Analysis and Synthesis of Nonlinear Controllers for Input Constrained Systems Using Semidefinite Programming Optimization

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***Abstract*— This work studies the problem of analyzing and, subsequently, optimizing the stabilization capabilities of a class of controllers for input constrained nonlinear systems. The proposed techniques apply to continuous, state feedback controllers which are defined in a subset of the state space where the time derivative of a known, candidate Control Lyapunov Function (CLF) can be made negative definite under the input constraints. This set is associated with the domain of this class of controllers and depends on the CLF. The analysis problem concerns approximating the domain and is posed via appropriately formulated set containment relationships through the generalized S-procedure with sum of squares (SOS) constraints. The optimization problem is concerned with the adjustment or enlargement of the domain and constitutes a way of controller synthesis. These objectives are pursued via optimizing over the coefficients of polynomial CLFs, through a sequence of semidefinite programming (SDP) problems with SOS constraints. By partitioning the state space based on the structure of the input value set and building upon earlier results on SOS methods, the SDP problems are subject to only convex constraints, rendering thus the proposed techniques computationally viable. The capabilities of the proposed algorithms are demonstrated through numerical examples.**

I. Introduction

Control design based on Control Lyapunov Functions (CLFs) constitutes a common way to stabilize a nonlinear system [1]. For instance, by utilizing the backstepping algorithm [2], a wide class of nonlinear systems can be brought into the control affine form; using an appropriate CLF, the system can then be stabilized to the origin. Numerous extensions have also been proposed, covering topics such as robustness to disturbances and uncertainties [3], inverse optimality [3], [4], connections to receding horizon control [5], and constrained systems [6], [7].

Constraints in the system's input constitute a common nonlinearity, originating from the physical limitations of most systems' actuation mechanisms. A closed loop system may suffer a performance degradation or even become unstable, when its input constraints become active. In this paper, we focus our attention on the class of continuous feedback control laws for systems with nonlinear dynamics subject to polytopic input constraints. For this case, [6] proposed non-predictive control laws which are based on an appropriately formulated, state-dependent Quadratic Programming (QP) optimization problem, given a CLF and certain performance parameters. The subset of the state space where this QP problem is pointwisely feasible depends on the particular dynamics and the CLF; not on the performance objective. It is possible to associate a particular CLF with all such control laws based on it, and obtain in this way a family of controllers. Two critical tasks arise with regards to the *domain* of such a QP-based controller, which is defined as the set where the QP problem is and remains feasible, and the controller is guaranteed to work and stabilize the system:
(i) Calculating the domain of the controller family, for a given CLF. This *analysis* problem has connections to the problem of characterizing the *region of attraction* of a control system.
(ii) CLF techniques are powerful, yet, they may yield conservative results. Properties of the closed loop system, such as the domain of the family of controllers, can vary significantly as functions of the particular CLF. The second question of interest concerns the optimization of CLFs in order to reshape or enlarge the domain, for a given system and input value set. In this way, one can better utilize the system's resources and capabilities. Starting with some initial CLF and optimizing over the set of CLFs can result in new families of controllers, with improved properties (at least in terms of their domain); one can, therefore, regard this process as a method of controller *design* or *synthesis*.

Studying the behavior of dynamical systems with numerical means is often necessary in order to verify attributes of interest. Such problems often involve the search over the set of nonnegative functions with certain properties; this is an (at least) NP-complete problem. Parameterizing, though, the set of nonnegative functions of interest as multivariate polynomials which are sums of squares (SOS) can yield semidefinite programming (SDP) problems with linear matrix inequality (LMI) constraints; such problems are convex and can be solved in polynomial time [8]. Although SDP problems, such as the ones resulting from the parsing of SOS constraints, can still be of rather high dimension and, thus, be computationally demanding, the decrease in computational complexity is remarkable and has resulted in a significant body of work in the field. SOS methods have been used in Lyapunov-based analysis and verification of autonomous systems [8], [9], [10], [11], while results on a control synthesis problem, without, however, input constraints, have been presented in [12], [13].

In this paper, we study the problems of analyzing and, subsequently, synthesizing CLF-based controllers for nonlinear systems with polynomial dynamics subject to polytopic input constraints. To this end, we formulate appropriate SDP optimization problems with SOS constraints. Both the analysis and the synthesis problems refer to families of controllers

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(and as opposed to individual, polynomial control laws), which are parameterized based on the CLF. To the best of the authors' knowledge, the present work is the first application of SDP optimization with SOS constraints to analysis and synthesis of input constrained nonlinear controllers based on CLFs. The input itself is not a variable of the SDP problems; only the values of the vertices of the input value set are utilized, following an appropriate partitioning of the state space. In this way, we avoid significantly large SDP problems or potentially intractable bilinearities, which could both occur otherwise. The efficacy and tractability of the proposed techniques are illustrated through numerical examples.

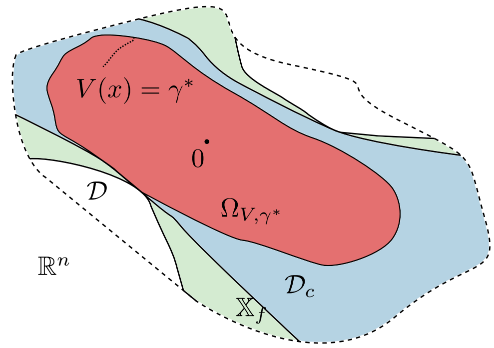

Fig. 1: Subsets of the state-space $\mathbb{R}^n$ of interest for problem.

II. Preliminaries

A. Notation

The set of n-dimensional real vectors is denoted by $\mathbb{R}^n$. The set of polynomials in $x \in \mathbb{R}^n$ of a maximum degree $2d$ which are SOS is Σ^{2d}. We denote by $Q \succ 0$ and $Q \succeq 0$ a (symmetric) positive definite and semidefinite matrix, respectively. For a $\mathcal{C}^2$ function $h : \mathbb{R}^n \to \mathbb{R}$, ∇h denotes its gradient (row vector), while $\nabla^2 h$ is the Hessian matrix. I_n is the $n \times n$ identity matrix. For any $x \in \mathbb{R}^n$, we denote the 2-norm by $\|x\|$, while given $y \in \mathbb{R}^n$ we denote the respective element-wise inequality by $x \succeq y$. The boundary and the interior of a set $\mathbb{S}$ are denoted by $\partial\mathbb{S}$ and $\text{Int}(\mathbb{S})$, respectively. $\mathbb{B}_r := \{x \in \mathbb{R}^n : \|x\| < r\}$ is the open ball around $x = 0$ with radius $r > 0$.

B. Nonlinear control based on Control Lyapunov Functions

We consider nonlinear control systems with dynamics

$$\dot{x} = f(x) + g(x)u, \qquad x(0) = x_0, \tag{1}$$

where $x \in \mathcal{D}$ is the state vector at time t with initial value x_0, and $\mathcal{D} \subseteq \mathbb{R}^n$ is an open set with $0 \in \text{Int}(\mathcal{D})$. Also, u is the input vector with $u(t) \in \mathbb{U} \subseteq \mathbb{R}^m$ for all $t \geq 0$, and $f : \mathcal{D} \to \mathbb{R}^n$, $g : \mathcal{D} \to \mathbb{R}^{n\times m}$ are vector and matrix valued polynomial functions of the state x, respectively, with $f(0) = 0$. Given a continuous, feedback control law $u_c : \mathcal{D}_c \to \mathbb{U}$, where $\mathcal{D}_c \subseteq \mathcal{D}$ with $0 \in \text{Int}(\mathcal{D}_c)$, the solution of (1) at time t is denoted by $\phi(t; x_0, u_c(\cdot))$; note that for ϕ to be well defined, it is assumed that $\phi(\tau; x_0, u_c(\cdot)) \in \mathcal{D}_c$ for $\tau \in [0, t)$. We assume that $\mathbb{U}$ is a convex, compact polytope, with vertices v_i for $i = 1 \ldots q$ and $0 \in \text{Int}(\mathbb{U})$, which can be parameterized by $\mathbb{U} := \{u \in \mathbb{R}^m : Au \preceq b\}$, where $A \in \mathbb{R}^{p\times m}$ and $b \in \mathbb{R}^p$ with $b \succ 0$.

We are interested in stabilizing (1) with control designs and methods based on Control Lyapunov Functions.

Definition 1: A $\mathcal{C}^1$ function $V : \mathbb{R}^n \to \mathbb{R}$ is a *Control Lyapunov Function* (CLF) for system (1), if it is positive definite, radially unbounded, and there is a set $\mathbb{X}_f \subseteq \mathcal{D}$ with $0 \in \text{Int}(\mathbb{X}_f)$ such that V satisfies

$$\inf_{u\in\mathbb{U}} \psi(x, u) < 0, \tag{2}$$

for all $x \in \mathbb{X}_f \setminus \{0\}$, where $\psi : \mathcal{D} \times \mathbb{U} \to \mathbb{R}$ with $\psi(x, u) := \nabla V(x) (f(x) + g(x)u)$.

Depending on the dynamics, $\mathbb{U}$, and the choice of V, (2) may only be satisfied in a subset of $\mathcal{D} \setminus \{0\}$, hence the need to consider the set $\mathbb{X}_f := \{x \in \mathcal{D} : \inf_{u\in\mathbb{U}} \psi(x, u) < 0\} \cup \{0\}$ where a CLF-based controller is pointwisely feasible. We are interested in mappings $u_c : \mathcal{D}_c \to \mathbb{U}$, where $\mathcal{D}_c \subseteq \mathbb{X}_f$ with $0 \in \text{Int}(\mathcal{D}_c)$, such that (2) holds *recursively* for all trajectories emanating from a positively invariant, compact subset of the control law's domain $\mathcal{D}_c$. A convenient parametrization of such a set can be given in terms of sublevel sets of the CLF V. First, let $\Omega_{V,\gamma} := \{x \in \mathbb{R}^n : V(x) \leq \gamma\}$. We also denote the largest γ such that $\Omega_{V,\gamma} \subseteq \mathbb{X}_f$ by γ^*. In this work it is assumed that such a *finite* γ^* exists.

Lemma 1: Let $\gamma > 0$ be such that $\Omega_{V,\gamma} \subseteq \mathbb{X}_f$. There exists a feedback control law $u_c : \Omega_{V,\gamma} \to \mathbb{U}$ such that $\dot{V} < 0$ holds for all $x \in \Omega_{V,\gamma} \setminus \{0\}$. Moreover, $\phi(t; x_0, u_c(\cdot)) \in \Omega_{V,\gamma}$ for all $t \geq 0$ and all $x_0 \in \Omega_{V,\gamma}$, and $\lim_{t\to\infty} \phi(t; x_0, u_c(\cdot)) = 0$.

Proof: Since $\Omega_{V,\gamma} \subseteq \mathbb{X}_f$, inequality (2) holds for all $x \in \Omega_{V,\gamma}\setminus\{0\}$. This implies that at every $x \in \Omega_{V,\gamma}$, there is a $u^* \in \mathbb{U}$ such that $\dot{V} < 0$. The invariance and attraction results follow from typical Lyapunov analysis arguments [14]. ■

The various subsets of $\mathbb{R}^n$ introduced are illustrated in Fig. 1. The reader should note that there may be trajectories emanating from $\mathcal{D}_c \setminus \Omega_{V,\gamma^*}$, which, still, enter Ω_{V,γ^*} and ultimately converge to the origin. Focusing only on Ω_{V,γ^*} introduces some conservatism in our results. Nevertheless, the advantage of obtaining a compact parametrization of the domain, such as $\Omega_{V,\gamma}$, can potentially outweigh this conservatism by allowing us to pose and solve the analysis and synthesis problems via a sequence of convex Semidefinite Programming problems with sum of squares constraints, which are introduced next.

C. Semidefinite Programming optimization

Semidefinite Programming (SDP) is concerned with the minimization of $c^\mathsf{T} x$, subject to linear matrix inequality (LMI) constraints, that is, $F_0 + \sum_{i=1}^n x_i F_i \succeq 0$, where $x = [x_1 \ldots x_n]^\mathsf{T} \in \mathbb{R}^n$ is the decision variable, and $c \in \mathbb{R}^n$, $F_i \in \mathbb{R}^{n\times n}$ with $F_i = F_i^\mathsf{T}$ for $i = 0 \ldots n$ are known. These problems are convex and can be solved efficiently with polynomial time algorithms, such as interior point methods [15], [16]. This work ultimately utilizes SDP *feasibility* problems, that is, a set of LMI constraints without an objective function.

D. Sum of squares polynomials

We briefly discuss polynomials which are sums of squares, as well as their applications in the subsequent derivations.

Definition 2: A polynomial $p : \mathbb{R}^n \to \mathbb{R}$ of degree no greater than $2d$ is a *sum of squares* (SOS), that is, $p(x) \in \Sigma^{2d}$ for all $x \in \mathbb{R}^n$, if $p(x) = \sum_{i=1}^{k} g_i^2(x)$, where $g_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1 \dots k$, are polynomials of degree no greater than d.

As a direct consequence of Definition 2, SOS polynomials are nonnegative for all $x \in \mathbb{R}^n$.

Lemma 2: [17] A polynomial $p(x)$ of degree no greater than $2d$ is SOS if and only if it can be written as $p(x) = z_{[d]}^{\mathsf{T}}(x) Q z_{[d]}(x)$, where $z_{[d]} : \mathbb{R}^n \to \mathbb{R}^{n_d}$ with $n_d = \frac{(n+d)!}{n!d!}$ is the vector of monomials of degree less than or equal to d and $Q \in \mathbb{R}^{n_d \times n_d}$ with $Q \succeq 0$.

The existence of such a Q for a given $p(x)$ is an LMI feasibility problem, as shown in [8]. The process of symbolically converting SOS constraints into LMI constraints and formulating corresponding SDP problems can be automated with specialized libraries [18], [19]. In this work, we utilize SOS constraints within SDP problems in two different ways: (a) in order to enforce that a particular polynomial has an SOS decomposition, and (b) as a means to prove appropriately formulated set containment relationships.

Next, we briefly describe the latter method, commonly referred to as the *generalized S-procedure* [8], [12]. Let $f_0 : \mathbb{R}^n \to \mathbb{R}$ and $f_i : \mathbb{R}^n \to \mathbb{R}$, for $i = 1 \dots m$, be known polynomial functions, and consider the set containment relationship defined by the inequality $f_0(x) \geq 0$ for all $x \in \mathbb{R}^n$ satisfying $f_i(x) \geq 0$. One can search for SOS polynomials $s_i(x) \in \Sigma^{2d}$ such that $f_0(x) - \sum_{i=1}^{m} s_i(x) f_i(x) \in \Sigma^{2d}$. The existence of such $s_i(x)$ for $i = 1 \dots m$ implies that $\bigcap_{i=1}^{m} \{x \in \mathbb{R}^n : f_i(x) \geq 0\} \subseteq \{x \in \mathbb{R}^n : f_0(x) \geq 0\}$. The S-procedure is a *sufficient* condition for the set containment relationship in question to hold, and it is known to introduce a certain level of conservatism, since the success of the method depends on the existence of the SOS decompositions involved. Also, the conservatism level depends not only on the specified maximum degree of the involved polynomials, but, at a more practical level, on the attributes and capabilities of the numerical libraries utilized, as well. However, the fact that proving complicated set containment relationships can be potentially achieved via solving a convex SDP problem, as opposed to a non-convex, nonlinear programming problem, offers significant computational advantages.

E. Problem statements

For the class of the CLF-based nonlinear controllers described before, the shape and size of $\mathcal{D}_c$ depend on the particular dynamics, the input value set $\mathbb{U}$ and the chosen CLF. In an effort to add some structure to the problem, we focus our attention to sublevel sets of the CLF which are contained in $\mathcal{D}_c$. For some $\hat{\gamma} > 0$ with $\hat{\gamma} \leq \gamma^*$, we define the *analysis* problem as the calculation of the *inner approximation estimate* $\Omega_{V,\hat{\gamma}} \subseteq \Omega_{V,\gamma^*} \subseteq \mathcal{D}_c$, given (1), $\mathbb{U}$ and a known CLF V. The *synthesis* problem is defined as the *modification* or *enlargement* of the set Ω_{V,γ^*}, via means of varying the CLF coefficients. For the synthesis problem to be well-posed in this case where the coefficients of the CLF are among the decision variables, we associate the adjustment or enlargement of Ω_{V,γ^*} with the maximization of the level set of an appropriately shaped, positive definite function, which is constrained to be contained in Ω_{V,γ^*}.

III. Partitioning the state-space

In this Section, we derive results which are necessary in order to subsequently pose and solve the analysis and synthesis problems as SDP optimization problems. In contrast to the earlier results in the literature on the analysis of autonomous systems using SOS polynomials, the S-procedure set containment relationships cannot be readily applied in the case of feedback control systems with dynamics given by (1) under input constraints. Here, we propose a solution which enables the use of the S-procedure. Our development relies on partitioning the set $\mathbb{X}_f$ based on the vertices of $\mathbb{U}$; the analysis and synthesis problems will contain a set containment relationship corresponding to each element of the partition. It will be shown why the proposed combination of set containment relationships can yield level sets of the CLF which are contained in $\mathbb{X}_f$. This result essentially solves the analysis problem and paves the way for developing an algorithm for the synthesis case.

The reader should note that the upcoming derivations and results do not suggest controlling the system using exclusively the vertices of $\mathbb{U}$; the use of the vertices herein is limited in the context of the analysis and synthesis computations. A corresponding control law u_c (either based on an existing CLF, or on an optimized one) may or may not saturate on the vertices of $\mathbb{U}$. This possibility depends on the desired performance and the particular CLF; the proposed partitioning scheme used for computational purposes here has no effect whatsoever. The results of the analysis and synthesis algorithms will guarantee, though, that if the state of the closed loop system is inside Ω_{V,γ^*} and the input saturates, this will be safe in the sense that the condition $\dot{V} < 0$, which serves as our indication of asymptotic stabilization, will still hold.

A. Partitioning $\mathbb{X}_f$ based on the vertices of $\mathbb{U}$

We will use a fundamental result from linear programming, which will, then, allow us to partition $\mathbb{X}_f$ into subsets corresponding to each vertex of $\mathbb{U}$.

Proposition 1: (Prop. 3.4.2, [20]) Let $\mathbb{P}$ be a polytope which has at least one extreme point. A linear function which is bounded from below over $\mathbb{P}$ attains a minimum at some extreme point of $\mathbb{P}$.

It is straightforward to apply Proposition 1 to the CLF property (2), yielding the following Lemma.

Lemma 3: Let $\mathbb{U}$ be a compact, convex polytope with vertices v_i, for $i = 1 \dots q$, and $\psi(x,u) := \nabla V(x)(f(x) + g(x)u)$. Then, $\mathbb{X}_f = \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$, where $\mathbb{X}_{f_i} := \{x \in \mathcal{D} : \psi(x, v_i) < 0\} \cup \{0\}$.

Proof: For a fixed $x^* \in \mathcal{D}$, ψ is linear in u. Moreover, since $u \in \mathbb{U}$, where $\mathbb{U}$ is a compact and convex polytope, ψ is bounded. If at the point x^* the inequality $\inf_{u \in \mathbb{U}} \psi(x^*, u) < 0$ holds true, then $x^* \in \mathbb{X}_f$. According to Proposition 1, though, $\inf_{u \in \mathbb{U}} \psi(x^*, u) = \min\{\psi(x^*, v_i),\ i = 1 \dots q\}$, that is, the infimum is attained and that can only happen at some vertex v_i. Therefore, x^* belongs to at least one of the $\mathbb{X}_{f_i}$ sets, that is, $\mathbb{X}_f \subseteq \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$. Now let $x^{**} \in \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$, that is, $x^{**} \in \mathbb{X}_{f_\ell}$ for $\ell \in \{1 \dots q\}$. Then, by definition, $x^{**} \in \mathbb{X}_f$ as well, which implies that $\bigcup_{i=1}^{q} \mathbb{X}_{f_i} \subseteq \mathbb{X}_f$. We therefore conclude that $\mathbb{X}_f = \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$. ∎

Under the assumption that $0 \in \text{Int}(\mathcal{D}_c)$, we deduce that the function $\lambda : \mathcal{D} \to \mathbb{R}$, where $\lambda(x) = \inf_{u \in \mathbb{U}} \psi(x,u)$, has to be negative in a neighborhood around $x = 0$. Depending on the particular dynamics and CLF, λ may increase along certain directions; this is typically the case when the unforced dynamics are neutrally stable or unstable. The details of the exact behavior of λ are not of interest for the results of this Section; it is only necessary to discuss the general structure of the sets involved, and, in particular, to qualitatively describe the boundaries of the $\mathbb{X}_{f_i}$ sets.

Let one consider the scalar, continuous function $\psi_i(x) := \psi(x, v_i)$ for any fixed v_i, the set $\Gamma_i := \{x \in \mathcal{D} : \psi_i(x) = 0\}$, which corresponds to the boundary of $\mathbb{X}_{f_i}$, $\partial\mathbb{X}_{f_i}$, and note that:

(i) Due to the continuity of ψ_i and the fact that $\psi_i(0) = 0$, since $\nabla V(0) = 0$, we deduce that there exists a hypersurface which satisfies $\psi_i(x) = 0$, passes through the origin and, by definition, belongs to Γ_i.

(ii) Since, by assumption, there exists a finite γ^* (corresponding to the *largest* $\Omega_{V,\gamma}$ to be contained in $\mathbb{X}_f$), there must exist some $x \in \mathcal{D} \setminus \{0\}$ such that $\lambda(x) = 0$. By definition, all x satisfying $\lambda(x) = 0$ form the boundary $\partial\mathbb{X}_f$; moreover, according to Proposition 1, if $\lambda(x) = 0$, then $u = v_\ell$ for some $\ell \in \{1 \dots q\}$. This implies that Γ_ℓ also consists of points $x \in \mathcal{D}$ away from $x = 0$.

We conclude, then, that each $\mathbb{X}_{f_i}$ set, for all $i \in \{1 \dots q\}$ is contained between a manifold passing through the origin and, potentially, other manifolds which do not contain the origin, as illustrated in Fig. 2.

B. *Further partitioning of the $\mathbb{X}_{f_i}$ sets*

One can expand $\psi(x, v_i)$ around $x = 0$ for a given v_i as a Taylor series; since $\psi(0, v_i) = 0$, this yields $\psi(x, v_i) = \nabla\psi(0, v_i)x + o(\|x\|)$, with $\lim_{x\to 0} o(\|x\|)/\|x\| = 0$. Let $\eta_i := \nabla\psi^\mathsf{T}(0, v_i)$, and note that $\eta_i = \nabla^2 V(0) g(0) v_i$.

Lemma 4: Let $\mathbb{X}_{0_i} := \{x \in \mathcal{D} : \eta_i^\mathsf{T} x \leq 0\}$. There exists $\mathbb{B}_r \subset \mathbb{X}_f$, such that $\mathbb{B}_r \cap \mathbb{X}_{f_i} \subseteq \mathbb{B}_r \cap \mathbb{X}_{0_i}$ for all $i = 1 \dots q$.

Proof: For sufficiently small $\|x\|$, $\psi(x, v_i) \cong \eta_i^\mathsf{T} x$. Let $\mathbb{B}_r$ be the open ball which contains all such x. If $x^* \in \mathbb{X}_{f_\ell}$ for some $\ell \in \{1 \dots q\}$, then $\psi(x^*, v_\ell) < 0$; if, additionally, $x^* \in \mathbb{B}_r \cap \mathbb{X}_{f_\ell}$, then $\psi(x^*, v_\ell) < 0$ implies $\eta_\ell^\mathsf{T} x < 0$, therefore $x^* \in \mathbb{B}_r \cap \mathbb{X}_{0_i}$, which proves the Lemma. ∎

Proposition 2: It is true that $\mathcal{D} = \bigcup_{i=1}^{q} \mathbb{X}_{0_i}$.

Proof: Let $x \in \mathcal{D} \setminus \{0\}$, and $y = \alpha x$, where $\alpha \in (0, 1]$. Since $0 \in \text{Int}(\mathbb{X}_f)$, there exists $r > 0$ such that $\mathbb{B}_r \subset \mathbb{X}_f$. For $\alpha < r/\|x\|$, we have $y \in \mathbb{B}_r$ and, therefore, $y \in \mathbb{X}_f$. Following Lemma 3, there is at least one $\ell \in \{1 \dots q\}$ such that $y \in \mathbb{X}_{f_\ell}$. For sufficiently small α, Lemma 4 holds and $y \in \mathbb{X}_{f_\ell}$ implies $y \in \mathbb{X}_{0_\ell}$, as well. As a consequence of the latter argument, $\eta_\ell^\mathsf{T} y < 0$, which in turn yields $\eta_\ell^\mathsf{T} x < 0$. Therefore, by definition of $\mathbb{X}_{0_\ell}$, it is true that $x \in \mathbb{X}_{0_\ell}$, which implies $\mathcal{D} \subseteq \bigcup_{i=1}^{q} \mathbb{X}_{0_i}$. The converse argument, $\bigcup_{i=1}^{q} \mathbb{X}_{0_i} \subseteq \mathcal{D}$, holds true by definition, proving the Proposition. ∎

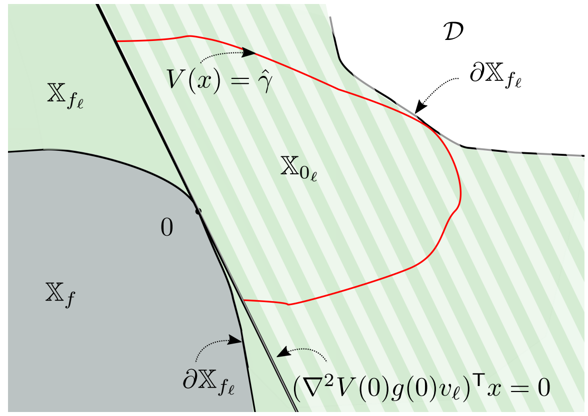

Fig. 2: A qualitative depiction of the $\mathbb{X}_{f_\ell}$, $\mathbb{X}_{0_\ell}$ sets, for some v_ℓ with $\ell \in \{1 \dots q\}$. For the analysis and synthesis problems, we are interested in containing the sublevel set Ω_{V,γ^*} in $\mathbb{X}_f = \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$.

C. *CLF level set containment in $\mathbb{X}_f$*

It is now possible to assemble a number of S-procedure set containment relationships which will subsequently allow us to prove containment of level sets of the CLF inside $\mathbb{X}_f$.

Proposition 3: Let one consider the S-procedure set containment relationships with SOS constraints given by

$$h_i(x) - s_{k_i}(x) k_i(x) - s_{p_i}(x) p(x) \in \Sigma^{2d}, \quad i = 1 \dots q,$$

where $s_{k_i}, s_{p_i} \in \Sigma^{2d}$, $p : \mathbb{R}^n \to \mathbb{R}$, and $h_i, k_i : \mathbb{R}^n \to \mathbb{R}$ for $i = 1 \dots q$ are polynomial functions, with $h_i(x) \geq 0$ for all $x \in \mathbb{X}_{f_i}$ and $k_i(x) \geq 0$ for all $x \in \mathbb{X}_{0_i}$. The existence of s_{k_i}, s_{p_i} such that all q set containment relationships are satisfied simultaneously implies that $\{x \in \mathcal{D} : p(x) \geq 0\} \subseteq \mathbb{X}_f$.

Proof: The existence of polynomials $s_{k_i}, s_{p_i} \in \Sigma^{2d}$ for some $i \in \{1 \dots q\}$ implies that $\{x \in \mathbb{X}_{0_i} : p(x) \geq 0\} \subseteq \mathbb{X}_{f_i}$. If there exist s_{k_i}, s_{p_i} for all $i = 1 \dots q$, then all q set containment relationships are satisfied simultaneously, and $\bigcup_{i=1}^{q} \{x \in \mathbb{X}_{0_i} : p(x) \geq 0\} \subseteq \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$. However, $\mathbb{X}_f = \bigcup_{i=1}^{q} \mathbb{X}_{f_i}$ and $\mathcal{D} = \bigcup_{i=1}^{q} \mathbb{X}_{0_i}$, according to Lemma 3 and Proposition 2, respectively. Therefore, $\{x \in \mathcal{D} : p(x) \geq 0\} \subseteq \mathbb{X}_f$, which proves the Proposition. ∎

IV. Analysis Algorithm

In the analysis problem, we are looking for the largest $\gamma^* > 0$ such that $\Omega_{V,\gamma^*} \subseteq \mathbb{X}_f$, given the dynamics (1), $\mathbb{U}$ and a (fixed) CLF V. To this end, one can build on the results of the previous Section, and, in particular, Proposition 3, in order to assemble a corresponding SDP problem.

We can approximate γ^* with the solution $\hat{\gamma}$ of the optimization problem of maximizing $\gamma > 0$, subject to $s_{p_i}(x), s_{k_i}(x) \in \Sigma^{2d}$ and

$$s_{p_i}(x)(V(x) - \gamma) + s_{k_i}(x)(\nabla^2 V(0) g(0) v_i)^\mathsf{T} x \\ - \nabla V(x)(f(x) + g(x) v_i) - \epsilon x^\mathsf{T} x \in \Sigma^{2d}, \quad (3)$$

for $i = 1 \dots q$, where $\epsilon x^\mathsf{T} x$ with $0 < \epsilon \ll 1$ enforces the strict inequality employed in the definition of $\mathbb{X}_{f_i}$.

This optimization problem is bilinear due to the products $s_{p_i}(x)\gamma$. Nevertheless, this particular bilinearity can be circumvented rather efficiently by employing a bisection algorithm. For a fixed γ we consider an SDP feasibility problem consisting of the constraints given by (3); we denote this problem by $\mathcal{SDP}(\gamma, f, g, \mathbb{U}, V)$. For some initial values $\gamma_l, \gamma_h > 0$ such that $\mathcal{SDP}(\gamma_l, f, g, \mathbb{U}, V)$ is feasible while $\mathcal{SDP}(\gamma_h, f, g, \mathbb{U}, V)$ is not, the bisection procedure proceeds as described by Algorithm 1.

Algorithm 1 Bisection based analysis algorithm

repeat
  $\gamma_m \leftarrow (\gamma_l + \gamma_h)/2$
  if $\mathcal{SDP}(\gamma_m, f, g, \mathbb{U}, V)$ is feasible **then** $\gamma_l \leftarrow \gamma_m$
  else $\gamma_h \leftarrow \gamma_m$
  end if
until $(\gamma_h - \gamma_l)/\gamma_l \leq \epsilon_{tol}$

The solution $\hat{\gamma}$ is known to be contained in the interval $[\gamma_l, \gamma_h)$, which Algorithm 1 can approximate to the desired relative accuracy level $\epsilon_{tol} = (\gamma_h - \gamma_l)/\gamma_l$. The correctness of the proposed Algorithm and its result, that is, the fact that $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_f$, follows from Proposition 3, with $p(x) = \gamma - V(x)$, $h_i(x) = -\nabla V(x)(f(x) + g(x)v_i) - \epsilon x^\mathsf{T} x$ and $k_i(x) = -(\nabla^2 V(0) g(0) v_i)^\mathsf{T} x$ for $i = 1 \ldots q$.

V. Synthesis algorithm

The objective of the synthesis problem is to adjust or enlarge Ω_{V,γ^*}. This can be achieved by parameterizing the CLF as an SOS polynomial of some fixed degree, and optimizing over its coefficients. The positive definiteness property, as well as (2), are enforced on this polynomial via appropriate constraints. Following the approach typically pursued in the literature of SOS methods for estimating the region of attraction of autonomous systems, we introduce the concept of a shape function $P : \mathbb{R}^n \to \mathbb{R}$, with $P(x) > 0$ for all $x \neq 0$, $P(0) = 0$, $P(x) \to \infty$ as $\|x\| \to \infty$, and require that $\Omega_{P,\mu} \subseteq \Omega_{V,\gamma}$. Maximizing μ can then yield the desired result in terms of reshaping or enlarging Ω_{V,γ^*}.

By appropriately augmenting the analysis problem from Section IV, we can now define the optimization problem for the synthesis case as maximizing $\mu > 0$, subject to $s_\mu(x) \in \Sigma^{2d}$, $s_{p_i}(x), s_{k_i}(x) \in \Sigma^{2d}$ for $i = 1 \ldots q$, $V(x) - \epsilon x^\mathsf{T} x \in \Sigma^{2d}$, $V(0) = 0$ and

$$s_{p_i}(x)(V(x) - \hat{\gamma}) + s_{k_i}(x)(\nabla^2 V(0) g(0) v_i)^\mathsf{T} x - \nabla V(x)(f(x) + g(x)v_i) - \epsilon x^\mathsf{T} x \in \Sigma^{2d},\ i = 1 \ldots q, \quad (4)$$

$$s_\mu(x)(P(x) - \mu) - (V(x) - \gamma) \in \Sigma^{2d}, \quad (5)$$

where the $\epsilon x^\mathsf{T} x$ term is utilized as in Section IV. In this problem, the scalar $\hat{\gamma} > 0$ is a constant, as we will explain next. This problem is bilinear in its decision variables, due to the products $s_{p_i}(x)V(x)$, $s_{k_i}(x)(\nabla^2 V(0))$ and $s_\mu(x)\mu$. Although the latter can be circumvented with a bisection scheme, the two other cases enforce a different approach for the synthesis algorithm.

For a fixed μ, we define two separate SDP feasibility problems, which are affine in their decision variables:

- $\mathcal{SDP}_\text{S}(\mu, f, g, \mathbb{U}, V, \hat{\gamma}, P)$, with decision variables s_μ, s_{p_i} and s_{k_i}, subject to (4) and (5);
- $\mathcal{SDP}_\text{V}(\mu, f, g, \mathbb{U}, \hat{\gamma}, P)$, with decision variables s_μ and V, subject to (4) and (5), given fixed $s_{p_i}, s_{k_i} \in \Sigma^{2d}$.

It is assumed that an initial CLF V is known for the system, and that the analysis algorithm from Section IV has yielded a $\hat{\gamma}$, such that $\Omega_{V,\hat{\gamma}} \subseteq \mathbb{X}_f$. A shape function P is subsequently chosen according to the designer's objectives, as well as an initial μ, such that $\Omega_{P,\mu} \subseteq \Omega_{V,\hat{\gamma}}$. The two SDP problems are then solved iteratively, following Algorithm 2.

Algorithm 2 Iterative synthesis algorithm

repeat
  if $\mathcal{SDP}_\text{S}(\mu, f, g, \mathbb{U}, V, \hat{\gamma}, P)$ is feasible **then**
    update s_{p_i}, s_{k_i} multipliers, **increase** μ
  else if $\mathcal{SDP}_\text{V}(\mu, f, g, \mathbb{U}, \hat{\gamma}, P)$ is feasible **then**
    update V, **increase** μ
  end if
until $\mathcal{SDP}_\text{S}$ and $\mathcal{SDP}_\text{V}$ both unfeasible

VI. Numerical examples

We consider a system with dynamics

$$\dot{x}_1 = x_2, \qquad \dot{x}_2 = -0.5x_1^2 - x_2 + u,$$

with $-2 \leq u \leq 4$. There exist trajectories $\phi(t; x_0, 0)$ which diverge away from the origin rapidly. This is due to the $-x_1^2$ term in the dynamics, which can lead to solutions with a finite escape time for various $x_0 \neq 0$. An unconstrained control law could trivially cancel the pathological term and globally stabilize the system. For the case where the input is constrained, though, this is typically not possible. We employ the proposed methods in order to examine, reshape or enlarge the subsets of $\mathbb{R}^2$ inside which asymptotic stabilization with a CLF-based controller is guaranteed to be possible.

We consider SOS polynomials of maximum degree 8. Algorithm 1 calculated that the largest $\Omega_{V,\gamma}$ contained in $\mathbb{X}_f$ for the CLF $V_1(x) = 1.7x_1^2 + 2x_1x_2 + 1.7x_2^2$ was given for $\hat{\gamma} = 12.896$. Figure 3 illustrates $\partial\mathbb{X}_f$ and the level set $V_1(x) = \hat{\gamma}$. One can see that there is no conservatism at all in this calculation, since $\partial\Omega_{V_1,\hat{\gamma}}$ appears to locally coincide with a segment of $\partial\mathbb{X}_f$. Algorithm 2 was then invoked three times, with the shape functions $P_2(x) = x^\mathsf{T} x$, $P_3(x) = 4x_1^2 + x_2^2$ and $P_4(x) = x_1^2 + 4x_2^2$, respectively. The iterative optimization results to $\hat{\mu}_2 = 6.578$, $\hat{\mu}_3 = 16.891$ and $\hat{\mu}_4 = 7.580$, for the new CLFs V_2, V_3 and V_4, respectively. The level sets of each CLF and the corresponding boundaries $\partial\mathbb{X}_f$ are illustrated in Fig. 3. The problem was processed using YALMIP, and solved with MOSEK and SDPT3.

The shape function choice can be motivated by the requirement to include particular regions of the state space into the controller's domain. For instance, requiring to extend the domain towards larger values of x_2 would result in a shape function similar to P_3, which, apparently, did not only influence the domain, but resulted in an increase of the area enclosed by $\Omega_{V,\hat{\gamma}}$, as well.

VII. Discussion and Conclusion

We have built on the fundamental SOS results in order to analyze and, subsequently, synthesize families of CLF-based controllers for input constrained, nonlinear systems.

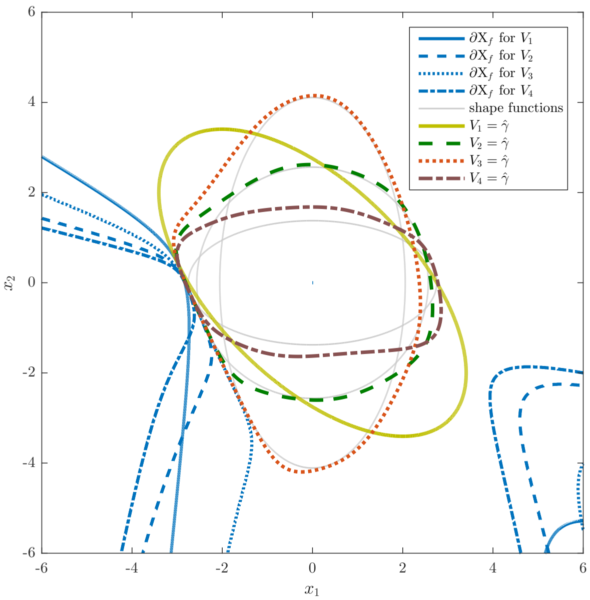

Fig. 3: $\partial\mathbb{X}_f$ for each CLF and the largest level sets $V(x) = \hat{\gamma}$ contained within them. The largest level sets of each shape function contained by the respective $\Omega_{V,\hat{\gamma}}$ are also illustrated.

The numerical examples presented indicate satisfactory performance, in terms of indistinguishable conservatism and success in adjusting the controller's domain. The main questions which arise with regards to the proposed methods and the prospects for further developments are three, namely, the *computational tractability*, the *conservatism* imposed by the SOS decompositions and, finally, the *applicability* of this work to broad classes of dynamical systems.

The tractability issue becomes relevant as the order of the underlying dynamical system or the degree of the involved S-procedure multipliers grow. Replacing SOS polynomials with alternative nonnegative functions, such as diagonally-dominant SOS polynomials which result in linear or second order cone programming problems [21], may be an interesting extension of this work.

The existence of an SOS decomposition is only a sufficient condition for the nonnegativeness of a polynomial, potentially increasing, thus, the conservatism in the solution of the corresponding problem. The same conclusion is also true in the case where SOS polynomials are used within an S-procedure expression to show set containments. Increasing the degree of the involved polynomials can lead to less conservative results. Nevertheless, this benefit can be outweighed by the increasing size of the corresponding SDP problem, leading to the tractability issue mentioned above. The appropriate balance can be dictated by the designer's requirements, objectives and available computational resources.

SOS methods are inherently applicable to systems with polynomial dynamics, such as the ones studied herein. Extensions which apply SOS methods to polynomial time-delay [22] and hybrid [23] dynamics have been proposed. Certain non-polynomial systems have been considered in [24]. Combining such results with the methods developed in this paper would be an interesting direction for future work.

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