{G} = \mathbf{G} + \mathbf{HF}(\mathbf{I}-\mathbf{HF})^{-1}\mathbf{G}, \tag{15}$$
$$\mathbf{P}_{\boldsymbol{u}} := \mathbf{F}(\mathbf{I}-\mathbf{HF})^{-1}\mathbf{G}, \tag{16}$$

and

$$\boldsymbol{\chi} := (\mathbf{I}-\mathbf{HF})^{-1}\boldsymbol{x}_0 = \boldsymbol{x}_0 + \mathbf{HF}(\mathbf{I}-\mathbf{HF})^{-1}\boldsymbol{x}_0, \tag{17}$$
$$\boldsymbol{\nu} := \mathbf{F}(\mathbf{I}-\mathbf{HF})^{-1}\boldsymbol{x}_0. \tag{18}$$

Note that the inverse of $\mathbf{I}-\mathbf{HF}$ is always well defined given that $\mathbf{T} := \mathbf{HF}$ is a block lower triangular matrix and in addition, its block diagonal elements are zero matrices (this follows readily from the structure of the block matrix $\mathbf{H}$). The performance index can be written as follows:

$$\begin{aligned} J(\pi) &= \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}(t)u(t)\Big] \\ &= \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t) + x(t)^{\mathrm{T}}\mathbf{K}(t)^{\mathrm{T}}\mathbf{R}(t)\mathbf{K}(t)x(t)\Big] \\ &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\boldsymbol{u}\Big], \end{aligned} \tag{19}$$

where $\boldsymbol{\mathcal{Q}} := \mathrm{diag}(\mathbf{Q}(0),\dots,\mathbf{Q}(N-1),\mathbf{0})$ and $\boldsymbol{\mathcal{R}} := \mathrm{diag}(\mathbf{R}(0),\dots,\mathbf{R}(N-1))$. In view of Eqs. (14)–(18), we can write the cost function as follows:

$$\begin{aligned} J(\pi) &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}\boldsymbol{u}\Big] \\ &= \mathbb{E}\Big[(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{w}+\boldsymbol{\chi})^{\mathrm{T}}\boldsymbol{\mathcal{Q}}(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{w}+\boldsymbol{\chi}) \\ &\quad + (\mathbf{P}_{\boldsymbol{u}}\boldsymbol{w}+\boldsymbol{\nu})^{\mathrm{T}}\boldsymbol{\mathcal{R}}(\mathbf{P}_{\boldsymbol{u}}\boldsymbol{w}+\boldsymbol{\nu})\Big], \end{aligned} \tag{20}$$

which, in view of Eqs. (2)-(3) and the fact that x_0 and $w(k)$ are mutually independent for all $k \in \{0,\dots,N-1\}$, can be written:

$$\begin{aligned} J(\pi) &= \mathrm{trace}(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{\mathcal{Q}}\mathbf{P}_{\boldsymbol{x}}^{\mathrm{T}} + \mathbf{P}_{\boldsymbol{u}}\boldsymbol{\mathcal{R}}\mathbf{P}_{\boldsymbol{u}}^{\mathrm{T}}) \\ &\quad + \mathrm{trace}(\boldsymbol{\mathcal{Q}}\mathbb{E}\big[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}\big] + \boldsymbol{\mathcal{R}}\mathbb{E}\big[\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}\big]). \end{aligned} \tag{21}$$

As is highlighted in [16], it is not clear at this point whether $J(\pi)$ is convex in $\mathbf{F}$. To overcome this problem, we will make use of an intuitive and straightforward bilinear transformation suggested in [16], which will allow us to express the performance index as a convex function of a new decision variable, which is denoted as $\mathbf{\Psi}$ and is defined as follows:

$$\mathbf{\Psi} := \mathbf{F}(\mathbf{I}-\mathbf{HF})^{-1}. \tag{22}$$

Using similar arguments as those in the discussion following Eq. (18), we conclude that $\mathbf{\Psi}$ is always well-defined and it is actually a block lower triangular matrix. It follows immediately from (22) that

$$\mathbf{F} = (\mathbf{I}+\mathbf{\Psi}\mathbf{H})^{-1}\mathbf{\Psi}, \tag{23}$$

where the right hand side of Eq. (23) is well defined based again on similar arguments as those in the discussion following Eq. (18). In view of (23), (15)–(16) and (17)–(18) become, respectively,

$$\mathbf{P}_{\boldsymbol{x}} := (\mathbf{I}+\mathbf{H}\mathbf{\Psi})\mathbf{G}, \qquad \mathbf{P}_{\boldsymbol{u}} := \mathbf{\Psi}\mathbf{G}, \tag{24}$$

and

$$\boldsymbol{\chi} := (\mathbf{I}+\mathbf{H}\mathbf{\Psi})\boldsymbol{x}_0, \qquad \boldsymbol{\nu} := \mathbf{\Psi}\boldsymbol{x}_0. \tag{25}$$

Thus, we have that

$$\mathbb{E}\big[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}\big] = (\mathbf{I}+\mathbf{H}\mathbf{\Psi})\mathbf{\Gamma}\mathbf{\Sigma}_0\mathbf{\Gamma}^{\mathrm{T}}(\mathbf{I}+\mathbf{H}\mathbf{\Psi})^{\mathrm{T}}, \tag{26}$$
$$\mathbb{E}\big[\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}\big] = \mathbf{\Psi}\mathbf{\Gamma}\mathbf{\Sigma}_0\mathbf{\Gamma}^{\mathrm{T}}\mathbf{\Psi}^{\mathrm{T}}, \tag{27}$$

where in the previous derivations, we have used the following identity

$$\mathbb{E}\big[\boldsymbol{x}_0\boldsymbol{x}_0^{\mathrm{T}}\big] = \mathbf{\Gamma}^{\mathrm{T}}\mathbb{E}\big[x_0x_0^{\mathrm{T}}\big]\mathbf{\Gamma} = \mathbf{\Gamma}^{\mathrm{T}}\mathbf{\Sigma}_0\mathbf{\Gamma}.$$

Therefore, in view of Eqs. (21), (24) and (26)-(27), and the fact that the quantities $\mathbf{P}_{\boldsymbol{x}}$ and $\mathbf{P}_{\boldsymbol{u}}$ are affine functions of the new decision variable $\mathbf{\Psi}$, it follows that the performance index $J(\pi)$ is a convex quadratic function of $\mathbf{\Psi}$.

Similarly, the function $h(\cdot)$ can be written as follows:

$$\begin{aligned} h(\boldsymbol{x},\boldsymbol{u}) &:= \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}_c(t)x(t) + u(t)^{\mathrm{T}}\mathbf{R}_c(t)u(t)\Big] \\ &= \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}_c(t)x(t) + x(t)^{\mathrm{T}}\mathbf{K}(t)^{\mathrm{T}}\mathbf{R}_c(t)\mathbf{K}(t)x(t)\Big] \\ &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_c\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}_c\boldsymbol{u}\Big], \end{aligned} \tag{28}$$

where $\boldsymbol{\mathcal{Q}}_c := \mathrm{diag}(\mathbf{Q}_c(0), \ldots, \mathbf{Q}_c(N-1), \mathbf{0})$ and $\boldsymbol{\mathcal{R}}_c := \mathrm{diag}(\mathbf{R}_c(0), \ldots, \mathbf{R}_c(N-1))$, or equivalently,

$$\begin{aligned}
h(\boldsymbol{x}, \boldsymbol{u}) &= \mathbb{E}\Big[\boldsymbol{x}^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_c\boldsymbol{x} + \boldsymbol{u}^{\mathrm{T}}\boldsymbol{\mathcal{R}}_c\boldsymbol{u}\Big] \\
&= \mathbb{E}\Big[(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{w} + \boldsymbol{\chi})^{\mathrm{T}}\boldsymbol{\mathcal{Q}}_c(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{w} + \boldsymbol{\chi}) \\
&\qquad + (\mathbf{P}_{\boldsymbol{u}}\boldsymbol{w} + \boldsymbol{\nu})^{\mathrm{T}}\boldsymbol{\mathcal{R}}_c(\mathbf{P}_{\boldsymbol{u}}\boldsymbol{w} + \boldsymbol{\nu})\Big] \\
&= \mathrm{trace}(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{\mathcal{Q}}_c\mathbf{P}_{\boldsymbol{x}}^{\mathrm{T}} + \mathbf{P}_{\boldsymbol{u}}\boldsymbol{\mathcal{R}}_c\mathbf{P}_{\boldsymbol{u}}^{\mathrm{T}}) \\
&\qquad + \mathrm{trace}(\boldsymbol{\mathcal{Q}}_c\mathbb{E}\big[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}\big] + \boldsymbol{\mathcal{R}}_c\mathbb{E}\big[\boldsymbol{\nu}\boldsymbol{\nu}^{\mathrm{T}}\big]), \qquad (29)
\end{aligned}$$

where in the last derivation, we have used Eqs. (2)-(3). Using similar arguments with those in the discussion on the convexity of the performance index $J(\pi)$ as function of the new decision variable $\boldsymbol{\Psi}$, we conclude that $h(\boldsymbol{x}, \boldsymbol{u})$ corresponds to a convex quadratic function of $\boldsymbol{\Psi}$.

Next, we investigate whether the matrix equality constraint on the terminal state covariance $\mathbb{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big] - \boldsymbol{\Sigma}_{\mathsf{f}} = \mathbf{0}$ can also be written as a convex quadratic equality constraint in $\boldsymbol{\Psi}$. To this aim, we will express $\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big]$ as a function of $\boldsymbol{\Psi}$. Specifically, in view of Eqs. (2)-(3) and (14)–(18), and the fact that $x(0)$ is independent of $w(k)$ for all $k \in \{0, \ldots, N-1\}$, it is straightforward to show that

$$\begin{aligned}
\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big] &= \mathbb{E}\big[(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{w} + \boldsymbol{\chi})(\mathbf{P}_{\boldsymbol{x}}\boldsymbol{w} + \boldsymbol{\chi})^{\mathrm{T}}\big] \\
&= \mathbf{P}_{\boldsymbol{x}}\mathbb{E}\big[\boldsymbol{w}\boldsymbol{w}^{\mathrm{T}}\big]\mathbf{P}_{\boldsymbol{x}}^{\mathrm{T}} + \mathbb{E}\big[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}\big] \\
&= \mathbf{P}_{\boldsymbol{x}}\mathbf{P}_{\boldsymbol{x}}^{\mathrm{T}} + \mathbb{E}\big[\boldsymbol{\chi}\boldsymbol{\chi}^{\mathrm{T}}\big]. \qquad (30)
\end{aligned}$$

In view of (24) and (26), Eq. (30) can be written as follows:

$$\begin{aligned}
\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big] &= (\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})(\mathbf{G}\mathbf{G}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2} \\
&\qquad \times (\mathbf{G}\mathbf{G}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2}(\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})^{\mathrm{T}}. \qquad (31)
\end{aligned}$$

Now, because $x_{\mathsf{f}} = x(N) = \boldsymbol{\mathcal{P}}_N\boldsymbol{x}$, where

$$\boldsymbol{\mathcal{P}}_N := [\mathbf{0} \ldots \mathbf{I}] \in \mathbb{R}^{n \times (N+1)n},$$

we can write

$$\mathbb{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big] = \boldsymbol{\mathcal{P}}_N\mathbb{E}\big[\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}}\big]\boldsymbol{\mathcal{P}}_N^{\mathrm{T}} = \mathbf{Z}\mathbf{Z}^{\mathrm{T}},$$

where $\mathbf{Z} := \boldsymbol{\mathcal{P}}_N(\mathbf{I} + \mathbf{H}\boldsymbol{\Psi})(\mathbf{G}\mathbf{G}^{\mathrm{T}} + \boldsymbol{\Gamma}\boldsymbol{\Sigma}_0\boldsymbol{\Gamma}^{\mathrm{T}})^{1/2}$. Note that $\mathbf{Z}$ is itself an affine function of the new decision variable, $\boldsymbol{\Psi}$. Now, in view of Definition 6.6.44 in [17], the symmetric matrix-valued function $f(\cdot) : \mathbb{R}^{n \times n} \to \mathbb{S}_n$, where $f(\mathbf{Z}) := \mathbf{Z}\mathbf{Z}^{\mathrm{T}} - \boldsymbol{\Sigma}_{\mathsf{f}}$, will be convex in $\mathbf{Z}$ if and only if:

$$f(\alpha\mathbf{Z}_1 + (1-\alpha)\mathbf{Z}_2) \preceq \alpha f(\mathbf{Z}_1) + (1-\alpha)f(\mathbf{Z}_2), \qquad (32)$$

for any $\alpha \in [0, 1]$ and for any $\mathbf{Z}_1$ and $\mathbf{Z}_2 \in \mathbb{R}^{n \times n}$. Establishing the validity of (32) is rather straightforward. Thus, we conclude immediately that, by definition, the function $f(\cdot)$ is convex, and in particular, it is a convex quadratic function of $\mathbf{Z}$. Because $\mathbf{Z}$ is in turn an affine function of $\boldsymbol{\Psi}$, we conclude that the function $f(\cdot)$ is a convex quadratic function of $\boldsymbol{\Psi}$. However, the fact that $f(\cdot)$ is a convex quadratic function of $\boldsymbol{\Psi}$ does not necessarily imply that the $n(n+1)/2$ quadratic, scalar equalities that result from the matrix equation $f(\mathbf{Z}) = \mathbf{0}$ are necessarily convex. An easy way to see why this is the case is to consider an example with $\mathbf{Z} = \left[\begin{smallmatrix} z_1 & z_2 \\ z_3 & z_4 \end{smallmatrix}\right] \in \mathbb{R}^2$, in which case

$$f(\mathbf{Z}) := \left[\begin{smallmatrix} z_1^2 + z_2^2 & z_1z_3 + z_2z_4 \\ z_1z_3 + z_2z_4 & z_3^2 + z_4^2 \end{smallmatrix}\right] - \boldsymbol{\Sigma}_{\mathsf{f}},$$

which gives two convex quadratic equalities, namely the ones corresponding to the diagonal elements of $f(\mathbf{Z})$, and one which is non-convex, namely the one corresponding to the off-diagonal elements of $f(\mathbf{Z})$. Therefore, in general, Problem 1 cannot be associated with a convex program. Instead of addressing the NLP, which is known to pose significant challenges in general, one can associate the NLP with a nonlinear program by employing convex relaxation techniques [2]. Perhaps, the most direct, convex relaxation technique is to substitute the equality constraint $f(\mathbf{Z}) = \mathbf{0}$ with the following convex constraint: $f(\mathbf{Z}) \preceq \mathbf{0}$. In particular, the inequality $f(\mathbf{Z}) \preceq \mathbf{0}$, which is equivalent to $\boldsymbol{\Sigma}_{\mathsf{f}} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}} \succeq \mathbf{0}$, can be written as a (convex) positive semi-definite constraint:

$$\boldsymbol{\mathcal{X}} := \begin{bmatrix} \boldsymbol{\Sigma}_{\mathsf{f}} & \mathbf{Z} \\ \mathbf{Z}^{\mathrm{T}} & \mathbf{I} \end{bmatrix} \succeq \mathbf{0},$$

given that $\boldsymbol{\Sigma}_{\mathsf{f}} - \mathbf{Z}\mathbf{Z}^{\mathrm{T}}$ is the Schur complement of $\mathbf{I}$ in $\boldsymbol{\mathcal{X}}$.

Because, the terminal state covariance $\boldsymbol{\Sigma}_{\mathsf{f}}$ can be viewed as a measure of the dispersion of the endpoints of a representative sample of state trajectories of the close-loop system, the relaxation of the non-convex equality constraint $f(\mathbf{Z}) = \mathbf{0}$ to the convex constraint $f(\mathbf{Z}) \preceq \mathbf{0}$ leads to desirable results in practice. This is because, when $f(\mathbf{Z}) \preceq \mathbf{0}$, or equivalently, $\mathbf{0} \preceq \mathbf{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big] \preceq \boldsymbol{\Sigma}_{\mathsf{f}}$, the endpoints, x_{f}, of a representative sample of trajectories associated with the convex program may not follow exactly the goal terminal Gaussian distribution $\mathcal{N}(0, \boldsymbol{\Sigma}_{\mathsf{f}})$ (this would be possible, if the equality, but non-convex, constraint $f(\mathbf{Z}) = \mathbf{0}$ was enforced instead). However, the same endpoints are more likely to be concentrated near the origin than those drawn from the goal Gaussian distribution (here, $\mathbb{E}\big[x_{\mathsf{f}}x_{\mathsf{f}}^{\mathrm{T}}\big]$ can be viewed as a measure of the dispersion of the terminal states x_{f} from a representative sample of trajectories).

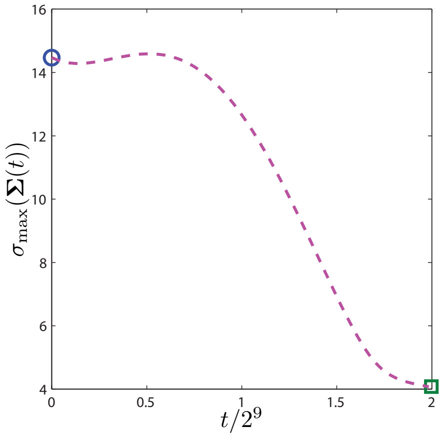
IV. Numerical Simulations

In this section, we present numerical simulations for a simple example in order to illustrate the key ideas of the previous sections. In particular, we consider the finite-horizon covariance control problem subject to the following difference equations:

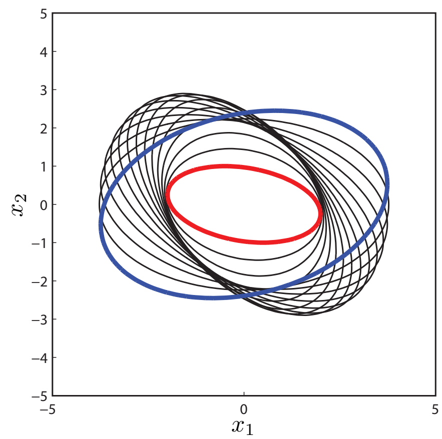
$$\begin{aligned}
&x_1(t+1) = x_1(t) + 2^{-9}x_2(t), \\
&x_2(t+1) = x_2(t) + 2^{-9}(-x_1(t) + u(t)) + 2^{-4.5}w(t),
\end{aligned}$$

with $N = 2^{10}$. We assume that the initial state, x_0, at stage $t = 0$ is drawn from the Gaussian distribution $\mathcal{N}(0, \boldsymbol{\Sigma}_0)$ and the objective is to drive the uncertain state of the system to a goal terminal state, x_{f}, at stage $t = 2^{10}$, which is drawn from the Gaussian distribution $\mathcal{N}(0, \boldsymbol{\Sigma}_{\mathsf{f}})$, where

$$\boldsymbol{\Sigma}_0 = \left[\begin{smallmatrix} 14 & 2 \\ 2 & 6 \end{smallmatrix}\right], \quad \boldsymbol{\Sigma}_{\mathsf{f}} = \left[\begin{smallmatrix} 4 & -0.5 \\ -0.5 & 1 \end{smallmatrix}\right],$$



(a) Time-evolution of $\sigma_{\max}(\mathbf{\Sigma}(t))$.



(b) Time-evolution of 1-level sets of $q(x;t) := x^{\mathrm{T}}\mathbf{\Sigma}(t)^{-1}x$.

Fig. 2. The values of $\sigma_{\max}(\mathbf{\Sigma}(t))$ at different stages $t \in \{0,\ldots,2^{10}\}$ allows us to observe the rate at which the state covariance $\mathbf{\Sigma}(t)$ converges to the $\mathbf{\Sigma}_{\mathrm{f}}$ at $t = N = 2^{10}$ (Fig. 2(a)). On the other hand, the evolution of the ellipsoids that correspond to the 1-level sets of the quadratic function $q(x;t) = x^{\mathrm{T}}\mathbf{\Sigma}(t)^{-1}x$ at different stages $t \in \{0,\ldots,2^{10}\}$ illustrates the trajectory that the state covariance follows until it converges to $\mathbf{\Sigma}_{\mathrm{f}}$ (Fig. 2(b)). In the latter figure, the blue and the red ellipsoids correspond to the 1-level sets of $q(x;0)$ and $q(x;2^{10})$, respectively.

while minimizing the performance index $\mathbb{E}\big[\sum_{t=0}^{2^{10}-1} u(t)^{\mathrm{T}}u(t)\big]$. Figure 2 illustrates the time-evolution of the state covariance matrix $\mathbf{\Sigma}(t)$ of the closed-loop system, when the latter is driven by the feedback control policy that solves Problem 1. In particular, the rate of convergence of the state covariance $\mathbf{\Sigma}(t)$ to its terminal value, $\mathbf{\Sigma}_{\mathrm{f}}$, is illustrated in Fig. 2(a) via the evolution of its maximum singular value, $\sigma_{\max}(\mathbf{\Sigma}(t))$, at different stages $t \in \{0,\ldots,2^{10}\}$. Furthermore, to better illustrate the "trajectory" of the covariance from $\mathbf{\Sigma}_0$ to its goal destination, $\mathbf{\Sigma}_{\mathrm{f}}$, in Fig. 2(b), we show the time-evolution of the ellipsoids that correspond to the 1-level sets of the quadratic form $q(x;t) := x^{\mathrm{T}}\mathbf{\Sigma}(t)^{-1}x$ at different stages $t \in \{0,\ldots,2^{10}\}$. (Note that this quadratic form appears in the expression of the density function of the multi-variate Gaussian distribution $\mathcal{N}(0,\mathbf{\Sigma}(t))$ from which the uncertain state of the system is drawn at each stage t).

V. Conclusion

In this work, we have addressed a covariance control problem for discrete-time stochastic linear systems, which we formulated as a stochastic optimal control problem subject to mean sum constraints involving quadratic functions of the state and the control input sequences. We have shown that the stochastic optimal control problem is equivalent to a deterministic nonlinear program, which, via a judicious choice of the decision variable, has a convex quadratic performance index and is subject to inequality and equality quadratic constraints, which, however, are not necessarily convex. We have shown that by employing a simple relaxation technique, this nonlinear program can be associated with a deterministic, convex program. Although the latter problem cannot yield feedback control policies that will force the endpoints of the closed-loop trajectories of the system to follow the goal terminal Gaussian distribution exactly, it will still lead to the generation of desirable trajectories that will be concentrated near the origin. In the future, we plan to extend the numerical techniques presented herein to covariance control problems with partial and imperfect state information.

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