Optimal Synthesis of the Zermelo–Markov–Dubins Problem in a Constant Drift Field[1]

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Abstract

We consider the optimal synthesis of the Zermelo–Markov–Dubins problem, that is, the problem of steering a vehicle with the kinematics of the Isaacs–Dubins car in minimum time in the presence of a drift field. By using standard optimal control tools, we characterize the family of control sequences that are sufficient for complete controllability and necessary for optimality for the special case of a constant field. Furthermore, we present a semi-analytic scheme for the characterization of a (nearly) optimal synthesis of the minimum-time problem. Finally, we establish a direct correspondence between the optimal syntheses of the Markov–Dubins and the Zermelo–Markov–Dubins problems by means of a discontinuous mapping.

Keywords: Markov-Dubins problem, optimal synthesis, Zermelo's navigation problem, nonholonomic systems

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1 Introduction

We consider the problem of guiding an aerial or marine vehicle with turning constraints to a prescribed terminal state in the presence of a constant drift field in minimum time. In particular, we assume that the vehicle travels in the plane with constant forward speed such that the direction of its forward velocity (heading) cannot be changed faster than a prescribed upper bound. Therefore, the kinematics of the vehicle in the absence of the drift field coincides with the kinematic model of the Isaacs-Dubins (ID for short) car [1–3]. The steering problem considered in this work is essentially a combination of two classical optimization problems, namely a problem posed by A. A. Markov in the late 1880's, dealing with the characterization of planar curves of minimal length of bounded curvature, and a problem posed by E. Zermelo in the early 1930's, dealing with the characterization of the planar minimum-time paths for a vehicle with single integrator kinematics traveling in a flow-field induced by local currents/winds [4]. Zermelo solved this problem for the general case of a both temporally and spatially varying drift field using "an extraordinary ingenious method" according to Carathéodory [5]. The problem posed by Markov was solved by Dubins [1]; henceforth, we shall refer to this problem as the Markov–Dubins (MD for short) problem as suggested by Sussmann [6]. For a discussion on the history of the MD problem, the reader is referred to [3, 7]. In addition, some interesting variations of the MD problem can be found in [8–16]. In this work, we refer to the combination of the Zermelo's navigation and the MD problems as the Zermelo–Markov–Dubins (ZMD for short) problem.

The ZMD problem for the special case of a constant drift field was first posed by McGee and Hedrick in [13]. The authors of [13] examined this special case of the ZMD indirectly, by interpreting the ZMD problem as a minimum-time intercept problem of a non-maneuvering target. They conjectured that, under some mild modifications, the family of extremals that is sufficient for optimality for the standard MD problem is sufficiently rich to provide feasible paths to the ZMD problem for an arbitrarily pair of boundary states. A numerical scheme for the computation of the Dubins-like paths proposed in [13], which may involve the solution of a set of coupled transcendental equations, has been proposed in [17]. A set of equations in triangular form that solves the same problem was presented in our previous work [18]. It is worth-mentioning that the equivalent formulation of the ZMD problem as a minimum-time intercept problem of a non-maneuvering target, as discussed in [13], is closely related to the intercept problem addressed by Glizer in [19, 20]. In particular, the author of [19, 20] considered an optimization problem where the hard input constraints were relaxed with the addition of a cost term penalizing the control effort, for which he characterized both exact and simpler approximate solutions in [19] and [20], respectively.

The objectives of this work are twofold. First, we revisit the ZMD problem for the special case of a constant drift field, and we rigorously characterize the structure of its extremals. Moreover, we highlight the existence of extremals of the ZMD problem that do not appear in the solution of the standard MD problem. The end result of our analysis is a family of control sequences that drive the vehicle from a given initial to an arbitrary (prescribed) terminal state in (nearly) minimum time. Second, we present a (nearly) optimal *synthesis* of the ZMD problem based on the proposed family of extremals. Furthermore, we establish the direct correspondence between the syntheses of the MD and the ZMD problems by means of a discontinuous mapping; something that significantly simplifies the characterization of the optimal synthesis of the ZMD problem. The detailed analysis and presentation of the optimal synthesis of the ZMD problem, which, to the best of the authors' knowledge, has never been addressed in the literature, along with its comparison with the synthesis of the standard MD problem presented in [21–23], are the main contributions of this work.

The rest of the paper is organized as follows. In Section 2, we formulate the ZMD problem as an optimal control problem and establish the existence of its solutions. In Section 3, we characterize the family of extremals for the ZMD problem that is sufficient for controllability and necessary for optimality. A nearly optimal synthesis of the ZMD problem is presented in Section 4. Finally, Section 5 provides some concluding remarks.

2 Kinematic Model and Problem Formulation

In this section, we introduce the kinematic model of the vehicle and examine its controllability. Subsequently, we formulate the minimum-time problem and examine its feasibility.

2.1 Kinematic Model and Problem Formulation

We consider an aerial/marine vehicle whose motion is described by the following set of equations

$$\dot{x} = \cos\theta + w_x, \qquad \dot{y} = \sin\theta + w_y, \qquad \dot{\theta} = \frac{u}{\rho}, \qquad t \geq 0, \tag{1}$$

where $(x, y) \in \mathbb{R}^2$ are the Cartesian coordinates of a reference point of the vehicle, $\theta \in \mathbb{S}^1$ is the direction (heading) of the vehicle's forward velocity, u is the control input, $w := (w_x, w_y)$ is the constant drift field induced by local winds/currents, and ρ is a positive constant. We write $w := \nu(\cos\phi, \sin\phi)$, where $\nu = |w|$ and $\phi \in \mathbb{S}^1$ is the direction of the drift. We assume that the set of admissible control inputs, denoted by $\mathcal{U}$, consists of all measurable functions defined on $[0, T]$, where $T \geq 0$, taking values in $U := [-1, 1]$. Next, we formulate the ZMD problem as a minimum-time problem for the system (1).

Problem 2.1 (ZMD Minimum-Time Problem)**.** *Given the system described by Eq.* (1) *and a state* $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathbb{R}^2 \times \mathbb{S}^1$*, determine a control input* $u^* \in \mathcal{U}$ *such that*

(i) *The trajectory* $\mathsf{x}^* : [0, T_\mathsf{f}] \mapsto \mathbb{R}^2 \times \mathbb{S}^1$ *generated by the control* u^* *satisfies the boundary conditions*

$$\mathsf{x}^*(0) = (0, 0, 0), \qquad \mathsf{x}^*(T_\mathsf{f}) = (x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}). \tag{2}$$

(ii) *The control* u^* *minimizes along the trajectory* x^* *the cost functional* $J(u) := T_\mathsf{f}$*, where* T_f *is the free final time.*

Note that if we assume, in addition, that the input value set is unbounded, that is, the input u can contain impulses and that both $\theta(0)$ and $\theta(T_\mathsf{f})$ are free, (in which case, θ acts as a control input), then Problem 2.1 reduces to the Zermelo's navigation problem.

Next, we present an intercept problem of a non-maneuvering target with a prescribed intercept angle, which yields an alternative formulation of Problem 2.1. In particular, the equations of motion of the interceptor are given by

$$\dot{x}_{\mathcal{P}} = \cos\theta_{\mathcal{P}}, \qquad \dot{y}_{\mathcal{P}} = \sin\theta_{\mathcal{P}}, \qquad \dot{\theta}_{\mathcal{P}} = \frac{u}{\rho}, \qquad t \geq 0, \tag{3}$$

where $(x_{\mathcal{P}}, y_{\mathcal{P}}) \in \mathbb{R}^2$ are the Cartesian coordinates of the interceptor with respect to an inertial frame attached to its initial position, and $\theta_{\mathcal{P}} \in \mathbb{S}^1$ is the direction of the interceptor's velocity. Note that the kinematics of the interceptor coincide with those of the ID car. Furthermore, the target motion is described by the following set of equations

$$\dot{x}_{\mathcal{T}} = -w_x, \qquad \dot{y}_{\mathcal{T}} = -w_y, \qquad t \geq 0, \tag{4}$$

where $(x_{\mathcal{T}}, y_{\mathcal{T}}) \in \mathbb{R}^2$ are the Cartesian coordinates of the non-maneuvering target measured with respect to an inertial frame attached to the initial position of the interceptor and $(-w_x, -w_y)$ are the components of the (constant) velocity of the target expressed in the same frame.

Problem 2.2. *Consider an interceptor and a non-maneuvering target, whose kinematics are described by Eq.* (3)*, and Eq.* (4)*, respectively, and let* $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathbb{R}^2 \times \mathbb{S}^1$ *be given. Determine an intercept control law* $u^* \in \mathcal{U}$ *such that*

(i) *The trajectory of the interceptor* $\mathsf{x}_{\mathcal{P}}^* : [0, T_{\mathsf{f}}] \mapsto \mathbb{R}^2 \times \mathbb{S}^1$*, where* $\mathsf{x}_{\mathcal{P}}^* := (x_{\mathcal{P}}^*, y_{\mathcal{P}}^*, \theta_{\mathcal{P}}^*)$*, generated by the control* u^* *and the (control-free) trajectory of the non-maneuvering target* $\mathsf{x}_{\mathcal{T}}^* : [0, T_{\mathsf{f}}] \mapsto \mathbb{R}^2$*, where* $\mathsf{x}_{\mathcal{T}}^* := (x_{\mathcal{T}}^*, y_{\mathcal{T}}^*)$*, satisfy the boundary conditions*

$$\mathsf{x}_{\mathcal{P}}^*(0) = (0, 0, 0), \qquad \mathsf{x}_{\mathcal{T}}(0) = (x_{\mathsf{f}}, y_{\mathsf{f}}), \tag{5}$$

$$x_{\mathcal{P}}^*(T_{\mathsf{f}}) = x_{\mathcal{T}}^*(T_{\mathsf{f}}), \qquad y_{\mathcal{P}}^*(T_{\mathsf{f}}) = y_{\mathcal{T}}^*(T_{\mathsf{f}}), \qquad \theta_{\mathcal{P}}^*(T_{\mathsf{f}}) = \theta_{\mathsf{f}}. \tag{6}$$

(ii) *The intercept control law* u^* *minimizes the intercept time.*

Next we show that Problems 2.1 and 2.2 are equivalent, in the sense that a control $u^* \in \mathcal{U}$ is a solution of Problem 2.1 if and only if is a solution of Problem 2.2, and vice versa. To this aim, let assume that the control input $u^* \in \mathcal{U}$ drives the system (1) from $\mathsf{x} = (0, 0, 0)$ to $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathbb{R}^2 \times \mathbb{S}^1$, in minimum time T_f. Next, let us apply the same control to the interceptor (3), which consequently reaches a state $(x_\mathcal{P}(T_\mathsf{f}), y_\mathcal{P}(T_\mathsf{f}), \theta(T_\mathsf{f}))$ at time $t = T_\mathsf{f}$. At the same time, the target reaches the point $(x_\mathcal{T}(T_\mathsf{f}), y(T_\mathsf{f})) = (x_\mathsf{f} - w_x T_\mathsf{f}, y_\mathsf{f} - w_y T_\mathsf{f})$. Let us consider the state transformation $\chi := x - x_\mathcal{P}$, $\psi := y - y_\mathcal{P}$, $\vartheta := \theta - \theta_\mathcal{P}$. It follows readily that

$$\dot{\chi} = \cos\theta + w_x - \cos\theta_\mathcal{P}, \qquad \dot{\psi} = \sin\theta + w_y - \sin\theta_\mathcal{P}, \qquad \dot{\vartheta} := u^* - u_\mathcal{P}, \tag{7}$$

where $\chi(0) = 0$, $\psi(0) = 0$, and $\theta(0) = \theta_\mathcal{P}(0) = 0$. Thus for $u_\mathcal{P} = u^*$, it follows that $\vartheta = \vartheta(0) = 0$, and thus $\theta = \theta_\mathcal{P}$, which implies that $\theta_\mathcal{P}(T_\mathsf{f}) = \theta_\mathsf{f}$ and, in addition, $\chi(T_\mathsf{f}) = w_x T_\mathsf{f}$, $\psi(T_\mathsf{f}) = w_y T_\mathsf{f}$. Therefore, $x_\mathcal{P}(T_\mathsf{f}) = x(T_\mathsf{f}) - w_x T_\mathsf{f} = x_\mathsf{f} - w_x T_\mathsf{f}$, and $y_\mathcal{P}(T_\mathsf{f}) = y(T_\mathsf{f}) - w_y T_\mathsf{f} = y_\mathsf{f} - w_y T_\mathsf{f}$. It follows that $x_\mathcal{P}(T_\mathsf{f}) = x_\mathcal{T}(T_\mathsf{f})$ and $y_\mathcal{P}(T_\mathsf{f}) = y_\mathcal{T}(T_\mathsf{f})$ and $\theta_\mathcal{P}(T_\mathsf{f}) = \theta_\mathsf{f}$. Thus at $t = T_\mathsf{f}$, the target is intercepted with the desired intercept angle θ_f. Now, let assume that there exists a control law $u'_\mathcal{P}$ different than $u_\mathcal{P} = u^*$ that steers the interceptor to the target at time $t = T'_\mathsf{f} < T_\mathsf{f}$. It is easy to show, by using a similar line of argument as before, that the control $u'_\mathcal{P}$ would steer the system (1) to $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f})$ at time $t = T'_\mathsf{f} < T_\mathsf{f}$, that is, faster than the minimum-time control u^*. Thus we have reached a contradiction and the equivalence of Problems 2.1 and 2.2 has been established.

At this point, it is worth mentioning that the ZMD problem was indirectly examined in [13], where the authors have analyzed the equivalent formulation of the ZMD problem as a minimum-time intercept problem of a non-maneuvering target (Problem 2.2). In this work, we will address the original formulation of the ZMD problem (Problem 2.1) directly, although in the subsequent analysis, we shall also employ the equivalent formulation of the ZMD problem as an intercept problem of a non-maneuvering target (Problem 2.2).

2.2 Controllability in the Case of a Constant Drift Field

Before proceeding to the solution of Problem 2.1, we examine its feasibility by studying the controllability of the system described by Eq. (1). The following proposition provides necessary and sufficient conditions for the complete controllability of the system described by Eq. (1).

Proposition 2.1. *Let $w = \nu(\cos\phi, \sin\phi)$ be a constant drift field. Then the system described by Eq. (1) is completely controllable if and only if $\nu < 1$.*

Proof. We show that for every $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathbb{R}^2 \times \mathbb{S}^1$, there exists an admissible control $u \in \mathcal{U}$ that will drive the system described by Eq. (1) from $(0, 0, 0)$, at time $t = 0$, to $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}})$, at time $t = t_{\mathsf{f}} < \infty$. First, we show sufficiency by using the interpretation of the ZMD problem as the minimum-time intercept Problem 2.2, as illustrated in Fig. 1. In particular, let σ be the ray emanating from the initial position $(x_{\mathsf{f}}, y_{\mathsf{f}})$ (point B in Fig. 1) of the target that is parallel to $\mathsf{e} := -w/|w| = -(\cos\phi, \sin\phi)$. Note that the target travels along σ with constant speed $\nu < 1$. Since the interceptor is a completely controllable system, there exists an admissible intercept strategy u that steers the interceptor, starting from the origin (point O in Fig. 1) to point B with $\theta_{\mathcal{P}} = \theta_{\sigma}$, where $\theta_{\sigma} = \pi + \phi \mod 2\pi$, at time $t = t_1 > 0$. Subsequently, the interceptor follows the target along σ. Since the interceptor is faster than the target, given that $\nu < 1$, then, at some time $t = t_2 > t_1$, it will reach a point $(x'_{\mathsf{f}}, y'_{\mathsf{f}})$ on σ sufficiently ahead of the target, say, at a distance $d \geq 0$. In addition, there exists an admissible control $u_d \in \mathcal{U}$ to drive the interceptor from (x, y, θ_{σ}) to $(x, y, \theta_{\mathsf{f}})$, for any $(x, y) \in \mathbb{R}^2$, after t_d units of time, where t_d is a function of ϕ and θ_{f} only. In particular, $(x_{\mathcal{P}}(t_{\mathsf{f}}), y_{\mathcal{P}}(t_{\mathsf{f}})) = (x_{\mathcal{P}}(t_2), y_{\mathcal{P}}(t_2)) = (x_{\mathcal{T}}(t_{\mathsf{f}}), y_{\mathcal{T}}(t_{\mathsf{f}}))$, and $\theta_{\mathcal{P}}(t_{\mathsf{f}}) = \theta_{\mathsf{f}}$, provided $d = \nu t_d$.

To show necessity, it suffices to observe that if $\nu \geq 1$, the target will travel at least as fast as the interceptor, and thus there exist boundary states for which no intercept will take place. □

2.3 Existence of Optimal Solutions

To show existence of an optimal solution to Problem 1, we apply Filippov's Theorem for minimum-time problems with prescribed initial and terminal states [24]. In particular, we observe that the right hand side of Eq. (1) defines a vector field $f : \mathbb{R}^3 \times U \mapsto \mathbb{R}^2 \times \mathbb{S}^1 \subset \mathbb{R}^3$, where $f(\theta, u) :=$

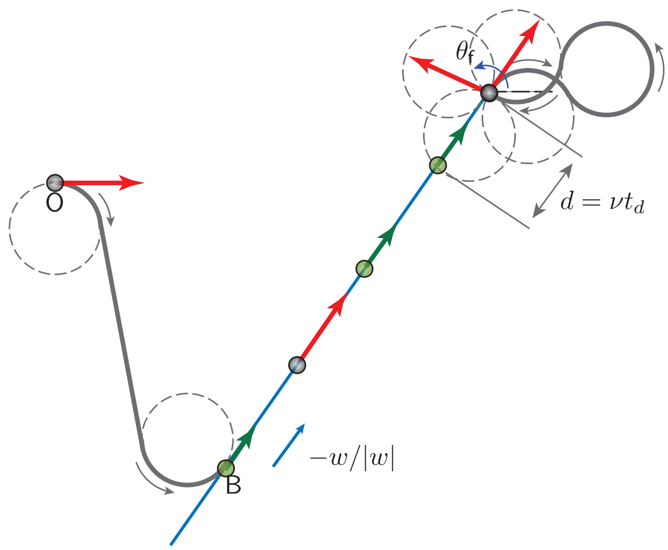


Figure 1: The system described by Eq. (1) is completely controllable if and only if $\nu < 1$.

$(\cos\theta + w_x, \sin\theta + w_y, u/\rho)$, which is continuous in u and continuously differentiable in θ, and the input value set $U = [-1, 1]$ is convex and compact. Furthermore, given that the vector field is affine in the control, and the input value set $U = [-1, 1]$ is convex, it follows that for a given $\theta \in \mathbb{S}^1$, the set $f(\theta, U)$ is convex. To prove the existence of optimal solutions for the ZMD problem it suffices, in light of Filippov's Theorem, to show that there exists a constant $c > 0$ such that

$$|\langle \mathsf{x}, f(\mathsf{x}, u)\rangle| \leq c(1 + |\mathsf{x}|^2), \quad \text{for all} \quad (\mathsf{x}, u) \in \mathbb{R}^2 \times \mathbb{S}^1 \times U, \tag{8}$$

where $\mathsf{x} := (x, y, \theta)$, and the inner product and the norm that appear in Eq. (8) are the standard scalar product and the Euclidean norm in $\mathbb{R}^3$, respectively. Furthermore, in light of the triangle inequality, the Cauchy-Schwartz inequality, and the inequalities $\sqrt{x^2 + y^2} + |\theta| \leq \sqrt{2}|\mathsf{x}|$ and $2|\mathsf{x}| \leq 1 + |\mathsf{x}|^2$, it follows that

$$\begin{aligned} |\langle \mathsf{x}, f(\mathsf{x}, u)\rangle| &\leq |x(\cos\theta + w_x) + y(\sin\theta + w_y)| + \frac{|u\theta|}{\rho} \\ &\leq \sqrt{x^2 + y^2}\sqrt{(\cos\theta + w_x)^2 + (\sin\theta + w_y)^2} + \frac{|\theta|}{\rho} \\ &\leq \frac{\sqrt{2}}{2} \max\left\{1 + \nu, \frac{1}{\rho}\right\}\left(1 + |\mathsf{x}|^2\right). \end{aligned} \tag{9}$$

Thus, all conditions of Filippov's Theorem are satisfied, leading us to the following two propositions.

Proposition 2.2. *Let $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathbb{R}^2 \times \mathbb{S}^1$ be given, and let assume that there exists an admissible control $u \in \mathcal{U}$ that drives the system described by Eq.* (1) *from* $(0,0,0)$*, at time $t = 0$, to $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}})$ after $0 \leq T < \infty$ units of time. Then, the minimum-time Problem 2.1 always has a solution.*

Proposition 2.3. *Let the drift field $w = \nu(\cos\phi, \sin\phi)$. If $\nu < 1$, then the minimum-time Problem 2.1 has a solution, for all $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathbb{R}^2 \times \mathbb{S}^1$.*

Proof. If $\nu < 1$, then it follows from Proposition 2.1 that the system (1) is completely controllable. Thus, there always exists a feasible path from $(0,0,0)$ to any $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathbb{R}^2 \times \mathbb{S}^1$, which furthermore implies, in light of Proposition 2.2, the existence of a minimum-time path between these two states. □

3 Analysis of the ZMD Minimum-Time Problem

In this Section, we revisit the ZMD problem posed in [13] and provide an in-depth examination of the structure of its solution.

3.1 Variational Analysis

To characterize the extremals of Problem 2.1, we consider its Hamiltonian

$$\mathcal{H} : \mathbb{R}^2 \times \mathbb{S}^1 \times \mathbb{R}^3 \times U \mapsto \mathbb{R}, \qquad \mathcal{H}(\mathsf{x}, \mathsf{p}, u) := p_0 + p_1 \cos\theta + p_2 \sin\theta + \frac{p_3 u}{\rho}, \tag{10}$$

where $\mathsf{p} := (p_1, p_2, p_3)$ and $p_0 \in \{0, 1\}$. By virtue of the Maximum Principle, if $\mathsf{x}^* := (x^*, y^*, \theta^*)$ is a minimum-time trajectory of the ZMD problem generated by the control $u^* \in \mathcal{U}$, then there exists a scalar $p_0^* \in \{0,1\}$ and an absolutely continuous function $\mathsf{p}^* : [0, T_{\mathsf{f}}] \mapsto \mathbb{R}^3$, where $\mathsf{p}^* := (p_1^*, p_2^*, p_3^*)$, known as the costate, such that

(i) $\|\mathsf{p}^*(t)\| + |p_0^*| \neq 0$, a.e. on $[0, T_{\mathsf{f}}]$,

(ii) $\mathsf{p}^*(t)$ satisfies, a.e. on $[0, T_\mathsf{f}]$, the canonical equation $\dot{\mathsf{p}}^* = -\dfrac{\partial \mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, u^*)}{\partial \mathsf{x}}$, that is,

$$\dot{p}_1^* = 0, \qquad \dot{p}_2^* = 0, \qquad \dot{p}_3^* = p_1^* \sin\theta^* - p_2^* \cos\theta^*, \tag{11}$$

(iii) $\mathsf{p}^*(T_\mathsf{f})$ satisfies the transversality condition

$$\mathcal{H}(\mathsf{x}^*(T_\mathsf{f}), \mathsf{p}^*(T_\mathsf{f}), u^*(T_\mathsf{f})) = 0. \tag{12}$$

Because the Hamiltonian does not depend explicitly on time, it follows from (12) that

$$\mathcal{H}(\mathsf{x}^*(t), \mathsf{p}^*(t), u^*(t)) = 0, \qquad \text{a.e. on} \ \ [0, T_\mathsf{f}]. \tag{13}$$

It follows, by virtue of (11), that $p_1^*(t) = p_1^*(0)$ and $p_2^*(t) = p_2^*(0)$, a.e. on $[0, T_\mathsf{f}]$, which furthermore implies, in light of (13), that

$$-p_0^* = p_1^*(0)(w_x + \cos\theta^*(t)) + p_2^*(0)(w_y + \sin\theta^*(t)) + \frac{p_3^*(t)u^*(t)}{\rho}, \qquad \text{a.e. on} \ \ [0, T_\mathsf{f}]. \tag{14}$$

Furthermore, the optimal control u^* necessarily minimizes the Hamiltonian evaluated along the optimal trajectory x^* and the corresponding costate vector p^*. Thus,

$$\mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, u^*) = \min_{v \in [-1,1]} \mathcal{H}(\mathsf{x}^*, \mathsf{p}^*, v), \qquad \text{a.e. on} \ \ [0, T_\mathsf{f}]. \tag{15}$$

It follows from (15) that

$$u^*(t) = \begin{cases} +1, & \text{if} \quad p_3^*(t) < 0, \\ \bar{u} \in [-1, 1], & \text{if} \quad p_3^*(t) = 0, \\ -1, & \text{if} \quad p_3^*(t) > 0. \end{cases} \tag{16}$$

The following proposition follows similarly to [25].

Proposition 3.1. *The only singular control of Problem 2.1 is* $u = 0$.

Thus, a minimum-time trajectory of Problem 2.1 corresponds necessarily to concatenations of singular arcs, when $u = 0$, and bang arcs, when $u = \pm 1$. Henceforth, we denote a bang and a singular arc by b and s, respectively; furthermore, we write b_α and s_α to denote, respectively, a bang and a singular arc traversed in α units of time. In addition, we write b_α^+ (resp., b_α^-) to denote the fact that the bang arc is generated with the application of the control input $u = +1$ (resp., $u = -1$) for α units of time. We denote by $\mathsf{b}_\alpha^\pm \mathsf{b}_\beta^\mp$ the concatenation of either a b_α^+ arc followed by a b_β^- arc or a b_α^- arc followed by a b_β^+ arc. Finally, we denote by Σ_α^n a chain of n bang arcs, that is, a concatenation of n consecutive bang arcs, traversed in α total units of time. We shall refer to the first and the last arc of a chain Σ_α^n as the boundary arcs, and to the rest of them as the intermediate arcs in the chain.

3.2 Structure of Candidate Optimal Paths

Next, we investigate the behavior of the switching function p_3^*. Subsequently, we examine the structure of the extremals of the ZMD problem. To this aim, let us consider an open interval $\mathcal{I} \subset [0, T_\mathsf{f}]$ such that $p_3^*(t) \neq 0$, for all $t \in \mathcal{I}$. The restriction of the optimal control u^* on $\mathcal{I}$ is a piecewise constant function, which may undergo a number of discontinuous jumps, and furthermore, $u^*(t) \in \{-1, +1\}$, for all $t \in \mathcal{I}$. By virtue of Eqs. (11) and (14), for any subinterval $\mathcal{I}_\mathsf{b}$ of $\mathcal{I}$, where $u^*(t)$ is constant, p_3^* satisfies the following differential equation

$$\ddot{p}_3^*(t) = -\frac{p_3^*(t)}{\rho^2} - \left(\frac{u^*(t)p_0^*}{\rho} + p_1^*(0)w_x + p_2^*(0)w_y\right), \quad \text{a.e. on } \mathcal{I}_\mathsf{b}. \tag{17}$$

The general solution of Eq. (17) restricted to the internal $\mathcal{I}_\mathsf{b}$ and its time derivative are given by

$$p_3^*(t) = C_1 \cos\frac{tu^*(t)}{\rho} + C_2 \sin\frac{tu^*(t)}{\rho} - \rho^2\left(p_1^*(0)w_x + p_2^*(0)w_y + \frac{u^*(t)p_0^*}{\rho}\right), \tag{18}$$

$$\dot{p}_3^*(t) = \frac{u^*(t)C_2}{\rho}\cos\frac{tu^*(t)}{\rho} - \frac{u^*(t)C_1}{\rho}\sin\frac{tu^*(t)}{\rho}, \tag{19}$$

where C_1, C_2 are real constants and $u^*(t) \equiv \pm 1$. It follows readily that

$$(\rho\dot{p}_3^*(t))^2 + (p_3^*(t) + u^*(t)p_0^*\rho + \varrho)^2 = C_1^2 + C_2^2, \quad \text{a.e. on } \mathcal{I}_\mathsf{b}, \tag{20}$$

where $\varrho = \rho^2(p_1^*(0)w_x + p_2^*(0)w_y)$.

Figures 2-3 illustrate the phase portrait $(p_3^*, \rho\dot{p}_3^*)$ of a chain of abnormal (when $p_0 = 0$) and normal (when $p_0 = 1$) bang arcs, respectively. In particular, as observed in Figs. 2(a)-2(b), the phase portrait of $(p_3^*, \rho\dot{p}_3^*)$ of a chain of abnormal bang arcs consists of a family of circles centered at a point K with coordinates $(\pm\varrho, 0)$ and radius r, where $r = \sqrt{C_1^2 + C_2^2}$, with parameterizations that trace them out clockwise at constant angular velocity $1/\rho$. Note that the control switches from $u^* = +1$ to $u^* = -1$ only if $|\varrho| \leq r$. Furthermore, as illustrated in Figs 2(a)-2(b), the time of motion along an abnormal bang arc of the ZMD problem is upper bounded by either $\pi\rho$ or $2\pi\rho$. This is in contrast to the standard MD problem, where the time of motion along an abnormal bang arc is always upper bounded by $\pi\rho$ [25]. On the other hand, the phase portrait of $(p_3^*, \rho\dot{p}_3^*)$ of a chain of normal bang arcs consists of two families of circles centered at points A and B, with coordinates $(\varrho - \rho, 0)$ and $(\rho + \varrho, 0)$, and radii r_+ (for $u^* = +1$) and r_- (for $u^* = -1$), respectively, with parameterizations that trace them out clockwise at constant angular velocity $1/\rho$; we denote these circles by $C(\mathsf{A}; r_+)$ and $C(\mathsf{B}; r_-)$, respectively. Note that a jump from $u^* = -1$ to $u^* = +1$, and vice versa, occurs only if $C(\mathsf{B}, r_-)$ intersects $C(\mathsf{A}, r_+)$ along the axis $p_3^* = 0$, that is, when $r_+ \geq |\varrho - \rho|$, $r_- \geq |\varrho + \rho|$, and $r_-^2 = r_+^2 + 4\varrho\rho$, as illustrated in Fig. 3. It is interesting to note that the phase portrait of $(p_3^*, \rho\dot{p}_3^*)$ is asymmetric with respect to the axis $p_3^* = 0$, in contrast to the symmetric phase portrait of the standard MD problem [26].

Next, we consider the optimality properties of a chain of bang arcs.

Proposition 3.2. *Let Σ_α^n be a chain of n bang arcs that is part of an optimal path of the ZMD problem from $(0,0,0)$ to some $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathbb{R}^2 \times \mathbb{S}^1$. If $n \geq 4$, then the total time along two consecutive, intermediate bang arcs $\mathsf{b}_{\alpha_i}^{\pm}\mathsf{b}_{\alpha_{i+1}}^{\mp}$ of Σ_α^n satisfies the lower bound*

$$\alpha_i + \alpha_{i+1} \geq 2\pi\rho, \quad \text{for all} \quad i \in \{2, \ldots, n-2\}. \tag{21}$$

Proof. Let $\mathsf{b}_{\alpha_i}^{+}\mathsf{b}_{\alpha_{i+1}}^{-} \in \Sigma_\alpha^n$. The case of a sub-path $\mathsf{b}_{\alpha_i}^{-}\mathsf{b}_{\alpha_{i+1}}^{+}$ can be treated similarly. If the two bang arcs are abnormal, then $\alpha_i + \alpha_{i+1}$ equals the time required for a particle with coordinates $(p_3^*, \rho p_3^*)$ to travel from point D to C, and subsequently, from C to D along the circle of radius r

centered at K, as illustrated in Fig. 2. We immediately conclude that $\alpha_i + \alpha_{i+1} = 2\pi\rho$, for all $i \in \{2, \ldots, n-2\}$. If the two bang arcs are normal, then $\alpha_i + \alpha_{i+1}$ equals the time required for a particle with coordinates $(p_3^*, \rho p_3^*)$ to travel from point D to C along $C(\mathsf{A}, r_+)$, and subsequently, from C to D along $C(\mathsf{B}, r_-)$ with angular velocity $1/\rho$, as illustrated in Fig. 3. There are two cases to consider. First, if $\alpha_i \geq \pi\rho$, for all $i \in \{2, \ldots, n-2\}$, then it follows that $\alpha_i + \alpha_{i+1} \geq 2\pi\rho$ (Fig. 3(a)). Second, if $0 < \alpha_i \leq \pi\rho$, for some $i \in \{2, \ldots, n-2\}$, then we observe that the time of motion from D to C along the circle $C(\mathsf{A}, r_+)$ is greater than the time of motion from D to C along the circle $C(\mathsf{B}, r_-)$ given that $\widehat{\mathsf{DAC}} > \widehat{\mathsf{DBC}}$ (Fig. 3(b)). Thus, it follows readily that $\alpha_i + \alpha_{i+1} \geq 2\pi\rho$, for all $i \in \{2, \ldots, n-2\}$. Therefore, in all cases, $\alpha_i + \alpha_{i+1} \geq 2\pi\rho$, for all $i \in \{2, \ldots, n-2\}$. □

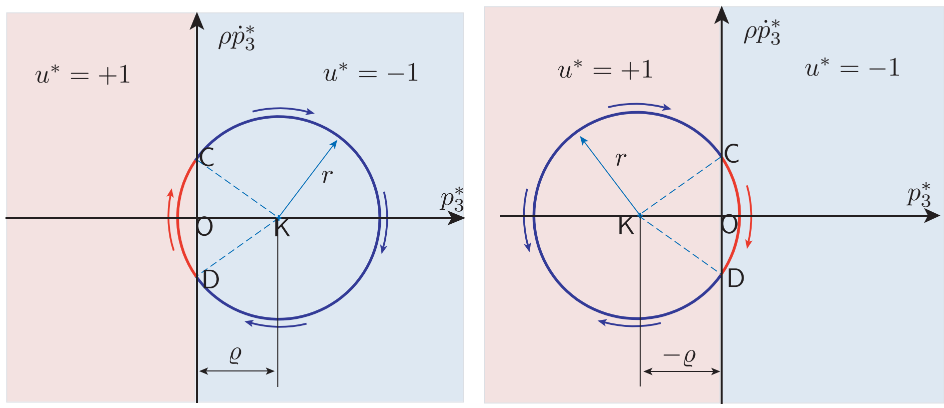
Next, we shall employ Proposition 3.2 to establish a basic property enjoyed by the min-time paths of the ZMD problem, namely, that infinite chattering (something known as the Füller phenomenon in optimal control theory [27]) cannot take place along them, that is, the number of bang arcs in every chain of bang arcs is necessarily finite.

Proposition 3.3. *Let the constant drift field $w = \nu(\cos\phi, \sin\phi)$, where $\nu < 1$. A chain of bang arcs Σ_α^n can be part of an optimal path of the ZMD problem only if it is finite.*

Proof. In light of Proposition 2.3, for all $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathbb{R}^2 \times \mathbb{S}^1$, there exists a minimum-time path of the ZMD problem from $(0, 0, 0)$ to $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}})$, and thus $T_{\mathsf{f}} < \infty$. Let assume, on the contrary, that a chain Σ_α^n, where $n \to \infty$, is part of a min-time path. By virtue of Proposition 3.2, the time of motion along the first $i+2$ bang arcs of Σ_α^n, where $i \in \{2, \ldots, n-2\}$, is lower bounded by $2i\pi\rho$. Then by taking $i \to \infty$, it follows that α grows unbounded. Consequently, $T_{\mathsf{f}} = \infty$, leading to a contradiction. □

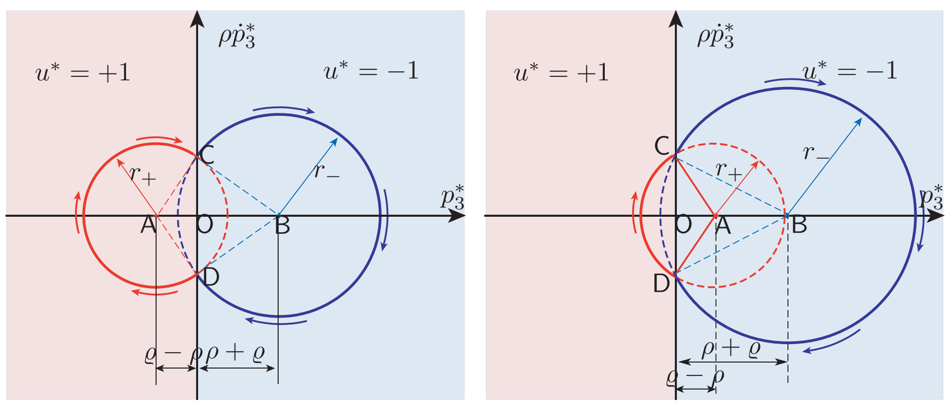
The following proposition highlights the existence of a type of extremals of the ZMD that does not belong to the sufficient for optimality family of extremals of the standard MD problem.

Proposition 3.4. *A b_α arc, where $\alpha = 2\pi\rho$, may be part of a minimum-time path of Problem 2.1.*



(a) $\varrho > 0$. (b) $\varrho < 0$.

Figure 2: Phase portrait $(p_3^*, \rho\dot{p}_3^*)$ of a chain of bang arcs composed of abnormal extremals ($p_0^* = 0$).



(a) $\alpha_i \geq \pi\rho$, for all i. (b) $\alpha_i \geq \pi\rho$, for some i.

Figure 3: Phase portrait $(p_3^*, \rho\dot{p}_3^*)$ of a chain of bang arcs composed of normal extremals ($p_0^* = 1$).

Proof. Let us consider the equivalent formulation of the ZMD problem as a min-time intercept problem (Problem 2.2). Let assume, without loss of generality, that $w = (\nu, 0)$ and let us consider the intercept problem with $\theta_{\mathsf{f}} = 0$, when the non-maneuvering target is located, at time $t = 0$, at a point T with coordinates $(-2\pi\rho\nu, 0)$, as illustrated in Fig. 4. By driving the interceptor with the control input $u = +1$ or $u = -1$, for all $0 \leq t \leq 2\pi\rho$, intercept will take place at O with $\theta_{\mathcal{P}} = 0$. We claim that the moving target cannot be intercepted faster than $2\pi\rho$ units of time. Let assume on the contrary that the target can be intercepted with $\theta_{\mathcal{P}} = \theta_{\mathsf{f}}$, at time $t = t_1 < 2\pi\rho$; which implies that intercept should take place in the interior of TO. Since O is aft T, it follows that the direction of the interceptor's velocity necessarily changes from $\theta = 0$ to $\theta_{\mathsf{f}} = 0$, within the time interval $[0, t_1]$, while the interceptor is traversing a full loop. It follows readily that $t_1 \geq 2\pi\rho$, which is absurd. $\square$

Remark 3.1 Note that a $\mathsf{b}_{2\pi\rho}$ arc can be part of an optimal path of the ZMD problem but not of the standard MD problem [25]. As we shall see shortly later, the previous fact will explain the existence of new types of extremals of the ZMD problem that do not appear in the solution of the MD problem.

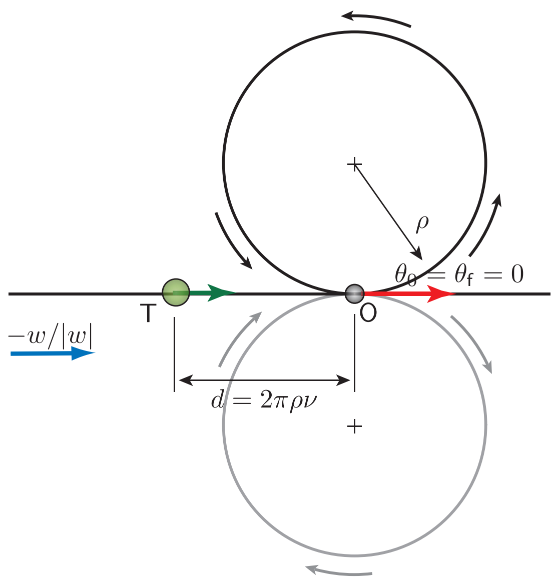


Figure 4: In contrast to the MD problem, a b_α arc, where $\alpha = 2\pi\rho$, may be part of an optimal solution of the ZMD problem. Consequently, there might exist candidate solutions of the ZMD that cannot be part of a solution of the standard MD problem.

The next proposition provides lower and upper bounds on the time of motion along a bang arc of a chain of bang arcs.

Proposition 3.5. *Let the constant drift $w = \nu(\cos\phi, \sin\phi)$, where $\nu < 1$, and let assume that a chain of n bang arcs Σ_α^n is part of a minimum-time path of the ZMD problem. If b_{α_i}, where $i \in \{1, \ldots, n\}$, is part of Σ_α^n, then*

(i) $\alpha_i \in [0, \pi\rho]$ or $\alpha_i \in [\pi\rho, 2\pi\rho]$, for all $i \in \{1, \ldots n\}$,

(ii) $\alpha_i + \alpha_{i+1} \geq 2\pi\rho,\quad$ for all $\quad i \in \{2, \ldots n-2\}$.

Proof. It suffices to observe that, if b_{α_i} is an abnormal bang arc ($\varrho = 0$), then α_i corresponds to the travel time of a particle with coordinates $(p_3^*, \rho\dot{p}_3^*)$ from point D (resp., C) to C (resp., D) along a circle centered at K with constant angular velocity $1/\rho$, as illustrated in Fig. 2. It follows $\alpha_i \in [0, \pi\rho]$ (resp., $\alpha_i \in [\pi\rho, 2\pi\rho]$), $\alpha_{i+1} \in [\pi\rho, 2\pi\rho]$ (resp., $\alpha_i \in [0, \pi\rho]$). If b_{α_i} is a normal bang arc, then α_i corresponds to the travel time of a particle with coordinates $(p_3^*, \rho\dot{p}_3^*)$ from point D (resp., C) to C (resp., D) along the circle $C(\mathsf{A}; r_+)$ (resp., $C(\mathsf{B}; r_-)$) with constant angular velocity $1/\rho$. The situation is illustrated in Figs. 3(a)-3(b). In particular, if $\varrho > 0$ and $\varrho < \rho$, then, as illustrated in Fig. 3(a), $\alpha_i \in [\pi\rho, 2\pi\rho]$. In Fig. 3(b), we observe that, given two consecutive, intermediate bang arcs $\mathsf{b}_{\alpha_i}^{\pm}\mathsf{b}_{\alpha_{i+1}}^{\mp}$, for $i \in \{2, \ldots, n-2\}$, if $\varrho > 0$ and $\varrho > \rho$ (the case when $\varrho < 0$ and $\varrho < \rho$ can be treated similarly), then either $\alpha_i \in [0, \pi\rho]$ and $\alpha_{i+1} \in [\pi\rho, 2\pi\rho]$ or both α_i and $\alpha_{i+1} \in [\pi\rho, 2\pi\rho]$. The rest of the proof follows readily from Proposition 3.2. $\square$

Remark 3.2 Note that the time of motion along an intermediate b_{α_i} arc of an optimal chain of bang arcs of the standard MD problem satisfies $\alpha_i \in]\pi\rho, 2\pi\rho[$ (see for example [25]). We henceforth denote a b_{α_i} arc, where $\alpha_i \in [0, \pi\rho]$, of an optimal chain of bang arcs of the ZMD problem by $\tilde{\mathsf{b}}_{\alpha_i}$.

Next, we investigate the structure of paths that consist of both singular and bang arcs. Because along a singular arc $p_3^* = 0$, which furthermore implies that $\dot{p}_3^* = 0$, it follows that any s arc corresponds to the origin of the phase portrait $(p_3^*, \rho\dot{p}_3^*)$. First, we show that optimal paths that consist of both singular and bang arcs do not involve infinite chattering.

Proposition 3.6. *Let the constant drift filed $w = \nu(\cos\phi, \sin\phi)$, where $\nu < 1$. An optimal path of the ZMD is necessarily a concatenation of a finite number of bang and singular arcs.*

Proof. In Proposition 3.3, we have shown that an optimal trajectory of the ZMD problem does not involve infinite chattering between bang-bang control inputs. Next, we show that both the total number of singular and bang arcs of an optimal path of the ZMD problem is necessarily finite, as well. In particular, we observe in Fig. 2-3 (note that now the points C and D coincide with the origin O of the phase portrait $(p_3^*, \rho\dot{p}_3^*)$) that a transition from a s_α arc to a different singular arc, say s_γ, may occur only via a finite chain of bang arcs $\Sigma^n_{2n\pi\rho}$. Thus, the time of motion along an optimal path that contains two s arcs is necessarily lower bounded by $2n\pi\rho$. The rest of the proof follows similarly to the proof of Proposition 3.3. □

Proposition 3.7. *Let $w = \nu(\cos\phi, \sin\phi)$, where $\nu < 1$. Paths of type* (i) $\mathsf{b}^\pm\mathsf{s}\mathsf{b}^\pm$, (ii) $\mathsf{b}^\pm\mathsf{s}\mathsf{b}^\mp$, (iii) $\mathsf{b}^\mp\mathsf{s}\mathsf{b}^\mp$, (iv) $\mathsf{s}\mathsf{b}^\mp\mathsf{s}$, (v) $\mathsf{b}^\pm\mathsf{b}^\mp\mathsf{s}$, *and* (vi) $\mathsf{s}\mathsf{b}^\pm\mathsf{b}^\mp$ *may be part of a minimum-time path of the ZMD problem.*

Remark 3.3 The fact that paths of type (iv)-(vi) may be part of an optimal solution of the ZMD problem is an immediate consequence of Proposition 3.4. However, as one of the reviewers pointed out, if a path of type (iv)-(vi) solves the ZMD problem, then there exists a path of type (i)-(iii) which is also a minimum-time path of the ZMD problem.

3.3 Sufficient for Controllability and Necessary for Optimality Family of Extremals of the ZMD Minimum-Time Problem

Next, we propose a family of candidate optimal paths of the ZMD problem that consist of all admissible concatenations of singular and bang arcs that steer the system described by Eq. (1) to an arbitrary terminal state in (nearly) minimum time. Since the trajectory of the system described by Eq. (1) uniquely determines the control that generates it, and vice versa, we can associate each of the candidate optimal paths with their corresponding control sequence. For example, a path $\mathsf{b}^\pm_\alpha\mathsf{s}_\beta\mathsf{b}^\pm_\gamma$ corresponds to the control sequence $\{\pm1, 0, \pm1\}$.

Theorem 3.1. *Any minimum-time path of the ZMD problem contains at least one of the following extremal paths*

(i) $\mathsf{b}^\pm_\alpha\mathsf{s}_\beta\mathsf{b}^\pm_\gamma$, $\mathsf{b}^\pm_\alpha\mathsf{s}_\beta\mathsf{b}^\mp_\gamma$, *where* $\alpha \in [0, 2\pi\rho]$, $\beta \in [0, \infty[$, *and* $(\pm\alpha/\rho \pm \gamma/\rho) \bmod 2\pi = \theta_\mathsf{f}$, $(\pm\alpha/\rho \mp \gamma/\rho)$

mod $2\pi = \theta_{\mathsf{f}}$, *respectively,*

(ii) $\mathsf{b}_{\alpha}^{\pm}\mathsf{b}_{\beta}^{\mp}\mathsf{b}_{\gamma}^{\pm}$, *where* $\alpha \in [0, 2\pi\rho]$, $\beta \in [\pi, 2\pi\rho]$, *and* $(\pm\alpha/\rho \mp \beta/\rho \pm \gamma/\rho)$ mod $2\pi = \theta_{\mathsf{f}}$,

(iii) $\mathsf{b}_{\alpha}^{\pm}\tilde{\mathsf{b}}_{\beta}^{\mp}\mathsf{b}_{\gamma}^{\pm}$, *where* $\alpha \in [0, 2\pi\rho]$, $\beta \in [0, \pi\rho]$, *and* $(\pm\alpha/\rho \mp \beta/\rho \pm \gamma/\rho)$ mod $2\pi = \theta_{\mathsf{f}}$.

We denote this family of paths by $\mathcal{P}_{\mathrm{ZMD}}^{*}$. *Let, furthermore,* $\mathsf{U}_{\mathrm{ZMD}}^{*}$ *be the corresponding family of control sequences that generate the paths of* $\mathcal{P}_{\mathrm{ZMD}}^{*}$. *Then* $\mathsf{U}_{\mathrm{ZMD}}^{*}$ *is sufficient for the complete controllability of the system described by Eq.* (1)*.*

Proof. In [13] was shown that the extremal paths (i)-(iii) suffice to ensure complete controllability of the system described by Eq. (1). In addition, the fact that $\mathcal{P}_{\mathrm{ZMD}}^{*}$ is a subset of the sufficient for optimality family of extremals of the ZMD problem follows readily from Propositions 3.1-3.7. □

Remark 3.4 In [13], it is claimed, but not rigorously proved, that the paths types (i)-(iii) given in Theorem 3.1 are sufficient for optimality. Based on the previous analysis, a more precise statement would be that the paths types given in Theorem 3.1 form a subset of the sufficient for optimality family of extremal paths of the ZMD problem. In addition, it can be conjectured, in light of Propositions 3.2-3.7, that the optimal paths of the ZMD that consist of more than three arcs, if such optimal paths can exist at all, correspond to a rather trivial set of boundary conditions. Thus, for the analysis of the synthesis of the ZMD problem, one may only consider the path types (i)-(iii) given in Theorem 3.1, which are sufficient for complete controllability, without a significant loss in optimality, and thus, characterize a nearly optimal synthesis of the ZMD problem.

4 Time-Optimal Synthesis

In this section, we present in detail the steps for the characterization of a nearly optimal synthesis of the ZMD problem.

4.1 Reachability Analysis

First, we carry out the reachability analysis for the system described by Eq. (1), when the admissible control is constrained to be an element in $\mathsf{U}^*_{\mathrm{ZMD}}$. To simplify the presentation, and with no loss in generality, we henceforth consider the minimum-time trajectories of (1) from $(0, 0, 0)$ to $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in P(\theta_{\mathsf{f}})$, where $P(\theta_{\mathsf{f}}) := \{(x, y, \theta) \in \mathbb{R}^2 \times \mathbb{S}^1 : \ \theta = \theta_{\mathsf{f}}\}$, as suggested in [22]. Furthermore, we denote the reachable set that corresponds to the control sequence $u \in \mathsf{U}^*_{\mathrm{ZMD}}$ as $\mathfrak{R}_{\mathrm{ZMD}}(u; \theta_{\mathsf{f}})$. Finally, we denote the corresponding reachable set of the standard MD problem by $\mathfrak{R}_{\mathrm{MD}}(u; \theta_{\mathsf{f}})$.

Next, we demonstrate how to characterize the reachable set $\mathfrak{R}_{\mathrm{ZMD}}(u; \theta_{\mathsf{f}})$, for $u \in \mathsf{U}^*_{\mathrm{ZMD}}$, by briefly presenting the main steps for the construction of $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{sb}^+; \theta_{\mathsf{f}})$. In particular, we observe that the coordinates of any state in $P(\theta_{\mathsf{f}})$ that can be reached by means of the control sequence $\{+1, 0, +1\}$, or equivalently, a $\mathsf{b}^+_\alpha \mathsf{s}_\beta \mathsf{b}^+_\gamma$ path, can be expressed in terms of the time of motion along each of the three arcs of the path, namely α, β, and γ. In particular, it follows readily that $\gamma(\alpha; \theta_{\mathsf{f}}) = \rho\hat{\theta}_{\mathsf{f}} - \alpha$, where $\hat{\theta}_{\mathsf{f}} = \theta_{\mathsf{f}}$ if $\alpha \leq \rho\theta_{\mathsf{f}}$ and $\hat{\theta}_{\mathsf{f}} = (2\pi + \theta_{\mathsf{f}})\rho$, otherwise. In addition, it follows after routine calculations similarly to [7] that

$$x_{\mathsf{f}}(\alpha, \beta) = \rho \sin\theta_{\mathsf{f}} + \beta \cos\frac{\alpha}{\rho} + w_x T_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+), \tag{22}$$

$$y_{\mathsf{f}}(\alpha, \beta) = \rho(1 - \cos\theta_{\mathsf{f}}) + \beta \sin\frac{\alpha}{\rho} + w_y T_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+), \tag{23}$$

where $T_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+) = \alpha + \beta + \gamma(\alpha; \theta_{\mathsf{f}})$.

Conversely, given a point $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{sb}^+; \theta_{\mathsf{f}})$, we can determine the corresponding pairs $(\alpha, \beta) \in [0, 2\pi\rho] \times [0, \infty[$. In particular, after some algebraic manipulation, it follows that

$$(1 - \nu^2)\beta^2 + 2(A(x_{\mathsf{f}}, \theta_{\mathsf{f}})w_x + B(y_{\mathsf{f}}, \theta_{\mathsf{f}})w_y)\beta - A(x_{\mathsf{f}}, \theta_{\mathsf{f}})^2 - B(y_{\mathsf{f}}, \theta_{\mathsf{f}})^2 = 0, \tag{24}$$

where $A(x_{\mathsf{f}}, \theta_{\mathsf{f}}) = x_{\mathsf{f}} - \rho \sin\theta_{\mathsf{f}} - w_x\rho\hat{\theta}_{\mathsf{f}}$, $B(y_{\mathsf{f}}, \theta_{\mathsf{f}}) = y_{\mathsf{f}} + \rho(\cos\theta_{\mathsf{f}} - 1) - w_y\rho\hat{\theta}_{\mathsf{f}}$. Note that Eq. (24), which is decoupled from α, admits at most two solutions. Given a solution β of (24), then α is determined with back substitution in Eqs. (22)-(23). In particular, after some algebraic manipulation, it follows that $\alpha(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) = \hat{\alpha}(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f})\rho$, where $\hat{\alpha} \in [0, 2\pi]$ satisfies

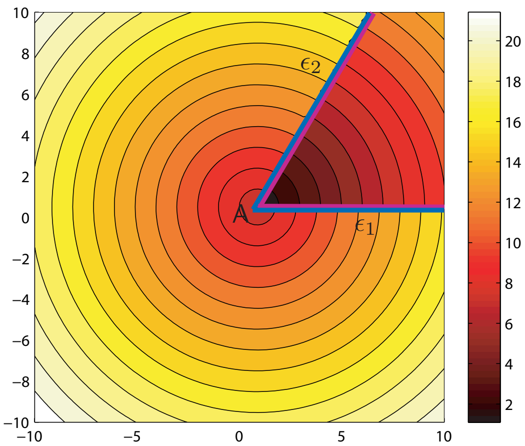
$$\cos\hat{\alpha}(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) = \frac{\rho A(x_\mathsf{f}, \theta_\mathsf{f})}{\beta} - w_x, \qquad \sin\hat{\alpha}(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) = \frac{\rho B(y_\mathsf{f}, \theta_\mathsf{f})}{\beta} + w_y, \tag{25}$$

when $\beta \neq 0$, whereas $\alpha(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) = \rho\theta_\mathsf{f}$, otherwise. In this way, for a given $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$, we obtain two pairs (α, β) and the corresponding final time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+) = \alpha + \beta + \gamma(\alpha; \theta_\mathsf{f})$. Subsequently, we associate the state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$ with the pair (α^*, β^*) that yields the minimum of the time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$, denoted by $T_\mathsf{f}^*(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$, where $T_\mathsf{f}^*(\mathsf{b}^+\mathsf{s}\mathsf{b}^+) := \alpha^* + \beta^* + \gamma(\alpha^*; \theta_\mathsf{f})$.

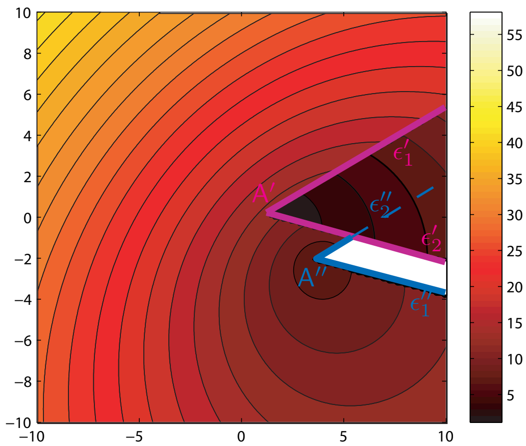
The previous procedure can be applied mutatis mutandis for the rest of the control sequences of $\mathsf{U}^*_{\mathrm{ZMD}}$, thus obtaining equations that yield α and β as functions of x_f and y_f, and vice versa. A system of equations which are either decoupled or in triangular form, which admit straightforward numerical or, in some cases, analytical solutions, is presented in Appendix A.

Next, we proceed with the characterization of the reachable set $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f})$ along with the level sets of the minimum-time $T_\mathsf{f}^*(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$. In particular, the reachable set $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f})$ consists of all points $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$, where x_f and y_f are computed from Eqs. (22)-(23), by taking $\alpha \in [0, 2\pi\rho]$ and $\gamma \in [0, 2\pi\rho]$ such that $(\alpha/\rho + \gamma/\rho) \bmod 2\pi = \theta_\mathsf{f}$ and $0 \leq \alpha + \gamma(\alpha) \leq (4\pi - \theta_\mathsf{f})\rho$. On the other hand, the minimum time $T_\mathsf{f}^*(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$ is easily determined from Eqs. (24)-(25). The reachable sets $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f})$, along with the contours of the minimum time $T_\mathsf{f}^*(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)$, when $0 \leq \alpha + \gamma(\alpha) \leq (4\pi - \theta_\mathsf{f})\rho$, for the standard MD and the ZMD problems are illustrated, respectively, in Figs. 5(a)-5(b). We observe that $\mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f}) = P(\theta_\mathsf{f})$, whereas $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f}) \subset P(\theta_\mathsf{f})$. In particular, the white region in Fig. 5(b) corresponds to the set of states $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$ that cannot be reached by means of a $\mathsf{b}^+_\alpha\mathsf{s}_\beta\mathsf{b}^+_\gamma$ path, when $0 \leq \alpha + \gamma(\alpha) \leq (4\pi - \theta_\mathsf{f})\rho$. It is worth-mentioning that $\mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f}) = P(\theta_\mathsf{f})$, when $0 \leq \alpha + \gamma(\alpha) \leq (4\pi - \theta_\mathsf{f})\rho$, but $\mathfrak{R}^{\mathrm{MD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f}) \subset P(\theta_\mathsf{f})$, if we consider instead the stricter condition $0 \leq \alpha + \gamma(\alpha) \leq 2\pi\rho$, as illustrated in Fig. 6(a). It should be highlighted that the last condition on α and γ is actually the condition that a $\mathsf{b}^+_\alpha\mathsf{s}_\beta\mathsf{b}^+_\gamma$ path should satisfy in order to be a candidate optimal solution of the MD problem (Proposition 3.1 of [22]). The reachable set $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+; \theta_\mathsf{f})$, when $0 \leq \alpha + \gamma(\alpha) \leq 2\pi\rho$, is illustrated in Fig. 6(b). Note, in addition, that points in the white region of the $P(\theta_\mathsf{f})$ of the ZMD problem illustrated in Fig. 5(b) will be reachable by means of $\mathsf{b}^+\mathsf{s}\mathsf{b}^+$ paths only if α and/or γ is greater than $2\pi\rho$. Clearly these paths are suboptimal solutions of the ZMD problem. The fact that, for a particular $u' \in \mathsf{U}^*_{\mathrm{ZMD}}$, $\mathfrak{R}_{\mathrm{ZMD}}(u';\theta_\mathsf{f})$ is a proper subset of $P(\theta_\mathsf{f})$ has rather low significance for the analysis of the optimal synthesis in so far the union of the reachable sets $\mathfrak{R}_{\mathrm{ZMD}}(u;\theta_\mathsf{f})$, for all $u \in \mathsf{U}^*_{\mathrm{ZMD}}$, covers $P(\theta_\mathsf{f})$.

The reachability analysis for the remaining control sequences of $\mathsf{U}^*_{\mathrm{ZMD}}$ can be carried out mutatis mutandis. Due to space limitations, the details are left to the reader. Figure 7 illustrates the reachable sets $\mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})$ (Fig. 7(a)) and $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})$ (Fig. 7(b)), respectively, when $\alpha \in [0, 2\pi\rho]$, $\beta \in [0,\infty[$ and $(\pm\alpha/\rho \mp \gamma/\rho) \bmod 2\pi = \theta_\mathsf{f}$.



(a) Reachable set $\mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})$.

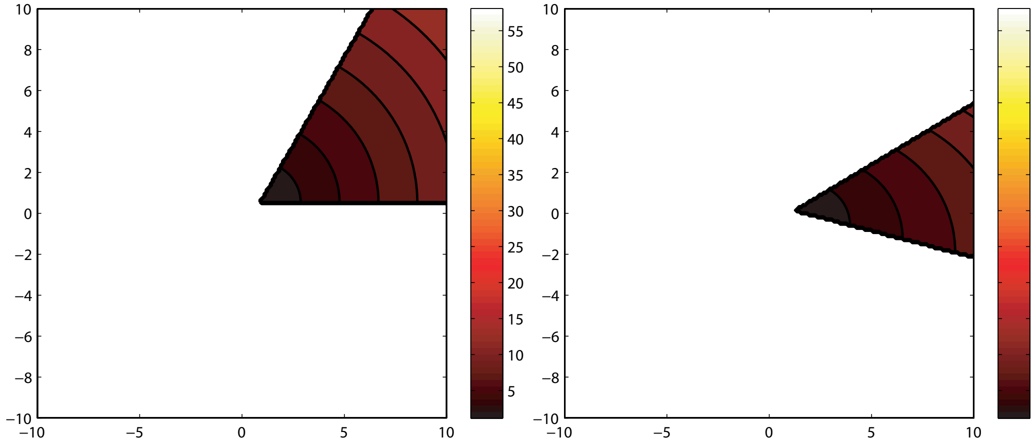


(b) Reachable set $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})$.

Figure 5: Reachable sets of the standard MD and the ZMD problems, when $0 \le \alpha + \gamma(\alpha) \le (4\pi - \theta_\mathsf{f})\rho$, $\theta_\mathsf{f} = \pi/3$, $\nu = 0.5$, and $\phi = 7\pi/4$.

4.2 The Direct Correspondence Between the Optimal Syntheses of the MD and the ZMD Problems

In this section, we introduce a discontinuous mapping that establishes a direct correspondence between the reachable sets of the MD and the ZMD problems. To this aim, let us consider, for a given $T \ge 0$, the mapping $\mathrm{H}_T : \mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f}) \mapsto \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})$, that maps a state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in$



(a) Reachable set $\mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$.

(b) Reachable set $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$.

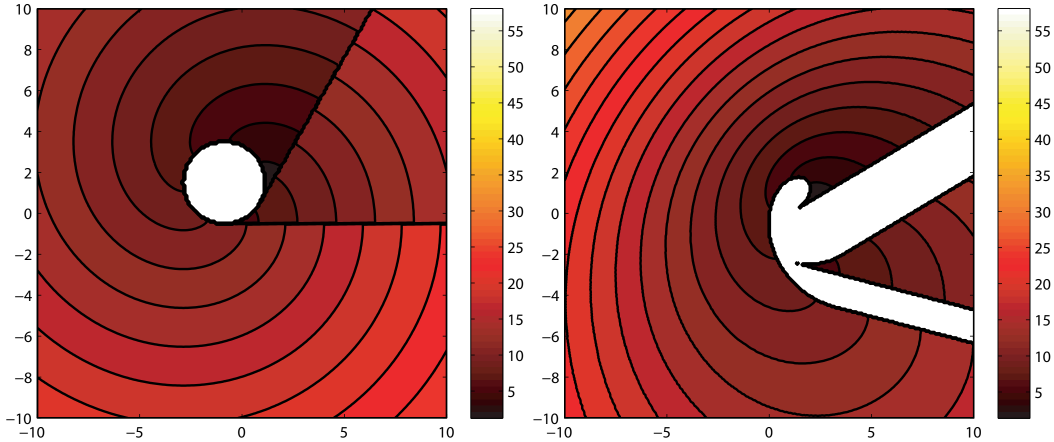
Figure 6: Reachable sets of the standard MD and the ZMD problems, when $0 \leq \alpha+\gamma(\alpha) \leq 2\pi\rho$, $\theta_{\mathsf{f}} = \pi/3$, $\nu = 0.5$, and $\phi = 7\pi/4$.

$\mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$ to a state $(X_{\mathsf{f}}, Y_{\mathsf{f}}, \Theta_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$, where

$$X_{\mathsf{f}} = x_{\mathsf{f}} + w_x T, \qquad Y_{\mathsf{f}} = y_{\mathsf{f}} + w_y T, \qquad \Theta_{\mathsf{f}} = \theta_{\mathsf{f}}. \tag{26}$$

The transformation H_T given in Eqs. (26) can be interpreted as follows: The system described by Eq. (3) can be steered with the application of a control input u, which corresponds to a control sequence $\{1, 0, 1\}$, from $(0, 0, 0)$ to $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$ after $T \geq 0$ units of time. Then, in the presence of a constant drift field (w_x, w_y), the system described instead by Eq. (1), will be steered by the same control input u to a state $(X_{\mathsf{f}}, Y_{\mathsf{f}}, \Theta_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$ after T units of time. By taking $T = T_{\mathsf{f}}^*(\mathsf{b}^+\mathsf{sb}^+)$, it follows that each state $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$ is mapped via the composite mapping $\mathrm{H}_{T_{\mathsf{f}}^*(\mathsf{b}^+\mathsf{sb}^+)}$ to a state $(X_{\mathsf{f}}, Y_{\mathsf{f}}, \Theta_{\mathsf{f}}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$. An important observation is that the time $T_{\mathsf{f}}^*(\mathsf{b}^+\mathsf{sb}^+)$ of the MD problem undergoes discontinuous jumps along the rays ϵ_1 and ϵ_2 emanating from the point A with coordinates $(x_{\mathsf{A}}, y_{\mathsf{A}}) = \rho(\sin\theta_{\mathsf{f}}, 1 - \cos\theta_{\mathsf{f}})$, where $\epsilon_1 := \{(x, y, \theta) : \ y = y_{\mathsf{A}},\ x \geq x_{\mathsf{A}},\ \theta = \theta_{\mathsf{f}}\}$, and $\epsilon_2 := \{(x, y, \theta) : \ y = y_{\mathsf{A}} + s\sin\theta_{\mathsf{f}},\ x = x_{\mathsf{A}} + s\cos\theta_{\mathsf{f}},\ \theta = \theta_{\mathsf{f}},\ \ s \geq 0\}$, as illustrated in Fig. 5(a).

Let now $\mathcal{K}(\theta_{\mathsf{f}}) \subset P(\theta_{\mathsf{f}})$ denote the cone with apex A defined by the rays ϵ_1 and ϵ_2, as illustrated in Fig. 8. It can be shown [7] that every state $(x_{\mathsf{f}}, y_{\mathsf{f}}, \theta_{\mathsf{f}}) \in \mathcal{K}(\theta_{\mathsf{f}}) \subset \mathfrak{R}_{\mathrm{MD}}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}}) = P(\theta_{\mathsf{f}})$ can



(a) Reachable set $\mathfrak{R}_{\rm MD}(\mathsf{b}^+\mathsf{sb}^-;\theta_{\mathsf{f}})$.

(b) Reachable set $\mathfrak{R}_{\rm ZMD}(\mathsf{b}^+\mathsf{sb}^-;\theta_{\mathsf{f}})$.

Figure 7: Reachable sets of the standard MD and the ZMD problems for $\theta_{\mathsf{f}}=\pi/3$, $\nu=0.5$, and $\phi=7\pi/4$.

be reached after $T^-(\mathsf{b}^+\mathsf{sb}^+)=T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)=\rho\theta_{\mathsf{f}}+\sqrt{(x_{\mathsf{f}}-\rho\sin\theta_{\mathsf{f}})^2+(y_{\mathsf{f}}+\rho\cos\theta_{\mathsf{f}}-\rho)^2}$, whereas the states in $P(\theta_{\mathsf{f}})\backslash\mathcal{K}(\theta_{\mathsf{f}})$ can be reached in minimum time $T^+(\mathsf{b}^+\mathsf{sb}^+)=T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)=\rho(2\pi+\theta_{\mathsf{f}})+\sqrt{(x_{\mathsf{f}}-\rho\sin\theta_{\mathsf{f}})^2+(y_{\mathsf{f}}+\rho\cos\theta_{\mathsf{f}}-\rho)^2}$. Thus, the minimum time $T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$ of the MD problem undergoes a discontinuous jump from $T^-(\mathsf{b}^+\mathsf{sb}^+)$ to $T^+(\mathsf{b}^+\mathsf{sb}^+)=T^-(\mathsf{b}^+\mathsf{sb}^+)+2\pi\rho$ along the rays ϵ_1 and ϵ_2. In Fig. 8, we observe that the rays ϵ_1 and ϵ_2 are mapped via $\mathrm{H}_{T^-(\mathsf{b}^+\mathsf{sb}^+)}$ to a new pair of rays, namely, ϵ_1' and ϵ_2', emanating from a point A' with coordinates $(x_{\mathsf{A}'},y_{\mathsf{A}'})=\mathrm{H}_{T^-(\mathsf{b}^+\mathsf{sb}^+)}(x_{\mathsf{A}},y_{\mathsf{A}})$. In addition, the rays ϵ_1 and ϵ_2 are mapped via $\mathrm{H}_{T^+(\mathsf{b}^+\mathsf{sb}^+)}$ to another pair of rays, namely, ϵ_1'' and ϵ_2'', emanating from a point A'' with coordinates $(x_{\mathsf{A}''},y_{\mathsf{A}''})=\mathrm{H}_{T^+(\mathsf{b}^+\mathsf{sb}^+)}(x_{\mathsf{A}},y_{\mathsf{A}})$. We henceforth denote by $\mathcal{K}'(\theta_{\mathsf{f}})$ and $\mathcal{K}''(\theta_{\mathsf{f}})$ the cones defined by the apexes A' and A'' and the pairs of rays ϵ_1', ϵ_2' and ϵ_1'', ϵ_2'', respectively. The situation is illustrated in Fig. 8. Owing to the discontinuity of $T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$ of the MD problem along the rays ϵ_1 and ϵ_2, the composite mapping $\mathrm{H}_{T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)}$ is also discontinuous along the rays ϵ_1 and ϵ_2. It is worth mentioning that the previously made observation that, when $0\leq\alpha+\gamma(\alpha)\leq(4\pi-\theta_{\mathsf{f}})\rho$, it holds that $\mathfrak{R}_{\rm ZMD}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})\subset P(\theta_{\mathsf{f}})$, whereas $\mathfrak{R}_{\rm MD}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})=P(\theta_{\mathsf{f}})$, can now be interpreted as a consequence of the discontinuity of the mapping $\mathrm{H}_{T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)}$. In particular, the mapping $\mathrm{H}_{T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)}$ turns out to be a non-surjective mapping of $\mathfrak{R}_{\rm MD}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$ to $\mathfrak{R}_{\rm ZMD}(\mathsf{b}^+\mathsf{sb}^+;\theta_{\mathsf{f}})$. Another important remark is that the time $T^*_{\mathsf{f}}(\mathsf{b}^+\mathsf{sb}^+)$ of the ZMD problem undergoes discontinuous jumps along the ray ϵ_2', the line segments $\mathsf{A'B}$ and $\mathsf{BA''}$, where B is the intersection point of ϵ_1' and ϵ_2'', and the ray ϵ_1''. Furthermore, the set of states in $P(\theta_\mathsf{f})$ that cannot be reached by means of a $\mathsf{b}_\alpha^+\mathsf{s}_\beta\mathsf{b}_\gamma^+$ path, when $0 \leq \alpha + \gamma(\alpha) \leq (4\pi - \theta_\mathsf{f})\rho$, corresponds to the set $\mathcal{K}''(\theta_\mathsf{f})\backslash(\mathcal{K}'(\theta_\mathsf{f}) \cap \mathcal{K}''(\theta_\mathsf{f}))$. The situation is illustrated in Figs. 5(b) and 8.

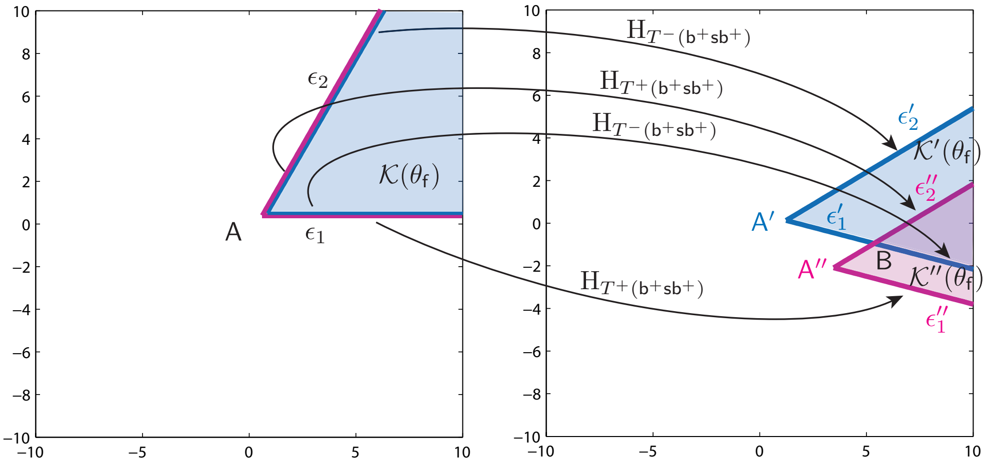


Figure 8: Owing to the discontinuity of the time-to-go, the rays ϵ_1 and ϵ_2 in $\mathfrak{R}_{\mathrm{MD}}(\mathsf{b^+sb^+};\theta_\mathsf{f})$ are mapped via $\mathrm{H}_{T_\mathsf{f}^*(\mathsf{b^+sb^+})}$ to two pairs of rays in $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b^+sb^+};\theta_\mathsf{f})$.

4.3 The Optimal Control Partition

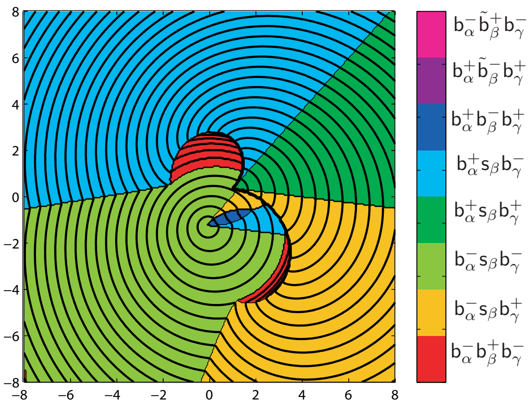
The next step involves the partitioning of $P(\theta_\mathsf{f})$ into a finite number of domains, which are henceforth denoted by $\mathfrak{R}^*_{\mathrm{ZMD}}(u;\theta_\mathsf{f})$, where $u \in \mathsf{U}^*_{\mathrm{ZMD}}$. The criterion that assigns a state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$ to a $u \in \mathsf{U}^*_{\mathrm{ZMD}}$ is the following: If $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}^*_{\mathrm{ZMD}}(u;\theta_\mathsf{f})$, then $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f})$ cannot be reached faster with the application of any other control sequence of $\mathsf{U}^*_{\mathrm{ZMD}}$ different than u, and vice versa. In particular, consider a state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b^+sb^+};\theta_\mathsf{f})$, and let $\mathsf{U}^c(\mathsf{b^+sb^+}) \subset \mathsf{U}^*_{\mathrm{ZMD}}$ denote the set of control sequences u different from $\mathsf{b^+sb^+}$ for which $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}_{\mathrm{ZMD}}(u;\theta_\mathsf{f})$. Then the state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}^*_{\mathrm{ZMD}}(\mathsf{b^+sb^+};\theta_\mathsf{f})$ if and only if $T_\mathsf{f}^*(\mathsf{b^+sb^+}) \leq \min_{u\in\mathsf{U}^c(\mathsf{b^+sb^+})} T_\mathsf{f}^*(u)$. We shall refer to this partition of $P(\theta_\mathsf{f})$ as the *optimal control partition*.

Figure 9 illustrates the optimal control partitions of $P(\theta_\mathsf{f})$, for $\theta_\mathsf{f} = \pi/3$, $\phi = 7\pi/4$, and different values of the magnitude of the drift field $\nu \in]0,1[$. In particular, we observe in Fig. 9(a) that, for $\nu = 0.2$, the structure of the optimal control partition of $P(\theta_{\mathsf{f}})$ as well as the level sets of the minimum time $T_{\mathsf{f}}^* = \min T_{\mathsf{f}}^*(u)$, where $u \in \mathsf{U}_{\mathsf{ZMD}}^*$, do not significantly differ from those of the standard MD problem presented in [21, 22]. The optimal control partition, as well as the level sets of the minimum time of the ZMD and MD problems, for higher values of ν, become, however, significantly different (Fig. 9(b)-9(d)). Furthermore, we observe that, as ν increases, the set $\mathfrak{R}_{\mathrm{ZMD}}^*(\mathsf{b}^-\tilde{\mathsf{b}}^+\mathsf{b}^-;\theta_{\mathsf{f}})$ corresponds to a non-trivial portion of the optimal control partition (Figs. 9(c)-9(d)).

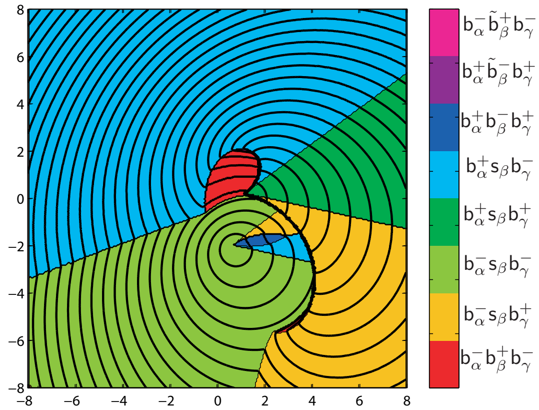
Figure 10 illustrates the optimal control partition of $P(\theta_{\mathsf{f}})$, for $\theta_{\mathsf{f}} = \pi/3$, $\nu = 0.5$ and different values of the drift direction ϕ. Figures 10(a)-10(d) illustrate how sensitive is the optimal control partition of $P(\theta_{\mathsf{f}})$ to variations of the drift direction for the ZMD problem. It is interesting to note that, for $\phi = 5\pi/4$, the set $\mathfrak{R}_{\mathrm{ZMD}}^*(\mathsf{b}^-\tilde{\mathsf{b}}^+\mathsf{b}^-;\theta_{\mathsf{f}})$ corresponds to a significant portion of the optimal control partition of $P(\theta_{\mathsf{f}})$ (Fig. 10(a)). Furthermore, we observe that, as we change the value of ϕ, some extremal paths of $\mathcal{P}_{\mathrm{ZMD}}^*$ become more favorable than others, in terms of minimizing the travel time. For example, when $\phi = 5\pi/4$ (Fig. 10(a)) and $\phi = 3\pi/4$ (Fig. 10(d)), then respectively, the sets $\mathfrak{R}_{\mathrm{ZMD}}^*(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-;\theta_{\mathsf{f}})$ and $\mathfrak{R}_{\mathrm{ZMD}}^*(\mathsf{b}^-\mathsf{s}\mathsf{b}^+;\theta_{\mathsf{f}})$ correspond to significantly larger portions of the optimal control partition of $P(\theta_{\mathsf{f}})$, when compared with the standard MD problem.

5 Conclusions

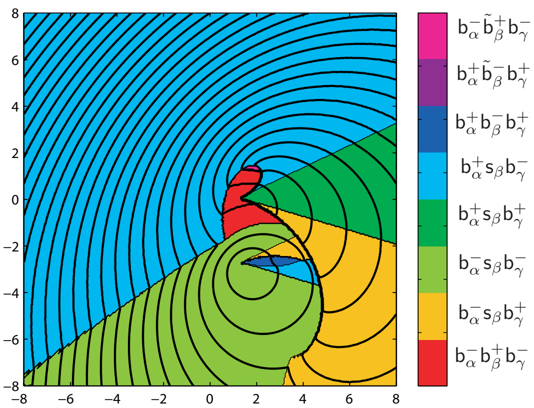
In this article, we have addressed a variation of the Markov–Dubins problem regarding the characterization of time-optimal trajectories for a vehicle with the kinematics of the Isaacs–Dubins car operating in a constant drift field. We have studied the optimality properties of the solution of the Zermelo–Markov–Dubins problem and subsequently characterized a nearly optimal synthesis of the problem. Our analysis has revealed similarities as well as some significant differences between the solutions of the Zermelo–Markov–Dubins and the standard Markov–Dubins problems.



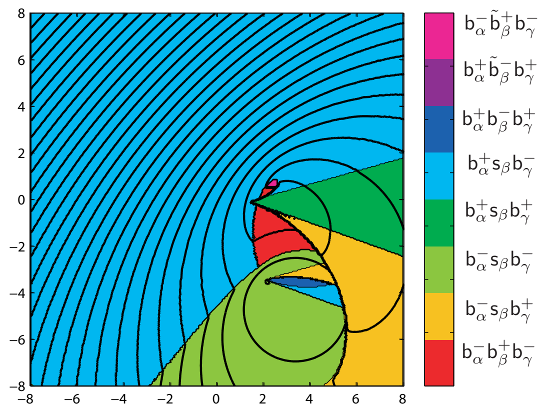
(a) $\nu = 0.2\text{m/s}$



(b) $\nu = 0.4\text{m/s}$



(c) $\nu = 0.6\text{m/s}$



(d) $\nu = 0.8\text{m/s}$

Figure 9: Optimal control partition of $P(\theta_{\mathsf{f}})$and contours of T^*_{f}, for $\theta_{\mathsf{f}} = \pi/3$, $\phi = 7\pi/4$ and different values of ν.

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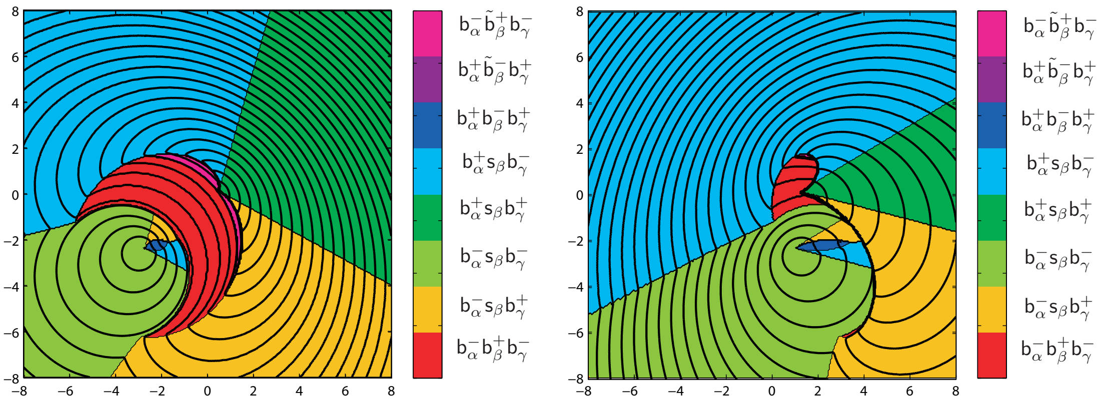
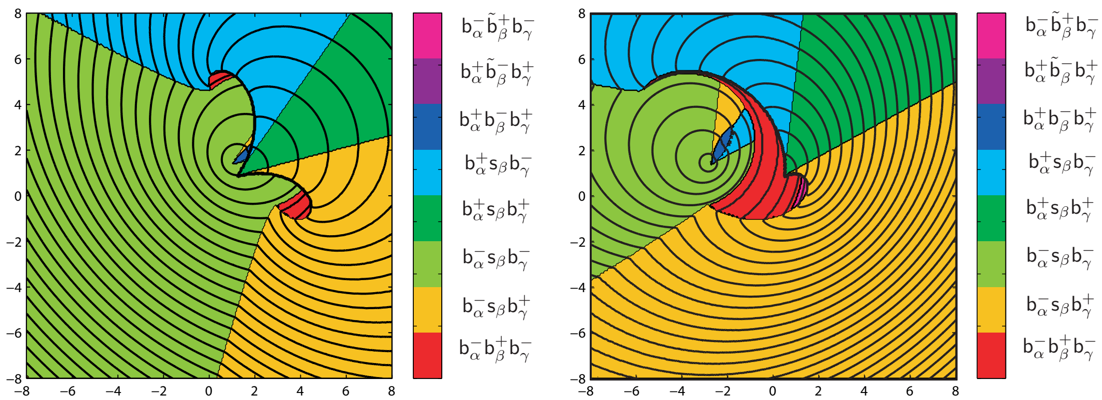


(a) $\phi = 5\pi/4$ (b) $\phi = 7\pi/4$



(c) $\phi = \pi/4$ (d) $\phi = 3\pi/4$

Figure 10: Optimal control partition of $P(\theta_\mathsf{f})$ and contours of T_f^*, for $\theta_\mathsf{f} = \pi/3$, $\nu = 0.5$ and different values of ϕ.

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A Appendix

A.1 $\mathsf{b}^+_\alpha \mathsf{s}_\beta \mathsf{b}^+_\gamma$ $\left[\mathsf{b}^-_\alpha \mathsf{s}_\beta \mathsf{b}^-_\gamma\right]$ Paths

The coordinates x_f, y_f of a state in $\mathfrak{R}_{\text{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})\left[\mathfrak{R}_{\text{ZMD}}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})\right]$, can be expressed as functions of the parameters α and β as follows

$$x_\mathsf{f} = [-]\rho \sin\theta_\mathsf{f} + \beta \cos\frac{\alpha}{\rho} + w_x T_\mathsf{f}, \tag{27}$$

$$y_\mathsf{f} = [-]\rho(1-\cos\theta_\mathsf{f}) + [-]\beta \sin\frac{\alpha}{\rho} + w_y T_\mathsf{f}, \tag{28}$$

where $T_\mathsf{f} = \alpha+\beta+\gamma$, and $\gamma/\rho = (\theta_\mathsf{f} - \alpha/\rho) \bmod 2\pi$ $\left[\gamma/\rho = (2\pi - \theta_\mathsf{f} - \alpha/\rho) \bmod 2\pi\right]$.

Conversely, given a state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}_{\text{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})\left[\mathfrak{R}_{\text{ZMD}}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})\right]$, we can determine $(\alpha,\beta) \in [0, 2\pi\rho] \times [0,\infty[$. In particular, after some algebraic manipulation, it follows that β satisfies the following quadratic equation, which is decoupled from α,

$$(1-\nu^2)\beta^2 + [-]2(A(x_\mathsf{f},\theta_\mathsf{f})w_x + B(y_\mathsf{f},\theta_\mathsf{f})w_y)\beta - (A^2(x_\mathsf{f},\theta_\mathsf{f}) + B^2(y_\mathsf{f},\theta_\mathsf{f})w_y)) = 0, \tag{29}$$

where $A(x_\mathsf{f},\theta_\mathsf{f}) = x_\mathsf{f} - [+]\rho\sin\theta_\mathsf{f} - w_x\rho\hat{\theta}_\mathsf{f}$, $B(y_\mathsf{f},\theta_\mathsf{f}) = [-]y_\mathsf{f} + \rho(\cos\theta_\mathsf{f} - 1) - [+]w_y\rho\hat{\theta}_\mathsf{f}$, and

$$\hat{\theta}_\mathsf{f} = \begin{cases} \theta_\mathsf{f} \;\; [2\pi - \theta_\mathsf{f}], & \text{if} \;\; \alpha \le \rho\theta_\mathsf{f} \;\; [\alpha \le (2\pi - \theta_\mathsf{f})\rho], \\ (2\pi + \theta_\mathsf{f})\rho \;\; [(4\pi - \theta_\mathsf{f})\rho], & \text{if} \;\; \alpha > \rho\theta_\mathsf{f} \;\; [\alpha > (2\pi - \theta_\mathsf{f})\rho]. \end{cases} \tag{30}$$

Note that for each $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}_{\text{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})\left[\mathfrak{R}_{\text{ZMD}}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})\right]$, there exist at most two solutions of (29). If β is one solution of (29), then α is determined with back substitution in Eqs. (27)-(28). In particular, after some algebraic manipulation, it follows that $\alpha = \hat{\alpha}\rho$, where $\hat{\alpha} \in [0, 2\pi]$ satisfies

$$\cos\hat{\alpha} = \frac{A(x_\mathsf{f},\theta_\mathsf{f})}{\beta} - w_x, \qquad \sin\hat{\alpha} = \frac{B(y_\mathsf{f},\theta_\mathsf{f})}{\beta} - [+]w_y, \tag{31}$$

when $\beta \neq 0$, whereas $\alpha = \rho\theta_\mathsf{f}$ $[\rho(2\pi - \theta_\mathsf{f})]$, otherwise. In this way, for a given $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$, we find pairs (α,β) and the corresponding final time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)[T_\mathsf{f}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-)] = \alpha+\beta+\gamma(\alpha)$, and subsequently, we associate the state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in P(\theta_\mathsf{f})$ with the pair (α^*, β^*) that gives the minimum time $T_\mathsf{f}(\mathsf{b}^+\mathsf{s}\mathsf{b}^+)[T_\mathsf{f}(\mathsf{b}^-\mathsf{s}\mathsf{b}^-)]$.

A.2 $\mathsf{b}^+_\alpha\mathsf{s}_\beta\mathsf{b}^-_\gamma$ $\left[\mathsf{b}^-_\alpha\mathsf{s}_\beta\mathsf{b}^+_\gamma\right]$ Paths

If $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})$ $\left[\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^-\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})\right]$, then

$$x_\mathsf{f} = 2\rho\sin\frac{\alpha}{\rho} + \beta\cos\frac{\alpha}{\rho} - [+]\rho\sin\theta_\mathsf{f} + w_x T_\mathsf{f}, \tag{32}$$

$$y_\mathsf{f} = [-]\rho(1+\cos\theta_\mathsf{f}) - [+]2\rho\cos\frac{\alpha}{\rho} + [-]\beta\sin\frac{\alpha}{\rho} + w_y T_\mathsf{f}, \tag{33}$$

where $T_\mathsf{f} = \alpha + \beta + \gamma,\ \gamma/\rho = (\alpha/\rho - \theta_\mathsf{f}) \bmod 2\pi\ \left[\gamma/\rho = (\alpha/\rho + \theta_\mathsf{f}) \bmod 2\pi\right]$.

Given a state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) \in \mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{s}\mathsf{b}^-;\theta_\mathsf{f})$ $\left[\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^-\mathsf{s}\mathsf{b}^+;\theta_\mathsf{f})\right]$, it can be shown that α satisfies the following transcendental equation (decoupled from β)

$$D(\alpha; x_\mathsf{f}, \theta_\mathsf{f})\sin\frac{\alpha}{\rho} + E(\alpha; y_\mathsf{f}, \theta_\mathsf{f})\cos\frac{\alpha}{\rho} = B(y_\mathsf{f}, \theta_\mathsf{f})w_x - [+]A(x_\mathsf{f}, \theta_\mathsf{f})w_y + 2\rho, \tag{34}$$

where, $A(x_\mathsf{f}, \theta_\mathsf{f}) = x_\mathsf{f} + [-]\rho\sin\theta_\mathsf{f} + [-]w_x\rho\hat{\theta}_\mathsf{f}$, $B(y_\mathsf{f}, \theta_\mathsf{f}) = [-]y_\mathsf{f} - \rho(\cos\theta_\mathsf{f} + 1) + w_y\rho\hat{\theta}_\mathsf{f}$, $D(\alpha; x_\mathsf{f}, \theta_\mathsf{f}) = A(x_\mathsf{f}, \theta_\mathsf{f}) - [+]2\rho(w_y + [-]w_x\alpha/\rho)$, $E(\alpha; y_\mathsf{f}, \theta_\mathsf{f}) = -B(y_\mathsf{f}, \theta_\mathsf{f}) - 2\rho(w_x - [+]w_y\alpha/\rho)$, and where

$$\hat{\theta}_\mathsf{f} = \begin{cases} \theta_\mathsf{f}, & \text{if} \quad \alpha \geq \rho\theta_\mathsf{f}\left[\alpha \leq \rho(2\pi - \theta_\mathsf{f})\right], \\ [-]2\pi + \theta_\mathsf{f}, & \text{if} \quad \alpha < \rho\theta_\mathsf{f}\left[\alpha > \rho(2\pi - \theta_\mathsf{f})\right]. \end{cases} \tag{35}$$

Furthermore, it can be shown that β satisfies the following equation

$$\begin{aligned}(1-\nu^2)\beta = &\left(A(x_\mathsf{f}, \theta_\mathsf{f}) - 2\rho\left(\sin\frac{\alpha}{\rho} + w_x\alpha\right)\right)\left(\cos\frac{\alpha}{\rho} - [+]w_x\right) \\ &+ \left(B(y_\mathsf{f}, \theta_\mathsf{f}) + 2\rho\left(\cos\frac{\alpha}{\rho} - [+]w_y\alpha\right)\right)\left(\sin\frac{\alpha}{\rho} - [+]w_y\right).\end{aligned} \tag{36}$$

A.3 $\mathsf{b}^+_\alpha\mathsf{b}^-_\beta\mathsf{b}^+_\gamma$ $\left[\mathsf{b}^-_\alpha\mathsf{b}^+_\beta\mathsf{b}^-_\gamma\right]$ and $\mathsf{b}^+_\alpha\tilde{\mathsf{b}}^-_\beta\mathsf{b}^+_\gamma$ $\left[\mathsf{b}^-_\alpha\tilde{\mathsf{b}}^+_\beta\mathsf{b}^-_\gamma\right]$ Paths

The coordinates of a state $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f})$ in $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+; \theta_\mathsf{f})$ $\left[\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-; \theta_\mathsf{f})\right]$ or $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\tilde{\mathsf{b}}^-\mathsf{b}^+; \theta_\mathsf{f})$ $\left[\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^-\tilde{\mathsf{b}}^+\mathsf{b}^-; \theta_\mathsf{f})\right]$ are given by

$$x_\mathsf{f} = 2\rho\left(\sin\frac{\alpha}{\rho} + \sin\frac{\beta-\alpha}{\rho}\right) + [-]\rho\sin\theta_\mathsf{f} + w_x T_\mathsf{f}, \tag{37}$$

$$y_\mathsf{f} = [-]\rho(1-\cos\theta_\mathsf{f}) - [+]2\rho\left(\cos\frac{\alpha}{\rho} - \cos\frac{\beta-\alpha}{\rho}\right) + w_y T_\mathsf{f}, \tag{38}$$

where, $T_\mathsf{f} = \alpha+\beta+\gamma,\ \gamma/\rho = (\theta_\mathsf{f} - \alpha/\rho + \beta/\rho) \bmod 2\pi$ $\left[\gamma/\rho = (-\theta_\mathsf{f} - \alpha/\rho + \beta/\rho) \bmod 2\pi\right]$.

Conversely, given $(x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f})$ in $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\mathsf{b}^-\mathsf{b}^+; \theta_\mathsf{f})$ $\left[\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^-\mathsf{b}^+\mathsf{b}^-; \theta_\mathsf{f})\right]$ or $\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^+\tilde{\mathsf{b}}^-\mathsf{b}^+; \theta_\mathsf{f})$ $\left[\mathfrak{R}_{\mathrm{ZMD}}(\mathsf{b}^-\tilde{\mathsf{b}}^+\mathsf{b}^-; \theta_\mathsf{f})\right]$, it follows after some algebra that β satisfies the following transcendental equation, which is decoupled from α,

$$K(\beta; x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) + 8\rho^2\left(\cos\frac{\beta}{\rho} - 1\right) = 0, \tag{39}$$

where $K(\beta; x_\mathsf{f}, y_\mathsf{f}, \theta_\mathsf{f}) = A^2(x_\mathsf{f}, \theta_\mathsf{f}) + B^2(y_\mathsf{f}, \theta_\mathsf{f}) + 4\nu^2\beta^2 + [-]4\beta(B(y_\mathsf{f}, \theta_\mathsf{f})w_y - [+]A(x_\mathsf{f}, \theta_\mathsf{f})w_x)$, $A(x_\mathsf{f}, \theta_\mathsf{f}) = x_\mathsf{f} - [+]\rho\sin\theta_\mathsf{f} - [+]w_x\rho\hat{\theta}_\mathsf{f}$, $B(y_\mathsf{f}, \theta_\mathsf{f}) = -[+]y_\mathsf{f} + \rho(1-\cos\theta_\mathsf{f}) + w_y\rho\hat{\theta}_\mathsf{f}$, and

$$\hat{\theta}_\mathsf{f} = \begin{cases} [-]\theta_\mathsf{f}, & \text{if } 0 \le [-]\theta_\mathsf{f} - \dfrac{\alpha}{\rho} + \dfrac{\beta}{\rho} < 2\pi, \\[2ex] -2\pi[+4\pi] + [-]\theta_\mathsf{f}, & \text{if } 2\pi[-4\pi] \le [-]\theta_\mathsf{f} - \dfrac{\alpha}{\rho} + \dfrac{\beta}{\rho} < 4\pi[-2\pi], \\[2ex] 2\pi + [-]\theta_\mathsf{f}, & \text{if } -2\pi \le [-]\theta_\mathsf{f} - \dfrac{\alpha}{\rho} + \dfrac{\beta}{\rho} < 0. \end{cases}$$

Given $\beta \in [0, 2\pi\rho]$, it follows after some algebraic manipulation that α satisfies

$$\begin{bmatrix} M(\beta; x_\mathsf{f}, \theta_\mathsf{f}) & N(\beta; y_\mathsf{f}, \theta_\mathsf{f}) \\ -[+]N(\beta; y_\mathsf{f}, \theta_\mathsf{f}) & [-]M(\beta; x_\mathsf{f}, \theta_\mathsf{f}) \end{bmatrix} \begin{bmatrix} \sin\dfrac{\alpha}{\rho} \\ \cos\dfrac{\alpha}{\rho} \end{bmatrix} = 2\rho \begin{bmatrix} 1 - \cos\dfrac{\beta}{\rho} \\ [-]\sin\dfrac{\beta}{\rho} \end{bmatrix}, \tag{40}$$

where $M(\beta; x_\mathsf{f}, \theta_\mathsf{f}) = A(x_\mathsf{f}, \theta_\mathsf{f}) - 2\beta w_x$, $N(\beta; y_\mathsf{f}, \theta_\mathsf{f}) = B(y_\mathsf{f}, \theta_\mathsf{f}) + [-]2\beta w_y$.