A Solution to the Minimum ℓ_1-Norm Controllability Problem for Discrete-Time Linear Systems via Iteratively Reweighted Least Squares

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***Abstract*— In this work, we study a fundamental controllability problem for discrete-time linear systems driven by sparse control sequences, that is, sequences comprised of a significant number of null elements, by utilizing an ℓ_1 optimal control problem formulation. It is well known that the solution to the latter problem, which we refer to as the minimum ℓ_1-norm controllability problem, enjoys, in general, nice sparsity properties in sharp contradistinction to the solution to the minimum ℓ_2-norm controllability problem for discrete-time linear systems. On the other hand, it is well known that the latter problem can be reduced to a convex quadratic program subject to linear equality constraints, whose solution can be characterized in closed form, in contrast with the minimum ℓ_1-norm controllability problem which lacks this analytic tractability. In this work, we propose an iterative approach that furnishes an approximate solution to the latter problem in closed form via the solution of a corresponding sequence of convex quadratic programs. Finally, we present numerical simulations from the application of the proposed approach to a space proximity operation problem.**

I. INTRODUCTION

This work deals with the characterization of a *sparse* control sequence that will steer a discrete-time linear system to the origin (or any other prescribed terminal state) in finite time. In this context, a sparse control sequence can be thought of as a sequence comprised of a small number of "large" inputs and many null inputs. The motivation for this problem stems from a number of applications in which the use of a small number of corrective / regulating control actions is more economical than the use of control sequences that result from the solution of, for instance, ℓ_2 optimal control problems, which are known to favor control sequences comprised primarily of non-null inputs of rather small magnitude.

The problem of finding the most sparse control sequence, that is, the control sequence with the minimum ℓ_0-norm, that will steer a discrete-time linear system to a prescribed state can be associated with the problem of finding the minimum 0-norm solution of a system of linear equations. In this context, the ℓ_0-norm of a finite-length sequence of vectors corresponds to the 0-norm of the vector that is formed by the concatenation of the vectors of this sequence. (The 0-norm of a vector, which is not a norm in the strict mathematical sense, is defined as the number of non-zero elements of the vector). Because the ℓ_0-norm minimization problem is not convex, it is not computationally tractable, in general. In addition, the ℓ_0-norm does not reflect an intuitive performance index for a dynamical system in contrast with other ℓ_p-norms. In this work, we will address the controllability problem for discrete-time linear systems in the class of sparse control sequences indirectly by associating it with an ℓ_1-norm minimization problem (thus, relaxing the requirement for sequences of minimum ℓ_0-norm). The motivation behind this approach stems from the fact that ℓ_1-norm minimization problems admit solutions that typically exhibit, under some technical assumptions, sparsity properties. It should be also highlighted that in contrast with the ℓ_0-norm, the ℓ_1-norm is used as the performance index of practical optimal control problems such as the *minimum-fuel* problem.

Literature Review: Control problems with sparsity constraints, have recently started to receive some attention in the controls' literature. The proposed approaches can be decomposed to direct approaches that seek for solutions of non-convex ℓ_0-norm minimization problems and indirect approaches that seek for sparse solutions of convex ℓ_1-norm minimization problems. The reader may refer to [1]–[6]. It is well known that ℓ_1-norm minimization problems are neither analytically nor computationally tractable. Consequently, the solution to such problems cannot be used in applications which require real-time computations of rather low complexity (such as problems on autonomous on-board guidance and navigation of spacecraft). On the other hand, the literature of compressive sensing is rich in iterative techniques, such as the so-called *iteratively reweighted least squares* (IRLS) algorithm as well as homotopy-based algorithms, which can characterize an approximation to the solution of an ℓ_1-norm minimization problem [7]–[11]. In particular, the main idea behind the IRLS algorithm is that a sparse approximation of the minimum ℓ_1-norm solution to a system of linear equations can be obtained as the limit of a sequence whose elements are the minimizers of a corresponding sequence of appropriately weighted ℓ_2-norm minimization problems, which are analytically tractable.

Main Contribution: In this work, we address a fundamental controllability problem for discrete-time linear systems, which is formulated as an ℓ_1 optimal control problem. We will refer to the latter problem as the minimum ℓ_1-norm controllability problem. In the proposed approach, the optimal control problem is first reduced to an 1-norm (vector norm) minimization problem subject to linear constraints, and subsequently, an approximation to the solution to the latter problem is found by means of the IRLS algorithm. The latter algorithm will in turn generate a sequence of vectors that are the minimizers of a corresponding sequence

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of convex QPs subject to linear equality constraints. Each of these QPs can be interpreted as a weighted minimum ℓ_2-norm controllability problem, whose solution can be characterized in closed form. In this way, the IRLS will serve as a "bridge" that will allow one to pass from the solution to an appropriately weighted minimum ℓ_2-norm controllability problem to (an approximation of) the solution to the minimum ℓ_1-norm controllability problem.

Structure of the paper: The rest of the paper is organized as follows. In Section II, we introduce some useful notation and briefly review some basic results from the theory of discrete-time linear systems. In Section III, we formulate the minimum ℓ_1-norm controllability problem and in Section IV, we present an iterative scheme for the characterization of an approximation of the solution to the latter problem that is based on the IRLS algorithm. Illustrative numerical simulations are presented in Section V. Finally, Section VI concludes the paper with a summary of remarks.

II. Preliminaries

A. Notation

We write $\mathbb{R}^n_{\geq 0}$ and $\mathbb{R}^n_{>0}$ to denote the set of n-dimensional vectors that have, respectively, non-negative and positive elements. We write $\mathbb{Z}^+$ and $\mathbb{Z}^{++}$ to denote the set of non-negative integers and strictly positive integers, respectively. Given z_α, $z_\beta \in \mathbb{Z}^+$ with $z_\alpha \leq z_\beta$, we denote the *discrete interval* from z_α to z_β as $[z_\alpha, z_\beta]_d$; note that $[z_\alpha, z_\beta]_d = [z_\alpha, z_\beta] \cap \mathbb{Z}^+$. Given a vector $a \in \mathbb{R}^n$, we denote by $a_{(i)}$ (or $(a)_{(i)}$ in cases in which we want to avoid the use of a double subscript notation) its i-th entry, that is, $a_{(i)} = \boldsymbol{e}_i^{\mathrm{T}} a$, where $\boldsymbol{e}_i$ denotes the unit vector whose only non-zero element is the i-th one, which is equal to one. We write $\pi^-(a)$ to denote the non-increasing arrangement of a, that is, if $v = \pi^-(a)$, then $v_{(1)} \geq \cdots \geq v_{(n)} \geq 0$, where $v_{(j)}$ is the j-th largest element of the set $\{|a_{(i)}|,\ i \in [1,n]_d\}$. In addition, we denote by $|a|_1$ and $|a|_2$ the 1-norm and the 2-norm of a, respectively; that is, $|a|_1 := \sum_{i=1}^n |a_{(i)}|$ and $|a|_2 := (\sum_{i=1}^n |a_{(i)}|^2)^{1/2}$. Given a finite-length (truncated) sequence of vectors $A_N := \{a(t) \in \mathbb{R}^n :\ t \in [0,N]_d\}$, we denote by $\|A_N\|_{\ell_1}$ and $\|A_N\|_{\ell_2}$ the ℓ_1 and ℓ_2 norms of A_N, respectively; that is, $\|A_N\|_{\ell_1} := \sum_{t=0}^N |a(t)|_1 = \sum_{t=0}^N \sum_{k=1}^n |a_{(k)}(t)|$ and $\|A_N\|_{\ell_2} := \big(\sum_{t=0}^N |a(t)|_2^2\big)^{1/2} = \big(\sum_{t=0}^N \sum_{k=1}^n |a_{(k)}(t)|^2\big)^{1/2}$. Furthermore, we denote by $\mathrm{diag}(a)$ the diagonal $n \times n$ matrix whose diagonal elements are the elements of a. We will write $\mathbf{1}$ to denote the vector whose elements are equal to one. Given a non-empty, discrete, finite point-set $\mathcal{S}$, we will denote by $\mathrm{card}(\mathcal{S})$ the number of points that comprise it. Finally, we write $\mathbf{0}_{m\times p}$ (or simply, $\mathbf{0}$) and $\mathbf{I}_m$ (or simply, $\mathbf{I}$) to denote the $m \times p$ zero matrix and the $m \times m$ identity matrix, respectively.

B. State Space Model

We consider a discrete-time linear system that is described by the following recursive equation:

$$x(t+1) = \mathbf{A}x(t) + \mathbf{B}u(t), \quad t \in [0, N-1]_d, \tag{1}$$

with $x(0) = x_0$, where $x(t) \in \mathbb{R}^n$ denotes that state of the system at time t, for $t \in [0,N]_d$, $u(t) \in \mathbb{R}^m$ denotes the input applied to the system at time t, for $t \in [0, N-1]_d$ and $\mathbf{A} \in \mathbb{R}^{n\times n}$, $\mathbf{B} \in \mathbb{R}^{n\times m}$ and $x_0 \in \mathbb{R}^n$ are given quantities.

For a given (finite-length) control input sequence $U_{N-1} := \{u(t) \in \mathbb{R}^m :\ t \in [0, N-1]_d\}$, the solution to (1) will satisfy the following equation:

$$x(t) = \mathbf{A}^t x_0 + \sum_{\tau=0}^{t-1} \mathbf{A}^{t-1-\tau}\mathbf{B}u(\tau), \tag{2}$$

for all $t \in [1,N]_d$. Equation (2), which can be written more compactly as follows:

$$\boldsymbol{x} = \mathbf{H}\boldsymbol{u} + \mathbf{\Gamma}x_0, \tag{3}$$

where $\boldsymbol{x} := [x(0)^{\mathrm{T}}, \ldots, x(N)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{(N+1)n}$, $\boldsymbol{u} := [u(0)^{\mathrm{T}}, \ldots, u(N-1)^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{Nm}$. Finally, $\mathbf{H} \in \mathbb{R}^{(N+1)n\times Nm}$ and $\mathbf{\Gamma} \in \mathbb{R}^{(N+1)n\times n}$ and in particular, $\mathbf{H}$ corresponds to a *block lower triangular* matrix of dimension $(N+1)\times N$ whose blocks, which are denoted by $\mathbf{H}_{i,j}$, are defined as follows:

$$\mathbf{H}_{i,j} := \begin{cases} \mathbf{A}^{i-1-j}\mathbf{B}, & \text{if} \quad i \geq j+1, \\ \mathbf{0}, & \text{if} \quad i < j+1, \end{cases}$$

for $(i,j) \in [1, N+1]_d \times [1,N]_d$, whereas $\mathbf{\Gamma} := \big[\mathbf{I}_n, \mathbf{A}^{\mathrm{T}}, \ldots, (\mathbf{A}^{N-1})^{\mathrm{T}}, (\mathbf{A}^N)^{\mathrm{T}}\big]^{\mathrm{T}}$.

III. Problem Formulation

A. A Quick Review of the Minimum ℓ_2-norm Controllability Problem and its Formulation as a Quadratic Program

In this section, we quickly review the process of reducing the minimum ℓ_2-norm controllability problem for discrete-time linear systems into an equivalent convex quadratic program subject to a set of linear equality constraints. First, we give the precise formulation of this controllability problem as an ℓ_2 optimal control problem.

Problem 1: Let $x_0 \in \mathbb{R}^n$ and $N \in \mathbb{Z}^{++}$ be given. Find an input sequence $U^\star_{N-1} := \{u^\star(t) \in \mathbb{R}^m :\ t \in [0, N-1]_d\}$ that will steer the system described by (1) from state $x = x_0$ at stage $t = 0$ to the origin $x = 0$ at stage $t = N$ while minimizing the performance index $J_2(U_{N-1}) := \|U_{N-1}\|^2_{\ell_2}$.

Next we reduce the minimum ℓ_2-norm controllability problem (Problem 1) to a linearly constrained convex quadratic program (QP) in terms of $\boldsymbol{u}$ (for more details on the reduction of quadratic optimal control problems for discrete-time linear systems to QPs, the reader may refer to [12, Chapter 5]). To this aim, we first note that

$$\begin{aligned} J_2(U_{N-1}) = \|U_{N-1}\|^2_{\ell_2} &= \sum_{t=0}^{N-1} u(t)^{\mathrm{T}}u(t) \\ &= \sum_{t=0}^{N-1}\sum_{k=1}^{m} |u_{(k)}(t)|^2 = |\boldsymbol{u}|_2^2 =: \mathcal{J}_2(\boldsymbol{u}). \end{aligned} \tag{4}$$

Next, we express the terminal constraint, $x(N) = 0$, as an equality constraint in terms of $\boldsymbol{u}$. In particular, we have that

$x(N) = \mathbf{\Pi}_N \boldsymbol{x}$, where $\mathbf{\Pi}_N$ is a block row vector comprised of $N+1$ blocks from which the first N ones are equal to $\mathbf{0}_{n\times n}$ and the last one is equal to $\mathbf{I}_n$. In view of (3), we have that $x(N) = \mathbf{\Pi}_N(\mathbf{H}\boldsymbol{u} + \mathbf{\Gamma} x_0)$ or, equivalently,

$$\boldsymbol{\mathcal{C}}_N \boldsymbol{u} = \boldsymbol{\beta}, \tag{5}$$

where $\boldsymbol{\mathcal{C}}_N \in \mathbb{R}^{n\times Nm}$ and $\boldsymbol{\beta} \in \mathbb{R}^n$ are defined as follows:

$$\boldsymbol{\mathcal{C}}_N := \mathbf{\Pi}_N \mathbf{H} = \left[\mathbf{A}^{N-1}\mathbf{B}, \dots, \mathbf{B}\right], \tag{6a}$$

$$\boldsymbol{\beta} := -\mathbf{\Pi}_N \mathbf{\Gamma} x_0 = -\mathbf{A}^N x_0. \tag{6b}$$

Problem 2: Find a vector $\boldsymbol{u}^\star \in \mathbb{R}^{Nm}$ that minimizes the convex quadratic performance index $\mathcal{J}_2(\boldsymbol{u}) := \boldsymbol{u}^{\mathrm{T}}\boldsymbol{u} = |\boldsymbol{u}|^2$ subject to the equality constraint $\boldsymbol{\mathcal{C}}_N \boldsymbol{u} = \boldsymbol{\beta}$, where $\boldsymbol{\mathcal{C}}_N$ and $\boldsymbol{\beta}$ are defined, respectively, in (6a)-(6b).

Proposition 1: Suppose that the matrix $\boldsymbol{\mathcal{C}}_N \in \mathbb{R}^{n\times Nm}$, which is defined in (6a) is full row rank, that is, $\mathrm{rank}(\boldsymbol{\mathcal{C}}_N) = n$. Then, Problem 2 admits a unique solution for all $x_0 \in \mathbb{R}^n$, which is denoted by $\boldsymbol{u}^\star$ and satisfies the following equation:

$$\boldsymbol{u}^\star = \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}(\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}})^{-1}\boldsymbol{\beta}, \tag{7}$$

where $\boldsymbol{\beta} \in \mathbb{R}^n$ is defined in (6b). Consequently, the control sequence $U_{N-1}^\star := \{u^\star(t) \in \mathbb{R}^m : \ t \in [0, N-1]_d\}$, where

$$u^\star(t) = \boldsymbol{\mathcal{P}}_t \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}(\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}})^{-1}\boldsymbol{\beta}, \qquad t \in [0, N-1]_d, \tag{8}$$

where $\boldsymbol{\mathcal{P}}_t \in \mathbb{R}^{m\times Nm}$ is a block row vector comprised of N blocks from which the $(t+1)$-th block is equal to $\mathbf{I}_m$ whereas all the other ones are equal to $\mathbf{0}_{m\times m}$, solves Problem 1.

Proof: Problem 2 seeks for the minimum 2-norm solution of the system of linear equations given in (5). By hypothesis, the matrix $\boldsymbol{\mathcal{C}}_N \in \mathbb{R}^{n\times Nm}$ is full row rank and thus the latter system of equations will admit at least one solution for any $\boldsymbol{\beta} \in \mathbb{R}^n$, and thus for any $x_0 \in \mathbb{R}^n$, in view of (6b). In addition, among all the solutions $\boldsymbol{u} \in \mathbb{R}^{Nm}$ of (5), the one that has the minimum 2-norm is given by (7) (see, for instance, Proposition 6.3 in [13]). ■

Note that for the evaluation of the right hand side of (7), one needs to multiply $\boldsymbol{\mathcal{C}}_N$ with $\boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$, which costs $\mathcal{O}(n^2 Nm)$ flops (here, $\mathcal{O}$ denotes the big-O Landau symbol). Instead of directly performing the multiplication between $\boldsymbol{\mathcal{C}}_N$ and $\boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$, we observe that

$$\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}} = \sum_{t=0}^{N-1} \mathbf{A}^t \mathbf{B}\mathbf{B}^{\mathrm{T}}(\mathbf{A}^{\mathrm{T}})^t. \tag{9}$$

Using (9) can significantly reduce the cost for the computation of $\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$ given that the latter is now expressed as the sum of N matrix products of the form $\boldsymbol{C}_t \boldsymbol{C}_t^{\mathrm{T}}$, where $\boldsymbol{C}_t := \mathbf{A}^t \mathbf{B} \in \mathbb{R}^{n\times m}$, for $t \in [0, N-1]_d$; the computation of each of these products requires $\mathcal{O}(n^2 m)$ flops. A more significant reduction over the cost of computing $\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$ can be achieved, if one computes the product $\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$ recursively. This is possible because $\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$ is actually equal to the so-called reachability Grammian [14] of the discrete-time linear system given in (1) evaluated at stage $t = N-1$, which is denoted by $\boldsymbol{\mathcal{R}}(N-1)$, where $\boldsymbol{\mathcal{R}}(t)$, $t \in [0, N-1]_d$ corresponds to the solution of the following recursive (or difference) Lyapunov equation:

$$\boldsymbol{\mathcal{R}}(t+1) = \mathbf{A}\boldsymbol{\mathcal{R}}(t)\mathbf{A}^{\mathrm{T}} + \mathbf{B}\mathbf{B}^{\mathrm{T}}, \quad t \in [0, N-2]_d, \tag{10}$$

with $\boldsymbol{\mathcal{R}}(0) = \mathbf{B}\mathbf{B}^{\mathrm{T}}$.

Finally, we note that the number of required flops for the inversion of $\boldsymbol{\mathcal{C}}_N \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}$ is $\mathcal{O}(n^3)$, which is independent of the number of stages, $N+1$.

B. The Minimum ℓ_1-norm Controllability Problem

The minimum ℓ_1-norm controllability problem can be formulated similarly to the minimum ℓ_2-norm controllability problem (Problem 1), after the necessary modifications have been carried out, as follows:

Problem 3: Let $x_0 \in \mathbb{R}^n$ and $N \in \mathbb{Z}^{++}$ be given. Find a control input sequence $U_{N-1}^\star := \{u^\star(t) \in \mathbb{R}^m : \ t \in [0, N-1]_d\}$ that will steer the system described by (1) from state $x = x_0$ at stage $t = 0$ to the origin $x = 0$ at stage $t = N$ while minimizing the performance index $J_1(U_{N-1}) := \|U_{N-1}\|_{\ell_1}$.

Next, we convert Problem 3 into an equivalent convex optimization problem. To this aim, we note that $J_1(U_{N-1})$ can be written as follows:

$$J_1(U_{N-1}) = \|U_{N-1}\|_{\ell_1} = \sum_{t=0}^{N-1} |u(t)|_1$$

$$= \sum_{t=0}^{N-1}\sum_{k=1}^{m} |u_{(k)}(t)| = |\boldsymbol{u}|_1 =: \mathcal{J}_1(\boldsymbol{u}). \tag{11}$$

Furthermore, the terminal constraint $x(N) = 0$ yields an equality constraint in terms of $\boldsymbol{u}$, which is given in (5).

Problem 4: Find a vector $\boldsymbol{u}^\star \in \mathbb{R}^{Nm}$ that minimizes the convex quadratic performance index $\mathcal{J}_1(\boldsymbol{u}) := |\boldsymbol{u}|_1$ subject to the equality constraint $\boldsymbol{\mathcal{C}}_N \boldsymbol{u} = \boldsymbol{\beta}$, where $\boldsymbol{\mathcal{C}}_N$ and $\boldsymbol{\beta}$ are defined, respectively, in (6a)-(6b).

One way to characterize the solution to Problem 4 is to employ the so-called modified Least Angle Regression (LARS) algorithm (also known as the *homotopy* method for ℓ_1-norm minimization problems). The LARS algorithm will generate a sequence of control input vectors $\{\boldsymbol{u}_k\}_{k\in\mathbb{Z}^+}$ that will eventually converge to the solution to Problem 4. In particular, for each $k \in \mathbb{Z}^+$, $\boldsymbol{u}_k$ is equal to a (global) minimizer of the following convex function $\mathfrak{J}_k(\cdot) : \mathbb{R}^{Nm} \to \mathbb{R}$ with

$$\mathfrak{J}_k(\boldsymbol{u}) := (1/2)|\boldsymbol{\mathcal{C}}_N \boldsymbol{u} - \boldsymbol{\beta}|_2^2 + \lambda_k |\boldsymbol{u}|_1, \quad k \in \mathbb{Z}^+, \tag{12}$$

where $\{\lambda_k\}_{k\in\mathbb{Z}^+}$ is a non-decreasing sequence of non-negative numbers that converges to zero (from above).

Proposition 2: Let $\{\lambda_k\}_{k\in\mathbb{Z}^+}$ be a non-decreasing sequence of non-negative numbers with $\lim_{k\to\infty} \lambda_k = 0$. Furthermore, let $\boldsymbol{u}_k \in \mathbb{R}^{Nm}$ denote the minimizer of the function $\mathfrak{J}_k(\cdot) : \mathbb{R}^{Nm} \to \mathbb{R}$, where $\mathfrak{J}_k(\boldsymbol{u})$ is defined in (12), for every $k \in \mathbb{Z}^+$. Then, the sequence $\{\boldsymbol{u}_k\}_{k\in\mathbb{Z}^+}$ will converge to a point $\bar{\boldsymbol{u}}$ that is a solution to Problem 4. If in addition, Problem 4 admits a unique solution, $\boldsymbol{u}^\star$, then $\lim_{k\to\infty} \boldsymbol{u}_k = \bar{\boldsymbol{u}} = \boldsymbol{u}^\star$.

Proof: The reader may refer to [15]. ■

Proposition 2 implies that the application of the LARS algorithm will give the solution to Problem 4 as the limit of a sequence of points formed by the global minimizers of a sequence of unconstrained convex optimization problems. Although, this approach can be implemented in practice given the proliferation of convex optimization algorithms and the relevant computational tools, it cannot furnish a solution to Problem 4 in closed form. To see why this is the case, it suffices to note that the convex function $\mathfrak{J}_k(\cdot)$ is not differentiable everywhere due to the existence of the term $\lambda|\boldsymbol{u}|_1$. Thus, the computation of the global minimizers of $\mathfrak{J}_k(\cdot)$ requires the characterization of its sub-differential, which is denoted by $\partial\mathfrak{J}_k(\cdot)$ and defined as the set-valued mapping $\boldsymbol{u} \mapsto \partial\mathfrak{J}_k(\boldsymbol{u})$ with $\partial\mathfrak{J}_k(\boldsymbol{u}) := \{\boldsymbol{z} \in \mathbb{R}^{Nm} : \ \mathfrak{J}_k(\boldsymbol{v}) - \mathfrak{J}_k(\boldsymbol{u}) \geq \boldsymbol{z}^{\mathrm{T}}(\boldsymbol{v}-\boldsymbol{u}), \text{ for all } \boldsymbol{v} \in \mathbb{R}^{Nm}\}$; note that $\partial\mathfrak{J}_k(\boldsymbol{u})$ is a convex and compact set. It follows that

$$\partial\mathfrak{J}_k(\boldsymbol{u}) = \{\boldsymbol{z} \in \mathbb{R}^{Nm} : \ \boldsymbol{z} = \boldsymbol{\mathcal{C}}_N^{\mathrm{T}}(\boldsymbol{\mathcal{C}}_N\boldsymbol{u} - \boldsymbol{\beta}) + \lambda_k\boldsymbol{\zeta}, \\ \boldsymbol{\zeta} \in \partial|\boldsymbol{u}|_1\},$$

where $\partial|\boldsymbol{u}|_1 = \{\boldsymbol{\zeta} \in \mathbb{R}^{Nm} : \ \boldsymbol{\zeta}_{(\ell)} \in \partial|\boldsymbol{u}_{(\ell)}|, \ \ell \in [1, Nm]_d\}$ with $\partial|\boldsymbol{u}_{(\ell)}| = \{\mathrm{sign}(\boldsymbol{u}_{(\ell)})\}$, when $\boldsymbol{u}_{(\ell)} \neq 0$, and $\partial|\boldsymbol{u}|_1 = [-1, 1]$, otherwise, for $\ell \in [1, Nm]_d$. We know that a vector $\boldsymbol{u}_k$ is a (global) minimizer of $\mathfrak{J}_k(\cdot)$, if and only if $\mathbf{0} \in \partial\mathfrak{J}_k(\boldsymbol{u}_k)$ [16] or equivalently,

$$\big(\boldsymbol{\mathcal{C}}_N^{\mathrm{T}}(\boldsymbol{\mathcal{C}}_N\boldsymbol{u}_k - \boldsymbol{\beta})\big)_{(\ell)} = \begin{cases} -\lambda_k\mathrm{sign}\big((\boldsymbol{u}_k)_{(\ell)}\big), & \text{if } (\boldsymbol{u}_k)_{(\ell)} \neq 0, \\ \nu \in [-\lambda_k, \lambda_k], & \text{otherwise}, \end{cases}$$

for all $\ell \in [1, Nm]_d$. It should be clear from the previous discussion that, in general, the characterization of the exact solution to Problem 4 in closed form, based on the LARS algorithm or any other direct solution approach, is practically impossible.

IV. Semi-Analytic Iterative Approach to the Minimum ℓ_1-Norm Controllability Problem

On the basis of the previous discussion, it is more prudent to look for a solution approach that will allow us to characterize in closed-form an approximation of the solution to Problem 4. The proposed approach will be based on the so-called *iteratively reweighted least squares* (IRLS) algorithm, which is a very popular tool for ℓ_1 optimization problems (also known as *basis pursuit* problems) and problems with sparsity constraints in the literature of compressive sensing [15] to yield a "proxy" to the optimal solution to Problem 4. The approximate solution to Problem 4 will in turn furnish a control sequence that is a proxy to the solution of the minimum ℓ_1-norm controllability problem (Problem 3), which can be characterized in closed form.

Next, we present the main steps of the proposed iterative approach for the characterization of a suboptimal solution to Problem 4 in analytic form. To this aim, we consider the following augmented performance index:

$$\begin{aligned}\tilde{\mathcal{J}}_1(\boldsymbol{u}, \boldsymbol{w}; \varepsilon) &:= \boldsymbol{u}^{\mathrm{T}}\mathrm{diag}(\boldsymbol{w})\boldsymbol{u} + \varepsilon^2\mathbf{1}^{\mathrm{T}}\boldsymbol{w} + \mathbf{1}^{\mathrm{T}}\boldsymbol{w}^{\dagger} \\ &= \sum_{t=0}^{N-1}\sum_{k=1}^{m} w_{(k)}(t)|u_{(k)}(t)|^2 \\ &\quad + \sum_{t=0}^{N-1}\sum_{k=1}^{m}\big(\varepsilon^2 w_{(k)}(t) + 1/w_{(k)}(t)\big), \end{aligned} \tag{13}$$

where $\boldsymbol{w} := \big[w(0)^{\mathrm{T}}, \ldots, w(N-1)^{\mathrm{T}}\big]^{\mathrm{T}} \in \mathbb{R}^{Nm}_{>0}$, that is, $w_k(t) > 0$ for all $t \in [0, N-1]_d$ and $k \in [1, m]_d$, $\boldsymbol{w}^{\dagger} := \big[w^{\dagger}(0)^{\mathrm{T}}, \ldots, w^{\dagger}(N-1)^{\mathrm{T}}\big]^{\mathrm{T}} \in \mathbb{R}^{Nm}_{>0}$ with $w^{\dagger}(t) = [1/w_{(1)}(t), \ldots, 1/w_{(m)}(t)]^{\mathrm{T}}$ for all $t \in [0, N-1]_d$, and finally, $\varepsilon \geq 0$. We observe that $\tilde{\mathcal{J}}_1(\boldsymbol{u}; \boldsymbol{w}, \varepsilon)$ can be also expressed as follows:

$$\tilde{\mathcal{J}}_1(\boldsymbol{u}, \boldsymbol{w}; \varepsilon) = \tilde{\mathcal{J}}_1^{\alpha}(\boldsymbol{u}; \boldsymbol{w}) + \tilde{\mathcal{J}}_1^{\beta}(\boldsymbol{w}; \varepsilon), \tag{14}$$

where $\tilde{\mathcal{J}}_1^{\alpha}(\boldsymbol{u}; \boldsymbol{w}) := \boldsymbol{u}^{\mathrm{T}}\mathrm{diag}(\boldsymbol{w})\boldsymbol{u}$ and $\tilde{\mathcal{J}}_1^{\beta}(\boldsymbol{w}; \varepsilon) := \varepsilon^2\mathbf{1}^{\mathrm{T}}\boldsymbol{w} + \mathbf{1}^{\mathrm{T}}\boldsymbol{w}^{\dagger}$. The reason why $\tilde{\mathcal{J}}_1^{\alpha}$ is considered to be a function of $\boldsymbol{u}$ whereas $\boldsymbol{w}$ is treated as a known parameter is because it will serve later on as the performance index of an optimization problem whose decision variable is $\boldsymbol{u}$. Similarly, $\tilde{\mathcal{J}}_1^{\beta}$ is considered to be a function of $\boldsymbol{w}$ only, for a given parameter ε.

Next, we describe the main steps of IRLS algorithm tailored to Problem 4. The discussion that will be given next will follow the exposition presented in [15, Chapter 15]. We will omit most proofs, which can be found in the relevant literature of compressive sensing. For the execution of the algorithm, we will assume that we are given the following data: $\gamma > 0$, $\sigma \in [1, Nm]_d$, $\epsilon_{\mathrm{tol}} > 0$ and $\bar{\varepsilon} > 0$.

Step 0: Set $\boldsymbol{w}^{[0]} := \mathbf{1}$, $\varepsilon^{[0]} := 1$ and $j := 0$.

Step 1: Set $\boldsymbol{u}^{[j+1]} := \boldsymbol{u}^{\star}_{\mathrm{IRLS}}$, where $\boldsymbol{u}^{\star}_{\mathrm{IRLS}}$ corresponds to the solution of the following convex QP problem:

$$\min \tilde{\mathcal{J}}_1^{\alpha}(\boldsymbol{u}; \boldsymbol{w}_j), \quad \text{subject to} \quad \boldsymbol{\mathcal{C}}_N\boldsymbol{u} = \boldsymbol{\beta}. \tag{15}$$

In view of Proposition 1 together with the following change of variables $\tilde{\boldsymbol{u}} = (\boldsymbol{\mathcal{W}}^{[j]})^{1/2}\boldsymbol{u}$, where $\boldsymbol{\mathcal{W}}^{[j]} := \mathrm{diag}(\boldsymbol{w}^{[j]})$, it follows readily that

$$\boldsymbol{u}^{\star}_{\mathrm{IRLS}} = (\boldsymbol{\mathcal{W}}^{[j]})^{-1}\boldsymbol{\mathcal{C}}_N^{\mathrm{T}}(\boldsymbol{\mathcal{C}}_N(\boldsymbol{\mathcal{W}}^{[j]})^{-1}\boldsymbol{\mathcal{C}}_N^{\mathrm{T}})^{-1}\boldsymbol{\beta}. \tag{16}$$

Step 2: Set $\varepsilon^{[j+1]} := \min\{\varepsilon^{[j]}, \gamma\boldsymbol{v}_{(\sigma+1)}\}$, where $\boldsymbol{v} := \pi^{-}(\boldsymbol{u}^{[j+1]})$.

Step 3: Set $\boldsymbol{w}^{[j+1]} := \boldsymbol{w}^{\star}_{\mathrm{IRLS}}$, where $\boldsymbol{w}^{\star}_{\mathrm{IRLS}}$ corresponds to the solution of the following optimization problem:

$$\min\big(\tilde{\mathcal{J}}_1^{\beta}(\boldsymbol{w}; \varepsilon^{[j+1]}) + \bar{\mathcal{J}}_1^{\alpha}(\boldsymbol{w})\big), \quad \boldsymbol{w} \in \mathbb{R}^{Nm}_{>0}, \tag{17}$$

where $\bar{\mathcal{J}}_1^{\alpha}(\boldsymbol{w}) := \tilde{\mathcal{J}}_1^{\alpha}(\boldsymbol{u}^{[j+1]}; \boldsymbol{w})$. It is not hard to show that the components of the vector of weights $\boldsymbol{w}^{\star}_{\mathrm{IRLS}}$ can be determined by the following equation:

$$(\boldsymbol{w}^{\star}_{\mathrm{IRLS}})_{(\ell)} = 1/\sqrt{(\boldsymbol{u}^{[j+1]}_{(\ell)})^2 + (\varepsilon^{[j+1]})^2}, \quad \ell \in [1, Nm]_d. \tag{18}$$

Step 4: Set $j := j+1$. If $j \leq j_{\max}$ and $\varepsilon^{[j]} \in [0, \bar{\varepsilon}]$, then report "success" and stop. If $\varepsilon^{[j]} \notin [0, \bar{\varepsilon}]$, then i) if $j < j_{\max}$, go to **Step 1**, and ii) if $j = j_{\max}$, report "failure."

The following proposition illuminates an important property that is enjoyed by the successive iterates of the previously described IRLS algorithm [15, Lemma 15.8]:

Proposition 3: Let $\boldsymbol{u}^{[j]}$ denote the vector generated at the j-th iteration of the IRLS algorithm. Then, $\lim_{j\to\infty}(\boldsymbol{u}^{[j]} - \boldsymbol{u}^{[j-1]}) = 0$.

It is very important to highlight that the fact that $\lim_{j\to\infty}(\boldsymbol{u}^{[j]} - \boldsymbol{u}^{[j-1]}) = 0$ does not necessarily imply that the sequence $\{\boldsymbol{u}^{[j]}\}_{j\in\mathbb{Z}^+}$ will also be Cauchy, and thus convergent (in view of the completeness of $\mathbb{R}^{Nm}$). In order to establish the convergence of the sequence $\{\boldsymbol{u}^{[j]}\}_{j\in\mathbb{Z}^+}$, which is generated with the application of the IRLS algorithm, to an s-sparse vector, that is, a vector with at most s non-zero elements, for a given $s \in [1, Nm]_d$, then we will need the following key assumption.

Assumption 1: There is a positive integer $s \in [1, Nm]_d$ such that matrix $\boldsymbol{\mathcal{C}}_N$ enjoys the *null space property of order* s, that is, for any subset $\mathcal{S}$ of $[1, Nm]_d$ with $\mathrm{card}(\mathcal{S}) \leq s$, it holds true that $|\boldsymbol{\rho}_{(\mathcal{S})}|_1 < |\boldsymbol{\rho}_{(\mathcal{S}^c)}|_1$, for all non-zero vectors $\boldsymbol{\rho}$ in the null space of $\boldsymbol{\mathcal{C}}_N$, $\mathrm{null}(\boldsymbol{\mathcal{C}}_N)$, where $\boldsymbol{\rho}_{(\mathcal{S})}$ (respectively, $\boldsymbol{\rho}_{(\mathcal{S}^c)}$) denotes the $\mathrm{card}(\mathcal{S})$-dimensional (respectively, $\mathrm{card}(\mathcal{S}^c)$-dimensional) vector formed by the components $\boldsymbol{\rho}_{(i)}$ of $\boldsymbol{\rho}$ with $i \in \mathcal{S}$ (resp., $i \in \mathcal{S}^c$).

Next, we give the main convergence result for the IRLS algorithm. The (rather lengthy) proof of this result can be found in the relevant literature [8], [15].

Theorem 1: Suppose that Assumption 1 holds true. Then, the sequence $\{\boldsymbol{u}^{[j]}\}_{j\in\mathbb{Z}^+}$ will converge to a point $\boldsymbol{u}_\circ$, that is, $\lim_{j\to\infty}\boldsymbol{u}^{[j]} = \boldsymbol{u}_\circ$. In addition, if $\lim_{j\to\infty}\varepsilon^{[j]} = 0$, then $\boldsymbol{u}_\circ$ is an s-sparse minimizer of Problem 4. If $\lim_{j\to\infty}\varepsilon^{[j]} = \varepsilon^\circ > 0$, then $\boldsymbol{u}_\circ$ is a global minimizer of the following problem:

$$\min \hat{\mathcal{J}}(\boldsymbol{u}), \quad \text{subject to} \;\; \boldsymbol{\mathcal{C}}_N\boldsymbol{u} = \boldsymbol{\beta},$$

where $\hat{\mathcal{J}}(\boldsymbol{u}) := \sum_{t=0}^{N-1}\sum_{k=1}^{m}\sqrt{|u_{(k)}(t)|^2 + (\varepsilon^\circ)^2}$.

Note that after the IRLS algorithm has converged to a vector $\boldsymbol{u}^\circ \in \mathbb{R}^{Nm}$ for a given vector $\boldsymbol{\beta} \in \mathbb{R}^n$, which is in turn determined by the given initial state $x_0 \in \mathbb{R}^n$ in accordance with Eq. (6b), then a control sequence that is an approximation to the solution to the minimum ℓ_1-norm controllability problem (Problem 3) from $x_0 \in \mathbb{R}^n$ at $t = 0$ to the origin $x = 0$ at $t = N$ can be characterized as follows:

$$\hat{U}^\star_{N-1} := \{\hat{u}^\star(t) = \boldsymbol{\mathcal{P}}_t\boldsymbol{u}^\circ : \quad t \in [0, N-1]_d\}, \tag{19}$$

where $\boldsymbol{\mathcal{P}}_t \in \mathbb{R}^{m\times Nm}$ is a block row vector comprised of N blocks from which the $(t+1)$-th block is equal to $\mathbf{I}_m$ whereas all the other ones are equal to $\mathbf{0}_{m\times m}$. Note that $\boldsymbol{u}^\circ$ can be written as follows:

$$\boldsymbol{u}^\circ = (\boldsymbol{\mathcal{W}}^{[j^\circ]})^{-1}\boldsymbol{\mathcal{C}}_N^{\mathrm{T}}(\boldsymbol{\mathcal{C}}_N(\boldsymbol{\mathcal{W}}^{[j^\circ]})^{-1}\boldsymbol{\mathcal{C}}_N^{\mathrm{T}})^{-1}\boldsymbol{\beta}, \tag{20}$$

where j° corresponds to the iteration at which the convergence criterion of the IRLS algorithm, which is given in **Step 4**, was met for the first time. In light of (20), we can claim that the control input sequence that serves as proxy to the solution to the minimum ℓ_1-norm controllability problem (Problem 3), provided that Assumption 1 holds true, can be characterized in closed form.

V. Numerical Simulations

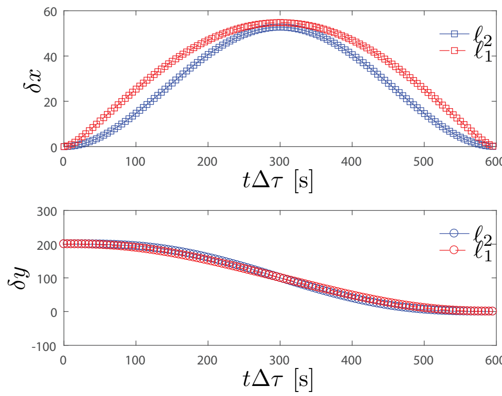
In this section, we will illustrate the applicability of the results presented so far in the class of space proximity operations. In these operations, the continuous use of "corrective" thrust maneuvers by a spacecraft can be too costly, in terms of fuel usage. It is well known that optimal maneuvers, such as minimum fuel maneuvers, may require the application of a very small number of "impulsive" corrective maneuvers (see for instance, [17]), which can be viewed as abstractions of "large" control inputs similar to those that typically appear in the solution of an ℓ_1 optimal control problem. We will consider, in particular, the so-called spacecraft rendezvous problem (a special class of space proximity operations) for two spacecraft moving along a geosynchronous circular orbit (of radius $\alpha = 42164\text{E+}03[\mathrm{m}]$).

The linearized relative motion of the second vehicle with respect to the first one (the "reference" vehicle) is described in continuous time by the so-called Clohessy Wiltshire (CW) equations:

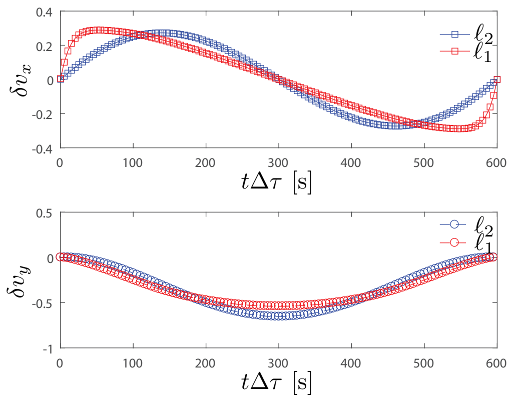
$$\dot{x}(t) = \mathbf{A}_c x(t) + \mathbf{B}_c u(t), \qquad x(0) = x_0, \tag{21}$$

where $x = [\delta x,\ \delta y,\ \delta v_x,\ \delta v_y]^{\mathrm{T}}$ and $\mathbf{A}_c = \begin{bmatrix}\mathbf{0}_{2\times 2} & \mathbf{I}_2\\ \mathbf{A}_3 & \mathbf{A}_4\end{bmatrix}$, $\mathbf{A}_3 = \left[\begin{smallmatrix}3n^2T^2 & 0\\ 0 & 0\end{smallmatrix}\right]$, $\mathbf{A}_4 = \left[\begin{smallmatrix}0 & 2nT\\ -2nT & 0\end{smallmatrix}\right]$ and $\mathbf{B}_c = (FT^2)/(mS)\begin{bmatrix}\mathbf{0}_{2\times 2} & \mathbf{I}_2\end{bmatrix}^{\mathrm{T}}$ and $u = [u_x,\ u_y]^{\mathrm{T}}$. In this model, $[\delta x,\ \delta y]^{\mathrm{T}}$ and $[\delta v_x,\ \delta v_y]^{\mathrm{T}}$ correspond to, respectively, the relative position vector and the relative velocity of the second spacecraft with respect to the first one at time t. Furthermore, m corresponds to the mass of the second vehicle and F, T, and S are normalization constants. The corresponding discrete-time model is described by the recursive equation given in (1) with $\mathbf{A} = \exp(\Delta\tau\mathbf{A}_c)$, $\mathbf{B} = \int_0^{\Delta\tau}\exp(s\mathbf{A}_c)\mathrm{d}s\mathbf{B}_c$, where $\Delta\tau > 0$ is the discretization step. For our simulations, we have used the following initial position and velocity vectors $[200,\ 0]^{\mathrm{T}}$ in $[\mathrm{m}]$ and $[0,\ 0]^{\mathrm{T}}$ in $[\mathrm{m/s}]$, respectively, (which means that the second vehicle is initially 200 m ahead of the reference vehicle while both of them travel along the same circular orbit with the same speed) and the following data: $n = 7.2922\text{E}{-}05$, $S = 100[\mathrm{m}]$, $T = 60[\mathrm{s}]$, $F = 0.1[\mathrm{N}]$, $m = 500[\mathrm{kg}]$, $\epsilon_{\mathrm{tol}} = 0.0065|\boldsymbol{u}^{[0]}|_1$, $\Delta\tau = 0.25[\mathrm{s}]$, $N = 2400$. The total duration of the proximity operation were taken to be 10 minutes. In addition, the values of the mass m of the second vehicle and the normalization constants F, T, and S were taken from [18]).

Figure 1 illustrates the evolution of the state components of system (21) driven by the control sequence that solves Problem 1 (minimum ℓ_2-norm controllability problem) and the control sequence that approximates the solution to Problem 4 (minimum ℓ_1-norm controllability problem), which is generated after ten iterations of the IRLS algorithm. Figure 2 illustrates the magnitude of the control input at each stage for both of the two utilized control sequences. We observe that the control sequence generated with the application of



(a) Time-evolution of the components of the position vector.



(b) Time-evolution of the components of the velocity vector.

Fig. 1. Time-evolution of the components of the state vector of the system driven by the control sequence that solves the ℓ_2 optimal control problem versus the control sequence that results from the application of the IRLS algorithm (approximate solution to the ℓ_1 optimal control problem).

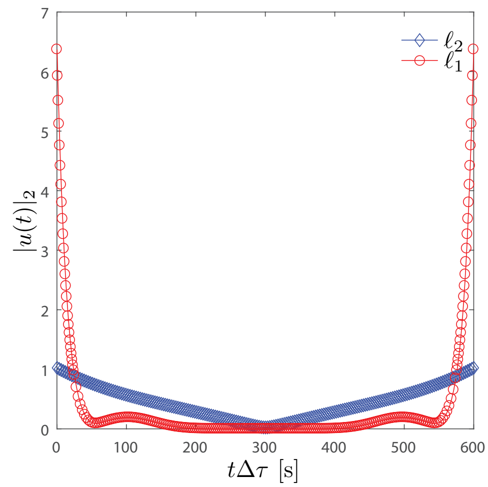


Fig. 2. Time-evolution of the 2-norm of the control input $u(t)$ from the control sequence that solves the ℓ_2 optimal control problem versus the one that results from the application of the IRLS algorithm (approximate solution to the ℓ_1 optimal control problem).

the IRLS algorithm consists of a significant number of null inputs and the magnitude of its control inputs is large only during a brief period at the beginning of the proximity operation and another one near its end. By contrast, the control sequence that solves Problem 1 is comprised mainly of non-zero control inputs of relatively small magnitude.

VI. Conclusion

In this work, we have presented an iterative scheme for the computation of an approximate solution to the minimum ℓ_1-norm controllability problem for discrete-time linear systems by solving a sequence of convex quadratic programs subject to linear constraints, each of which admits a closed form solution. The proposed approach is based on a popular algorithm from compressive sensing, namely the iteratively reweighted least square algorithm tailored to the control problem. In our future work, we will explore the minimum ℓ_1-norm controllability problem for continuous-time linear systems.

References

[1] S. Schuler, C. Ebenbauer, and F. Allgower, "ℓ_0-system gain and ℓ_1-optimal control," in *18th IFAC World Congress*, pp. 9230–9235, 2011.

[2] M. Gallieri and J. M. Maciejowski, "Lasso MPC: Smart regulation of over-actuated systems," in *2012 American Control Conference (ACC)*, pp. 1217–1222, June 2012.

[3] F. Lin, M. Fardad, and M. R. Jovanovic, "Design of optimal sparse feedback gains via the alternating direction method of multipliers," *IEEE Trans. Autom. Control*, vol. 58, no. 9, pp. 2426–2431, 2013.

[4] B. Polyak, M. Khlebnikov, and P. Shcherbakov, "An LMI approach to structured sparse feedback design in linear control systems," in *2013 European Control Conference (ECC)*, pp. 833–838, July 2013.

[5] R. P. Aguilera, G. Urrutia, R. A. Delgado, D. Dolz, and J. C. Aguero, "Quadratic model predictive control including input cardinality constraints," *IEEE Trans. Autom. Control*, vol. 62, no. 6, pp. 3068–3075, 2017.

[6] B. Demirel, E. Ghadimi, D. E. Quevedo, and M. Johansson, "Optimal control of linear systems with limited control actions: threshold-based event-triggered control," *Trans. Control Netw. Syst.*, 2017.

[7] D. Wipf and S. Nagarajan, "Iterative reweighted ℓ_1 and ℓ_2 methods for finding sparse solutions," *IEEE Journal of Selected Topics in Signal Processing*, vol. 4, no. 2, pp. 317–329, 2010.

[8] I. Daubechies, R. DeVore, M. Fornasier, and C. S. Gunturk, "Iteratively reweighted least squares minimization for sparse recovery," *Comm. Pure Appl. Math*, vol. 63, no. 1, pp. 1–38, 2010.

[9] R. Chartrand and W. Yin, "Iteratively reweighted algorithms for compressive sensing," in *2008 IEEE International Conference on Acoustics, Speech and Signal Processing*, pp. 3869–3872, March 2008.

[10] M. S. Asif and J. Romberg, "Fast and accurate algorithms for re-weighted ℓ_1-norm minimization," *IEEE Transactions on Signal Processing*, vol. 61, no. 23, pp. 5905–5916, 2013.

[11] M. S. Asif and J. Romberg, "Sparse recovery of streaming signals using ℓ_1-homotopy," *IEEE Transactions on Signal Processing*, vol. 62, no. 16, pp. 4209–4223, 2014.

[12] G. Goodwin, M. M. Seron, and J. A. de Dona, *Constrained Control and Estimation: An Optimisation Approach*. Springer Publishing Company, Incorporated, 1st ed., 2010.

[13] G. Calafiore and L. El Ghaoui, *Optimization Models*. Cambridge, UK: Cambridge University Press, 2014.

[14] P. Antsaklis and A. N. Michel, *Linear Systems*. Boston, MA: Birkhäuser, 2005.

[15] S. Foucart and H. Rauhut, *A Mathematical Introduction to Compressive Sensing*. Birkhauser Basel, 2013.

[16] S. Boyd and L. Vandenberghe, *Convex Optimization*. Cambridge University Press, 2004.

[17] A. Miele, M. W. Weeks, and M. Ciarcià, "Optimal trajectories for spacecraft rendezvous," *Journal of Optimization Theory and Applications*, vol. 132, no. 3, pp. 353–376, 2007.

[18] M. Gulino, "A homotopic approach to solve the fuel optimal spacecraft proximity operations problem," Master's thesis, University of Texas at Austin, 2016.