Minimum Time Control for a Newtonian Particle in a Spatiotemporal Flow Field

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***Abstract*— We address the problem of steering a Newtonian particle to a prescribed terminal position and velocity in a spatiotemporal flow field under an explicit constraint on the norm of its acceleration. The cases when either the terminal position or the terminal velocity of the particle is free are also considered. By employing standard techniques from optimal control theory, we characterize the structure of the candidate time-optimal control and subsequently reduce the original optimal control problem to a system of coupled nonlinear algebraic equations. Although the latter system of equations has to be solved numerically, in general, we show that, in some cases, it can be brought into a triangular form, whose solution does not require a significant computational effort. Numerical simulations that illustrate the theoretical developments are presented.**

I. Introduction

We address the problem of driving a Newtonian particle to a prescribed terminal position with a prescribed velocity in the presence of a spatiotemporal drift field. It is assumed that the control input of the particle is the rate of change of its air velocity, whose norm is bounded by an a priori known constant. The problem considered in this work can be put under the umbrella of minimum time control problems for inhomogeneous linear systems whose control input is "constrained to a hypersphere" [1]. The latter corresponds to a special class of minimum time control problems for linear systems with convex control input sets, which were originally studied independently by LaSalle and Krasovskii and their collaborators [2]–[4].

The problem of steering a simplified kinematic model of a vehicle to a prescribed position with either a prescribed or a free velocity in the presence of drift (due to, say winds or currents in its vicinity) has received a significant amount of attention in the literature. It should be mentioned, however, that the majority of the available results deal with the so-called Dubins vehicle [5], [6]. The latter corresponds to a kinematic model of a particle that is constrained to travel with constant forward speed via controlling the rate of change of the angle of its forward velocity, which is bounded by an a priori given bound. The minimum-time problem for the Dubins vehicle in a flow field has been studied in [7]–[9]. The results in these references deal exclusively with constant and, in some cases, time-varying, yet spatially invariant, flow fields. The problem of guiding the Dubins vehicle in the presence of a stochastic flow while minimizing the expected time of arrival at a given target set was recently addressed in [10]. In the latter references, the flow is modeled as additive stochastic noise, whose statistics are known a priori. The approach presented in [10] signifies an important departure from other standard deterministic techniques that are typically employed in the literature to address similar problems. It should be mentioned, however, that the statistics of the flow field in [10] are assumed to be spatially and temporally invariant.

The problem of guiding a Newtonian particle by means of a time-optimal controller under a constraint on the norm of its acceleration has received considerable attention in the literature of applied mechanics. The reader is referred to a series of papers by Akulenko and his collaborators [11]–[15]. The authors of [16] independently developed results that bear some similarities with some of the ones presented in [11]–[15]. None of the previous references deal, however, with the case when the Newtonian particle is traveling in the presence of a flow field.

The main contribution of this paper is the formulation and solution of a classical optimal control problem, which, to the best of our knowledge, has never been addressed in the literature. In particular, we consider the problem of characterizing a time-optimal control law that will steer a Newtonian particle of unit mass to a prescribed position with a prescribed velocity in the presence of a spatiotemporal flow field, where the latter is induced, for example, by local winds or currents. The cases when the terminal position or the velocity of the particle are, respectively, fixed and free, and vice versa, are also considered. In order to simplify the presentation and analysis, we will assume that the velocity of the flow is approximated by a time-varying inhomogeneous linear (affine) field. The minimum-time problem is addressed by means of standard techniques from optimal control theory, and in particular, a general formulation of the Minimum Principle (also known as the Maximum Principle) [17], which leads to the complete characterization of the structure of the candidate time-optimal control law. Subsequently, we reduce the original minimum-time control problem to a system of coupled nonlinear algebraic equations, which are solved numerically. We also show that, in some cases, this system of equations can be brought into a triangular form, whose solution does not require a significant computational effort.

The rest of the paper is organized as follows. The optimal control problem is formulated in Section II. The analysis

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of the problem and the characterization of the structure of its solution via standard optimal control techniques is presented in Section IV. Numerical simulations are presented in Section V. Finally, Section VI concludes the paper with a summary of remarks.

II. Problem Formulation

We consider a Newtonian particle traveling in the presence of a flow whose velocity varies both spatially and temporally. It is assumed that the velocity field $\boldsymbol{w}$ of the flow is approximated by an inhomogeneous time-varying linear field, that is,

$$\boldsymbol{w}(t,\boldsymbol{x}) \approx \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t),$$

where $\boldsymbol{x} \in \mathbb{R}^n$ and $t \in \mathbb{R}_+$ denote, respectively, the spatial and temporal variables. We assume that each element of the matrix $\mathbf{A}$ and the vector $\boldsymbol{f}$ is a piece-wise continuous function of time, and, in addition, that there exist positive constants $\overline{A}$, $\overline{f}$ such that, for any $T > 0$,

$$|||\mathbf{A}(t)||| \leq \overline{A}, \qquad \|\boldsymbol{f}(t)\| \leq \overline{f}, \qquad \text{for all} \ \ t \in [0,T], \tag{1}$$

where $|||\cdot|||$ and $\|\cdot\|$ denote, respectively, the induced matrix and vector 2-norms.

The motion of the Newtonian particle is then described by the following set of equations

$$\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t) + \boldsymbol{v}, \qquad \boldsymbol{x}(0) = \boldsymbol{x}_0, \tag{2a}$$
$$\dot{\boldsymbol{v}} = \boldsymbol{u}(t), \qquad \boldsymbol{v}(0) = \boldsymbol{v}_0, \tag{2b}$$

where $\boldsymbol{x} \in \mathbb{R}^n$ ($\boldsymbol{x}_0 \in \mathbb{R}^n$) and $\boldsymbol{v} \in \mathbb{R}^n$ ($\boldsymbol{v}_0 \in \mathbb{R}^n$) is, respectively, the position and the (forward) velocity vector of the particle at time t (time $t = 0$), and $\boldsymbol{u}(t)$ is the control input at time t. It is assumed that $\boldsymbol{u}(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ denotes the set of piecewise continuous functions $\boldsymbol{g} : \mathbb{R}_+ \mapsto \mathbb{R}^n$, where $\mathbb{R}_+$ denotes the set of non-negative numbers, that attain values on the set $U := \{\nu \in \mathbb{R}^n : \ |\nu| \leq \overline{u}\}$, where $\overline{u}$ is the maximum norm of the rate of change of the air velocity of the particle. Finally, we shall henceforth denote by $\boldsymbol{z}$ (respectively, $\boldsymbol{z}_0$) the composite state vector at time t (resp., $t = 0$), where $\boldsymbol{z} := [\boldsymbol{x}^{\mathrm{T}}, \ \boldsymbol{v}^{\mathrm{T}}]^{\mathrm{T}}$ (resp., $\boldsymbol{z}_0 := [\boldsymbol{x}_0^{\mathrm{T}}, \ \boldsymbol{v}_0^{\mathrm{T}}]^{\mathrm{T}}$). The equations of motion of the particle in terms of the state vector $\boldsymbol{z}$ are given by

$$\dot{\boldsymbol{z}} = \mathbf{F}(t)\boldsymbol{z} + \mathbf{G}\boldsymbol{u}(t) + \boldsymbol{\Gamma}\boldsymbol{f}(t), \tag{3}$$

where

$$\mathbf{F}(t) := \begin{bmatrix} \mathbf{A}(t) & \mathbf{I}_2 \\ \mathbf{0}_2 & \mathbf{0}_2 \end{bmatrix}, \quad \mathbf{G} := \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{I}_2 \end{bmatrix}, \quad \boldsymbol{\Gamma} := \begin{bmatrix} \mathbf{I}_2 \\ \mathbf{0}_2 \end{bmatrix}.$$

Next, we consider the following minimum-time problem:

Problem 1: Let $\boldsymbol{z}_0 := [\boldsymbol{x}_0^{\mathrm{T}}, \ \boldsymbol{v}_0^{\mathrm{T}}]^{\mathrm{T}}$, $\boldsymbol{z}_{\mathrm{f}} := [\boldsymbol{x}_{\mathrm{f}}^{\mathrm{T}}, \ \boldsymbol{v}_{\mathrm{f}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ be given. Then, find the control input $\boldsymbol{u}_\star(\cdot) \in \mathcal{U}$ that will transfer the system described by Eq. (2a)-(2b) from the prescribed initial state $\boldsymbol{z}_0$ to the prescribed terminal state $\boldsymbol{z}_{\mathrm{f}}$, in minimum (free) time t_{f}.

Definition 1: Let $\boldsymbol{z}_{\mathrm{f}} := [\boldsymbol{x}_{\mathrm{f}}^{\mathrm{T}}, \ \boldsymbol{v}_{\mathrm{f}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ be given. Then, we say that the system described by Eqs. (2a)-(2b) is completely controllable at $\boldsymbol{z}_{\mathrm{f}}$, if there exist a control input $\boldsymbol{u}(\cdot) \in \mathcal{U}$ and a time $\tau \in \mathbb{R}_+$ such that with the application of the control input $\boldsymbol{u}(t)$, for $t \in [0,\tau]$, the system will be transferred from any initial state $\boldsymbol{z}_0 := [\boldsymbol{x}_0^{\mathrm{T}}, \ \boldsymbol{v}_0^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ to the state $\boldsymbol{z}_{\mathrm{f}}$ at time $t = \tau$.

Next, we examine the problem of existence of (optimal) solutions to Problem 1.

Proposition 1: Let $\boldsymbol{z}_{\mathrm{f}} := [\boldsymbol{x}_{\mathrm{f}}^{\mathrm{T}}, \ \boldsymbol{v}_{\mathrm{f}}^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$ be given and let $\boldsymbol{w}(t,\boldsymbol{x}) = \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t)$, where $\mathbf{A}(t)$ and $\boldsymbol{f}(t)$ satisfy (1). Then, the system described by Eqs. (2a)-(2b) is completely controllable at $\boldsymbol{z}_{\mathrm{f}}$, if, and only if, Problem 1 admits a solution for any $\boldsymbol{z}_0 := [\boldsymbol{x}_0^{\mathrm{T}}, \ \boldsymbol{v}_0^{\mathrm{T}}]^{\mathrm{T}} \in \mathbb{R}^{2n}$.

Proof:

Next, we show that complete controllability implies existence of an optimal trajectory of Problem 1 from any $\boldsymbol{z}_0 \in \mathbb{R}^{2n}$ to the given $\boldsymbol{z}_{\mathrm{f}} \in \mathbb{R}^{2n}$ (the proof of the converse is trivial and is omitted). From Filippov's Theorem on the existence of solutions of minimum-time problems [18, pp.310-317], it suffices to prove that there exists $k > 0$ such that $\langle \dot{\boldsymbol{z}}, \boldsymbol{z} \rangle \leq k(1 + \|\boldsymbol{z}\|^2)$. We have that

$$\begin{aligned}
\langle \dot{\boldsymbol{z}}, \boldsymbol{z} \rangle &= \langle \dot{\boldsymbol{x}}, \boldsymbol{x} \rangle + \langle \dot{\boldsymbol{v}}, \boldsymbol{v} \rangle \\
&= \langle \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t) + \boldsymbol{v}, \boldsymbol{x} \rangle + \langle \boldsymbol{u}, \boldsymbol{v} \rangle \\
&= \langle \mathbf{A}(t)\boldsymbol{x}, \boldsymbol{x} \rangle + \langle \boldsymbol{f}(t), \boldsymbol{x} \rangle + \langle \boldsymbol{v}, \boldsymbol{x} \rangle + \langle \boldsymbol{u}, \boldsymbol{v} \rangle \\
&\leq \|\mathbf{A}(t)\boldsymbol{x}\|\|\boldsymbol{x}\| + \|\boldsymbol{f}(t)\|\|\boldsymbol{x}\| + \|\boldsymbol{v}\|\|\boldsymbol{x}\| \\
&\quad + \|\boldsymbol{u}\|\|\boldsymbol{v}\| \\
&\leq |||\mathbf{A}(t)|||\|\boldsymbol{x}\|^2 + \overline{f}\|\boldsymbol{x}\| + \|\boldsymbol{v}\|\|\boldsymbol{x}\| + \|\boldsymbol{u}\|\|\boldsymbol{v}\| \\
&\leq \overline{A}\|\boldsymbol{x}\|^2 + \overline{f}\|\boldsymbol{x}\| + \|\boldsymbol{v}\|\|\boldsymbol{x}\| + \overline{u}\|\boldsymbol{v}\| \\
&\leq \overline{A}\|\boldsymbol{x}\|^2 + \overline{f}\|\boldsymbol{x}\| + 1/2(\|\boldsymbol{v}\|^2 + \|\boldsymbol{x}\|^2) + \overline{u}\|\boldsymbol{v}\| \\
&\leq \overline{A}\|\boldsymbol{x}\|^2 + 1/2\overline{f}(1 + \|\boldsymbol{x}\|^2) \\
&\quad + 1/2(\|\boldsymbol{v}\|^2 + \|\boldsymbol{x}\|^2) + 1/2\overline{u}(1 + \|\boldsymbol{v}\|^2) \\
&\leq 1/2\left(1 + \overline{f} + 2\overline{A}\right)\|\boldsymbol{x}\|^2 + 1/2(\overline{u} + 1)\|\boldsymbol{v}\|^2 \\
&\quad + 1/2(\overline{u} + \overline{f}) \\
&\leq k(1 + \|\boldsymbol{x}\|^2 + \|\boldsymbol{v}\|^2) \\
&\leq k(1 + \|\boldsymbol{z}\|^2),
\end{aligned} \tag{4}$$

where $k := 1/2\max\left\{1 + \overline{f} + 2\overline{A}, \overline{u} + 1, \overline{u} + \overline{f}\right\}$, and where we have used the Cauchy Schwarz inequality along with the following inequalities: $2\|\boldsymbol{x}\|\|\boldsymbol{v}\| \leq \|\boldsymbol{x}\|^2 + \|\boldsymbol{v}\|^2$ and $2\|\boldsymbol{\gamma}\| \leq 1 + \|\boldsymbol{\gamma}\|^2$, for $\boldsymbol{\gamma} \in \{\boldsymbol{x}, \boldsymbol{v}\}$. ■

Remark 1 Proposition 1 highlights the fact that complete controllability implies existence of optimal solutions to Problem 1 for all initial states $\boldsymbol{z}_0 \in \mathbb{R}^{2n}$. Note that, in some special cases, one can check whether a system like the one described by Eqs. (2a)-(2b), whose control input attains values in a compact and convex set U, is completely controllable (or not) by using available tests or criteria. See for example [19], [20], for the case when the matrix $\mathbf{A}$ is constant and $\boldsymbol{f}(t) \equiv 0$.

Remark 2 Some of the assumptions used in the proof of Proposition 1 can be relaxed, and in particular the assumption on the uniform boundedness of $\mathbf{A}(t)$ and $\boldsymbol{f}(t)$. For example, one can assume instead that the elements of $\mathbf{A}(t)$ and $\boldsymbol{f}(t)$ are summable on bounded intervals of $[0, \infty)$ (see, for example, [4]).

III. Analysis of the Optimal Control Problem

Next, we employ a general formulation of the Minimum Principle [17] in order to characterize the structure of the optimal control $\boldsymbol{u}_\star(\cdot)$ that solves the minimum-time Problem 1. In particular, let $t \mapsto \boldsymbol{z}_\star(t)$, where

$$\boldsymbol{z}_\star(t) := \begin{bmatrix} \boldsymbol{x}_\star(t) \\ \boldsymbol{v}_\star(t) \end{bmatrix} \in \mathbb{R}^{2n},$$

denote the optimal trajectory generated with the application of the optimal control input $t \mapsto \boldsymbol{u}_\star(t)$, for $t \in [0, t_\mathrm{f}]$. Then, there exists a scalar $p_0^\star \in \{0, 1\}$ and an absolutely continuous function $t \mapsto \boldsymbol{p}_{\boldsymbol{z}}^\star(t)$, known as the costate, where

$$\boldsymbol{p}_{\boldsymbol{z}}^\star(t) := \begin{bmatrix} \boldsymbol{p}_{\boldsymbol{x}}^\star(t) \\ \boldsymbol{p}_{\boldsymbol{v}}^\star(t) \end{bmatrix} \in \mathbb{R}^{2n},$$

such that

(i) $\|\boldsymbol{p}_{\boldsymbol{x}}^\star(t)\| + \|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\| + |p_0^\star| \neq 0$, for all $t \in [0, t_\mathrm{f}]$[1],

(ii) For all $t \in [0, t_\mathrm{f}]$, $\boldsymbol{p}_{\boldsymbol{x}}^\star$ and $\boldsymbol{p}_{\boldsymbol{v}}^\star$ satisfy the following (canonical) differential equations

$$\dot{\boldsymbol{p}}_{\boldsymbol{x}}^\star = -\frac{\partial \mathcal{H}(t, \boldsymbol{z}_\star, \boldsymbol{p}_{\boldsymbol{z}}^\star, \boldsymbol{u}_\star, p_0^\star)}{\partial \boldsymbol{x}} = -\mathbf{A}^\mathrm{T}(t)\boldsymbol{p}_{\boldsymbol{x}}^\star, \quad \text{(5a)}$$

$$\dot{\boldsymbol{p}}_{\boldsymbol{v}}^\star = -\frac{\partial \mathcal{H}(t, \boldsymbol{z}_\star, \boldsymbol{p}_{\boldsymbol{z}}^\star, \boldsymbol{u}_\star, p_0^\star)}{\partial \boldsymbol{v}} = -\boldsymbol{p}_{\boldsymbol{x}}^\star, \quad \text{(5b)}$$

where $\mathcal{H}$ denotes the Hamiltonian, where

$$\mathcal{H}(t, \boldsymbol{z}, \boldsymbol{p}_{\boldsymbol{z}}, \boldsymbol{u}, p_0) := \langle \boldsymbol{p}_{\boldsymbol{x}}, \mathbf{A}(t)\boldsymbol{x} + \boldsymbol{f}(t) + \boldsymbol{v}\rangle + \langle \boldsymbol{p}_{\boldsymbol{v}}, \boldsymbol{u}\rangle + p_0. \quad (6)$$

(iii) The Hamiltonian $\mathcal{H}$ satisfies the following transversality condition at time $t = t_\mathrm{f}$

$$\mathcal{H}(t_\mathrm{f}, \boldsymbol{z}_\star(t_\mathrm{f}), \boldsymbol{p}_{\boldsymbol{z}}^\star(t_\mathrm{f}), \boldsymbol{u}_\star(t_\mathrm{f}), p_0^\star) = 0. \quad (7)$$

(iv) Furthermore, the optimal control $\boldsymbol{u}^\star$ necessarily minimizes the Hamiltonian evaluated along the optimal state and costate trajectories $t \mapsto \boldsymbol{z}_\star(t)$ and $t \mapsto \boldsymbol{p}_{\boldsymbol{z}}^\star(t)$, respectively, that is,

$$\boldsymbol{u}_\star(t) = \underset{\|\boldsymbol{\nu}\| \leq \overline{u}}{\arg\min}\, \mathcal{H}(t, \boldsymbol{z}_\star(t), \boldsymbol{p}_{\boldsymbol{z}}^\star(t), \boldsymbol{\nu}, p_0^\star), \quad (8)$$

for all $t \in [0, t_\mathrm{f}]$.

It is easy to show that Eq. (8) implies that the candidate optimal control satisfies the following equation

$$\boldsymbol{u}_\star(t) = \begin{cases} -\overline{u}\dfrac{\boldsymbol{p}_{\boldsymbol{v}}^\star(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\|}, & \text{if } \boldsymbol{p}_{\boldsymbol{v}}^\star(t) \neq 0, \\ \boldsymbol{\nu} \in U, & \text{otherwise.} \end{cases}$$

Lemma 1: Let $\boldsymbol{\Phi}(t, \tau)$, where t, $\tau \in \mathbb{R}$, denote the state transition matrix of the homogeneous linear system $\dot{\boldsymbol{x}} = \mathbf{A}(t)\boldsymbol{x}$. Let also $\boldsymbol{\Phi}_\mathcal{A}(t, \tau)$, where t, $\tau \in \mathbb{R}$, denote the state transition matrix of the adjoint system, which is described, in turn, by the following equation: $\dot{\boldsymbol{p}}_{\boldsymbol{x}} = -\mathbf{A}^\mathrm{T}(t)\boldsymbol{p}_{\boldsymbol{x}}$. Then,

$$\boldsymbol{\Phi}_\mathcal{A}(t, \tau) = \boldsymbol{\Phi}^{-\mathrm{T}}(t, \tau) = \boldsymbol{\Phi}^\mathrm{T}(\tau, t), \quad (9)$$

for all $t, \tau \in \mathbb{R}$.

Next, we integrate Equations (5a)-(5b), and obtain the following expressions

$$\boldsymbol{p}_{\boldsymbol{x}}^\star(t) = \boldsymbol{\Phi}_\mathcal{A}(t, 0)\boldsymbol{p}_{\boldsymbol{x}}^\star(0), \quad \text{(10a)}$$

$$\boldsymbol{p}_{\boldsymbol{v}}^\star(t) = \boldsymbol{p}_{\boldsymbol{v}}^\star(0) + \boldsymbol{\Psi}(t)\boldsymbol{p}_{\boldsymbol{x}}^\star(0), \quad \text{(10b)}$$

where $\boldsymbol{\Psi}(t) := -\int_0^t \boldsymbol{\Phi}_\mathcal{A}(\sigma, 0)\mathrm{d}\sigma = -\int_0^t \boldsymbol{\Phi}^\mathrm{T}(0, \sigma)\mathrm{d}\sigma$. Therefore,

$$\boldsymbol{u}_\star(t) = \begin{cases} -\overline{u}\dfrac{\boldsymbol{\beta} + \boldsymbol{\Psi}(t)\boldsymbol{\alpha}}{\|\boldsymbol{\beta} + \boldsymbol{\Psi}(t)\boldsymbol{\alpha}\|}, & \text{if } \boldsymbol{\beta} + \boldsymbol{\Psi}(t)\boldsymbol{\alpha} \neq 0, \\ \boldsymbol{\nu} \in U, & \text{otherwise,} \end{cases}$$

where $\boldsymbol{\alpha} := \boldsymbol{p}_{\boldsymbol{x}}^\star(0)$, $\boldsymbol{\beta} := \boldsymbol{p}_{\boldsymbol{v}}^\star(0)$, provided that $\boldsymbol{p}_{\boldsymbol{v}}^\star(t) \neq 0$.

Proposition 2: Let $0 < t_0 < t_1 < t_\mathrm{f}$. Then $\boldsymbol{p}_{\boldsymbol{v}}^\star(t) \neq 0$, for all $t \in]t_0, t_1[$.

Proof: Let us assume that there exists $0 < t_0 < t_1 < t_\mathrm{f}$ such that $\boldsymbol{p}_{\boldsymbol{v}}^\star(t) = 0$, for all $t \in]t_0, t_1[$. Then, we also have that $\dot{\boldsymbol{p}}_{\boldsymbol{v}}^\star(t) = 0$, for all $t \in]t_0, t_1[$, which implies that $-\boldsymbol{\Phi}_\mathcal{A}(t, 0)\boldsymbol{p}_{\boldsymbol{x}}^\star(0) = 0$. Therefore, $\boldsymbol{p}_{\boldsymbol{x}}^\star(0) = 0$, which implies, in light of Eq. (10a)-(10b) and the fact that $\boldsymbol{p}_{\boldsymbol{v}}^\star(t) = 0$, for all $t \in]t_0, t_1[$, that $\boldsymbol{p}_{\boldsymbol{v}}^\star(0) = 0$. Note that $\boldsymbol{p}_{\boldsymbol{x}}^\star(0) = \boldsymbol{p}_{\boldsymbol{v}}^\star(0) = 0$ implies that $\boldsymbol{p}_{\boldsymbol{x}}^\star(t) \equiv \boldsymbol{p}_{\boldsymbol{v}}^\star(t) \equiv 0$, which along with the transversality condition (7) yield $p_0^\star = 0$. Therefore,

$$\|\boldsymbol{p}_{\boldsymbol{x}}^\star(t)\| + \|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\| + |p_0^\star| = 0,$$

for all $t \in]t_0, t_1[$, which contradicts the Minimum Principle (condition (i)). ■

Remark 3 Proposition 2 implies that the time-optimal control law always attains its values on the boundary of the set U. No singular arcs appear in the solution to our problem.

Proposition 3: Let $\boldsymbol{z}_0$ and $\boldsymbol{z}_\mathrm{f} \in \mathbb{R}^{2n}$ be given and let us assume that Problem 1 admits a solution for this particular set of boundary conditions. Then, the time-optimal control law satisfies necessarily the following equation

$$\boldsymbol{u}_\star(t; \boldsymbol{\alpha}, \boldsymbol{\beta}) = -\overline{u}\frac{\boldsymbol{p}_{\boldsymbol{v}}^\star(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\|} = -\overline{u}\frac{\boldsymbol{\beta} + \boldsymbol{\Psi}(t)\boldsymbol{\alpha}}{\|\boldsymbol{\beta} + \boldsymbol{\Psi}(t)\boldsymbol{\alpha}\|}, \quad (11)$$

for all $t \in [0, t_\mathrm{f}]$, except, possibly, from a finite number of time instants $\tau \in [0, t_\mathrm{f}]$, where $\boldsymbol{\beta} + \boldsymbol{\Psi}(\tau)\boldsymbol{\alpha} = 0$.

Proof: It follows directly from Proposition 2. ■

Remark 4 We will henceforth write $\boldsymbol{u}_\star(t; \boldsymbol{\alpha}, \boldsymbol{\beta})$ to emphasize the direct dependence of $\boldsymbol{u}_\star$ on the parameters $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ (initial values of the costates $\boldsymbol{p}_{\boldsymbol{x}}$ and $\boldsymbol{p}_{\boldsymbol{v}}$). Note that the optimal control $\boldsymbol{u}_\star(\cdot)$ is always a continuous function of time; something, which is in contrast with the minimum-time control laws in problems where the control input attains values on a "hypercube." In the latter case, the minimum-time control laws are typically discontinuous functions of time (for example, bang-bang controllers) [1].

[1] We shall refrain from using the expressions like "for almost all $t \in [0, t_f]$" or "a.e. on $[0, t_f]$" throughout the manuscript to avoid any unnecessary distraction that they may cause to the reader.

A. Reduction of the Optimal Control Problem to a System of Nonlinear Equations

Note that for the complete characterization of the solution to Problem 1, we need to determine $(2n+1)$ unknowns, namely the components of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in \mathbb{R}^n$, and the free final time t_{f}. To this aim, we first integrate the equations of motion of the particle, which are given in (2a)-(2b), from $t=0$ to $t=t_{\mathrm{f}}$, for $\boldsymbol{u}(t)=\boldsymbol{u}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta})$. In this way, we obtain the corresponding candidate optimal trajectories $t \mapsto \boldsymbol{x}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta})$ and $t \mapsto \boldsymbol{v}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta})$, where

$$\begin{aligned} \boldsymbol{x}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta}) = \boldsymbol{\Phi}(t,0)\boldsymbol{x}_0 & \\ + \int_0^t \boldsymbol{\Phi}(t,\tau)(\boldsymbol{f}(\tau)+\boldsymbol{v}_0)\mathrm{d}\tau & \\ + \int_0^t \boldsymbol{\Phi}(t,\tau)\left(\int_0^\tau \boldsymbol{u}_\star(\sigma;\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{d}\sigma\right)\mathrm{d}\tau, & \end{aligned} \tag{12a}$$

$$\boldsymbol{v}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta}) = \boldsymbol{v}_0 + \int_0^t \boldsymbol{u}_\star(\sigma;\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{d}\sigma. \tag{12b}$$

Then, the boundary conditions

$$\boldsymbol{x}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta}) = \boldsymbol{x}_{\mathrm{f}}, \qquad \boldsymbol{v}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta}) = \boldsymbol{v}_{\mathrm{f}}$$

yield the following two vector equations

$$\begin{aligned} \boldsymbol{x}_{\mathrm{f}} = \boldsymbol{\Phi}(t_{\mathrm{f}},0)\boldsymbol{x}_0 & \\ + \int_0^{t_{\mathrm{f}}} \boldsymbol{\Phi}(t_{\mathrm{f}},\tau)(\boldsymbol{f}(\tau)+\boldsymbol{v}_0)\mathrm{d}\tau & \\ + \int_0^{t_{\mathrm{f}}} \boldsymbol{\Phi}(t_{\mathrm{f}},\tau)\left(\int_0^\tau \boldsymbol{u}_\star(\sigma;\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{d}\sigma\right)\mathrm{d}\tau, & \end{aligned} \tag{13a}$$

$$\boldsymbol{v}_{\mathrm{f}} = \boldsymbol{v}_0 + \int_0^{t_{\mathrm{f}}} \boldsymbol{u}_\star(\sigma;\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{d}\sigma. \tag{13b}$$

In addition, the transversality condition (7) yields the following (scalar) equation

$$\begin{aligned} 0 = p_0^\star + \langle \boldsymbol{p}_{\boldsymbol{x}}^\star(t_{\mathrm{f}}), \mathbf{A}(t_{\mathrm{f}})\boldsymbol{x}_{\mathrm{f}} + \boldsymbol{f}(t_{\mathrm{f}}) + \boldsymbol{v}_{\mathrm{f}}\rangle & \\ + \langle \boldsymbol{p}_{\boldsymbol{v}}^\star(t_{\mathrm{f}}), \boldsymbol{u}_\star(t_{\mathrm{f}};\boldsymbol{\alpha},\boldsymbol{\beta})\rangle, & \end{aligned}$$

where $\boldsymbol{p}_{\boldsymbol{x}}^\star(t_{\mathrm{f}}) = \boldsymbol{\Phi}_{\mathcal{A}}(t_{\mathrm{f}},0)\boldsymbol{\alpha}$, $\boldsymbol{p}_{\boldsymbol{v}}^\star(t_{\mathrm{f}}) = \boldsymbol{\beta} + \boldsymbol{\Psi}(t_{\mathrm{f}})\boldsymbol{\alpha}$. It follows that

$$\begin{aligned} 0 = p_0^\star + \langle \boldsymbol{\Phi}_{\mathcal{A}}(t_{\mathrm{f}},0)\boldsymbol{\alpha}, \mathbf{A}(t_{\mathrm{f}})\boldsymbol{x}_{\mathrm{f}} + \boldsymbol{f}(t_{\mathrm{f}}) + \boldsymbol{v}_{\mathrm{f}}\rangle & \\ - \overline{u}\|\boldsymbol{\beta} + \boldsymbol{\Psi}(t_{\mathrm{f}})\boldsymbol{\alpha}\|. & \end{aligned} \tag{14}$$

Therefore, Eqs. (13a)-(13b) and (14) form a system of $(2n+1)$ equations for $(2n+1)$ unknowns, namely, the components of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta}$ ($2n$ unknowns), and the free final time t_{f}. This system of nonlinear equations has to be solved, in general, numerically.

B. Fixed terminal position and free terminal velocity

Next, we consider the case when the terminal position and velocity vectors are, respectively, fixed and free. In this case, we have $\boldsymbol{p}_{\boldsymbol{v}}^\star(t_{\mathrm{f}}) = 0$, which implies, in light of Eqs. (10a)-(10b), that $\boldsymbol{p}_{\boldsymbol{v}}^\star(0) = -\boldsymbol{\Psi}(t_{\mathrm{f}})\boldsymbol{p}_{\boldsymbol{x}}^\star(0)$. Therefore,

$$\boldsymbol{p}_{\boldsymbol{v}}^\star(t) = (\boldsymbol{\Psi}(t) - \boldsymbol{\Psi}(t_{\mathrm{f}}))\boldsymbol{\alpha},$$

and the optimal control is now given by

$$\boldsymbol{u}_\star(t;\boldsymbol{\alpha}) = -\overline{u}\frac{\boldsymbol{p}_{\boldsymbol{v}}^\star(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\|} = -\overline{u}\frac{(\boldsymbol{\Psi}(t) - \boldsymbol{\Psi}(t_{\mathrm{f}}))\boldsymbol{\alpha}}{\|(\boldsymbol{\Psi}(t) - \boldsymbol{\Psi}(t_{\mathrm{f}}))\boldsymbol{\alpha}\|}. \tag{15}$$

In this case, we have $(n+1)$ unknowns, namely the n components of $\boldsymbol{\alpha} \in \mathbb{R}^n$ and t_{f}, which will be determined by the systems of $(n+1)$ equations formed by (13a) and (14) after replacing there the candidate optimal control $\boldsymbol{u}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta})$ with the right hand side of Eq. (15). It is interesting to note that by writing $\boldsymbol{\alpha} = \|\boldsymbol{\alpha}\|\hat{\boldsymbol{\alpha}}$, where $\hat{\boldsymbol{\alpha}}$ is a unit vector, the number of unknowns reduces to n; in this case, Eq. (14) can be ignored.

C. Free terminal position and fixed terminal velocity

Next, we consider the case when the terminal position and velocity vectors are, respectively, free and fixed. In this case, we have that $\boldsymbol{p}_{\boldsymbol{x}}^\star(t_{\mathrm{f}}) = 0$, which implies, in turn, that

$$\boldsymbol{p}_{\boldsymbol{x}}^\star(t) = 0, \quad \boldsymbol{p}_{\boldsymbol{v}}^\star(t) = \boldsymbol{p}_{\boldsymbol{v}}^\star(0) = \boldsymbol{\beta}, \quad \text{for all} \quad t \in [0, t_{\mathrm{f}}].$$

Therefore, the optimal control is now given by

$$\boldsymbol{u}_\star(t;\boldsymbol{\beta}) = -\overline{u}\frac{\boldsymbol{p}_{\boldsymbol{v}}^\star(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\|} = -\overline{u}\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|} = \boldsymbol{\zeta},$$

for $t \in [0, t_{\mathrm{f}}]$, where $\boldsymbol{\zeta} := -\bar{u}\boldsymbol{\beta}/\|\boldsymbol{\beta}\|$. Note that, in this case, the optimal control is a constant vector $\boldsymbol{\zeta} \in \mathbb{R}^n$, whose length is equal to $\overline{u}$. Therefore, we have essentially only n unknowns which satisfy Eq. (13b) (vector equation). In this case, one can ignore, for example, Eq. (14) (scalar equation).

IV. The Case of a Time-Varying Wind Field

A special case of interest is when the drift field is only a function of time. In this case, $\mathbf{A}(t) \equiv \mathbf{0}$; consequently, $\boldsymbol{\Phi}(t,0) = \mathbf{I}_2$ and $\boldsymbol{\Psi}(t) = -t\mathbf{I}_2$, for all $t \geq 0$. It follows readily that

$$\boldsymbol{p}_{\boldsymbol{x}}^\star(t) = \boldsymbol{\alpha}, \qquad \boldsymbol{p}_{\boldsymbol{v}}^\star(t) = \boldsymbol{\beta} - t\boldsymbol{\alpha}.$$

In addition, Eq. (11) now yields the following equation for the time-optimal control:

$$\boldsymbol{u}_\star(t;\boldsymbol{\alpha},\boldsymbol{\beta}) = -\overline{u}\frac{\boldsymbol{p}_{\boldsymbol{v}}^\star(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^\star(t)\|} = -\overline{u}\frac{\boldsymbol{\beta} - t\boldsymbol{\alpha}}{\|\boldsymbol{\beta} - t\boldsymbol{\alpha}\|},$$

for all $t \in [0, t_{\mathrm{f}}]$, except, possibly, from the time instant $\tau \in [0, t_{\mathrm{f}}]$, where $\boldsymbol{\beta} = \tau\boldsymbol{\alpha}$ (if such τ exists). Furthermore, Eqs. (13a)-(13b) become

$$\begin{aligned} \boldsymbol{x}_{\mathrm{f}} = \boldsymbol{x}_0 + \int_0^{t_{\mathrm{f}}} (\boldsymbol{f}(\tau) + \boldsymbol{v}_0)\mathrm{d}\tau & \\ + \int_0^{t_{\mathrm{f}}}\int_0^\tau \boldsymbol{u}_\star(\sigma;\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{d}\sigma\mathrm{d}\tau, & \end{aligned} \tag{16a}$$

$$\boldsymbol{v}_{\mathrm{f}} = \boldsymbol{v}_0 + \int_0^{t_{\mathrm{f}}} \boldsymbol{u}_\star(\sigma;\boldsymbol{\alpha},\boldsymbol{\beta})\mathrm{d}\sigma. \tag{16b}$$

Finally, the transversality condition (14) becomes

$$p_0^\star + \langle \boldsymbol{p}_{\boldsymbol{x}}^\star(t_{\mathrm{f}}), \boldsymbol{f}(t_{\mathrm{f}}) + \boldsymbol{v}(t_{\mathrm{f}})\rangle - \overline{u}\|\boldsymbol{\beta} - t_{\mathrm{f}}\boldsymbol{\alpha}\| = 0. \tag{17}$$

Therefore, Eqs. (16a)-(16b) and (17) form a system of $(2n+1)$ equations for the $(2n+1)$ unknowns, namely t_{f} and the components of the vectors $\boldsymbol{\alpha}$ and $\boldsymbol{\beta} \in \mathbb{R}^n$.

A. Fixed terminal position and free terminal velocity

Next, we consider the case when the terminal position is prescribed and the terminal velocity is free. In this case, we have that $\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t_{\mathrm{f}}) = 0$, which implies, in turn, that $\boldsymbol{\beta} = -t_{\mathrm{f}}\boldsymbol{\alpha}$. Therefore,

$$\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t) = (t_{\mathrm{f}} - t)\boldsymbol{\alpha},$$

and the optimal control is given by

$$\boldsymbol{u}_{\star}(t;\boldsymbol{\zeta}) = -\overline{u}\frac{\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t)\|} = -\overline{u}\frac{(t_{\mathrm{f}} - t)\boldsymbol{\alpha}}{\|(t_{\mathrm{f}} - t)\boldsymbol{\alpha}\|} = \boldsymbol{\zeta},$$

for $t \in [0, t_{\mathrm{f}}[$, where $\boldsymbol{\zeta} := -\overline{u}\boldsymbol{\alpha}/\|\boldsymbol{\alpha}\|$. Therefore, the optimal control, in this case, is a constant vector in $\mathbb{R}^n$; in particular, $\boldsymbol{u}_{\star}(t;\boldsymbol{\zeta}) \equiv \boldsymbol{\zeta}$, where $\|\boldsymbol{\zeta}\| = \overline{u}$ (note that we can set $\boldsymbol{u}_{\star}(t_{\mathrm{f}};\boldsymbol{\zeta}) = \boldsymbol{\zeta}$ regardless of the fact that $\|(t_{\mathrm{f}} - t)\boldsymbol{\alpha}\| = 0$ at $t = t_{\mathrm{f}}$). Therefore, instead of characterizing the vector $\boldsymbol{\alpha}$ (n unknowns), we can now find a vector $\boldsymbol{\zeta}$ of length $\overline{u}$ (that is, $(n-1)$ unknowns); consequently, we have a total of n unknowns instead of $(n + 1)$, in contradistinction with the corresponding case when the velocity of the flow is both spatially and temporally varying. In the case considered herein, we can, for example, ignore Eq. (14). It can be shown, that the components of $\boldsymbol{\zeta}$ and the free final time t_{f} form a system of equations in triangular form. In particular, after integrating Eq. (16a) using $\boldsymbol{u}_{\star}(t;\boldsymbol{\zeta}) = \boldsymbol{\zeta}$ for $t \in [0, t_{\mathrm{f}}]$, it follows that

$$\boldsymbol{x}_{\mathrm{f}} = \boldsymbol{x}_0 + \boldsymbol{v}_0 t_{\mathrm{f}} + t_{\mathrm{f}}^2/2\boldsymbol{\zeta} + \int_0^{t_{\mathrm{f}}} \boldsymbol{f}(t)\mathrm{d}t. \tag{18}$$

Eq. (18) can be written as follows

$$\boldsymbol{x}_{\mathrm{f}} - \boldsymbol{x}_0 - \boldsymbol{v}_0 t_{\mathrm{f}} - \int_0^{t_{\mathrm{f}}} \boldsymbol{f}(t)\mathrm{d}t = t_{\mathrm{f}}^2/2\boldsymbol{\zeta}, \tag{19}$$

which implies, after taking the square of the norm at both sides, that t_{f} is the smallest positive root of the following nonlinear equation

$$\begin{aligned} 0 = &-1/4\overline{u}^2 t_{\mathrm{f}}^4 + \|\boldsymbol{v}_0\|^2 t_{\mathrm{f}}^2 \\ &+ 2\langle \boldsymbol{v}_0, \boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}} + \int_0^{t_{\mathrm{f}}} \boldsymbol{f}(t)\mathrm{d}t\rangle t_{\mathrm{f}} + \|\boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}}\|^2 \\ &+ 2\langle \boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}}, \int_0^{t_{\mathrm{f}}} \boldsymbol{f}(t)\mathrm{d}t\rangle + \|\int_0^{t_{\mathrm{f}}} \boldsymbol{f}(t)\mathrm{d}t\|^2, \end{aligned} \tag{20}$$

whereas the optimal control $\boldsymbol{u}_{\star}(t;\boldsymbol{\zeta}) \equiv \boldsymbol{\zeta}$ is determined by the following equation

$$\boldsymbol{\zeta} = -\frac{2}{t_{\mathrm{f}}^2}\left(\boldsymbol{x}_0 - \boldsymbol{x}_{\mathrm{f}} + \boldsymbol{v}_0 t_{\mathrm{f}} + \int_0^{t_{\mathrm{f}}} \boldsymbol{f}(t)\mathrm{d}t\right). \tag{21}$$

Therefore, after we have characterized the smallest positive solution to Eq. (20), we can characterize the optimal control $\boldsymbol{u}_{\star}(t;\boldsymbol{\zeta}) \equiv \boldsymbol{\zeta}$ from Eq. (21) with back substitution. So essentially, we need to solve only one equation (the one for t_{f}). Note that a detailed treatment of the latter special case has recently appeared in [21].

B. Free terminal position and fixed terminal velocity

Next, we consider the case when the terminal position is free and the terminal velocity is prescribed. In this case, $\boldsymbol{\alpha} = \boldsymbol{p}_{\boldsymbol{x}}^{\star}(0) = \boldsymbol{p}_{\boldsymbol{x}}^{\star}(t_{\mathrm{f}}) = 0$, which implies that

$$\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t) = \boldsymbol{p}_{\boldsymbol{v}}^{\star}(0) = \boldsymbol{\beta}, \quad \text{for all} \quad t \in [0, t_{\mathrm{f}}].$$

Therefore, the optimal control is now given by

$$\boldsymbol{u}_{\star}(t;\boldsymbol{\beta}) = -\overline{u}\frac{\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t)}{\|\boldsymbol{p}_{\boldsymbol{v}}^{\star}(t)\|} = -\overline{u}\frac{\boldsymbol{\beta}}{\|\boldsymbol{\beta}\|} = \boldsymbol{\xi},$$

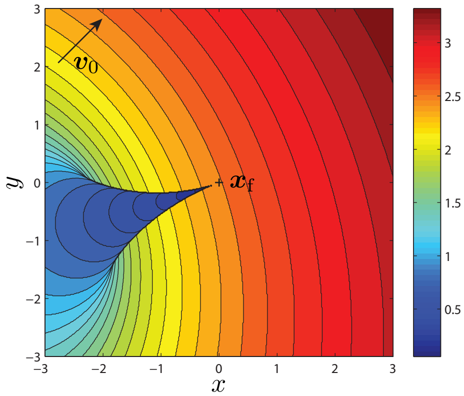
for $t \in [0, t_{\mathrm{f}}]$. Note that the optimal control is again a constant vector, call it $\boldsymbol{\xi} \in \mathbb{R}^n$, where $\|\boldsymbol{\xi}\| = \overline{u}$. We can then obtain a system of n equations and proceed as in the previous case.

V. Numerical Simulations

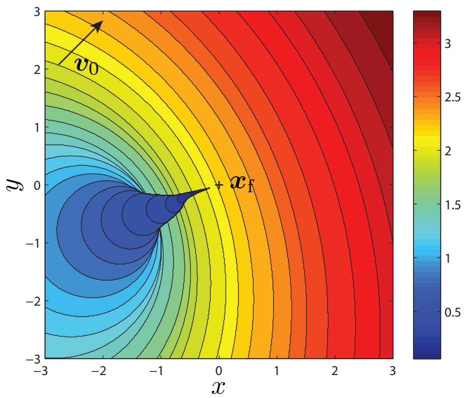
In this section, we present numerical simulations to illustrate the previous theoretical developments. In particular, we consider the motion of the Newtonian particle in the two-dimensional Euclidean plane ($n = 2$) in the presence of a flow with a time-varying and spatially invariant velocity $\boldsymbol{w}$, where $\boldsymbol{w}(t) = w_0(t)[1,\ 0]^{\mathrm{T}}$, where $w_0(t) := A_0\cos(\omega t) + B_0\sin(1.5\omega t) + C_0$. For our simulations, we have used the following data: $\bar{u} = 2$, $A_0 = 0.25$, $B_0 = 0.65$, $C_0 = 1.2$, and $\omega \in \{1, 10, 100\}$. Figure 1 illustrates the level sets ℓ_c of the minimum time function $\boldsymbol{x} \mapsto t_{\mathrm{f}}(\boldsymbol{x};\boldsymbol{v}_0)$ in the $x-y$ plane, where $\ell_c := \{\boldsymbol{x} \in \mathbb{R}^2 :\ t_{\mathrm{f}}(\boldsymbol{x};\boldsymbol{v}_0) = c\}$, and where $t_{\mathrm{f}}(\boldsymbol{x};\boldsymbol{v}_0)$ denotes the minimum time required to steer the particle to the origin ($\boldsymbol{x}_{\mathrm{f}} = 0$) with free terminal velocity, when the latter commences, at time $t = 0$, at a point $\boldsymbol{x}_0 = \boldsymbol{x}$, where $\boldsymbol{x} \in \mathbb{R}^2$, with the same initial velocity $\boldsymbol{v}_0 \in \mathbb{R}^2$. For our simulations, we consider $\boldsymbol{v}_0 = 1.2[\cos\pi/4,\ \ \sin\pi/4]^{\mathrm{T}}$. One important observation is that the minimum time function undergoes discontinuous jumps along the manifolds, which are denoted by the thick black lines in Fig. 1. We also observe that the origin in the $x-y$ plane is not necessarily an interior point of the set of points from which it can be reached by the particle at some time $t \in [0, \tau]$ (accesibility region of the origin), for all $\tau > 0$; that is, the kinematic model of the particle does not enjoy the so-called small-time local accessibility property.

VI. Conclusion

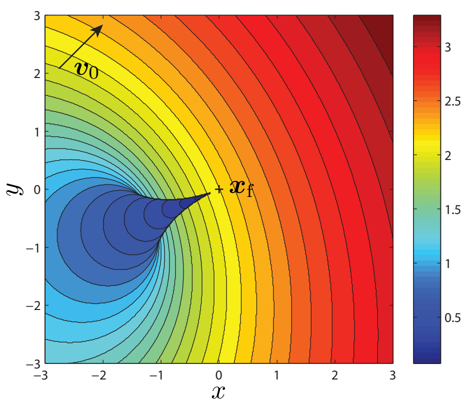
In this paper, we have addressed a classical minimum time problem. In particular, we have addressed the problem of characterizing the time-optimal control law that will steer a Newtonian particle to a prescribed terminal position with an either free or prescribed terminal velocity, and vice versa, in the presence of a spatiotemporal flow field. We have characterized the structure of the time-optimal control law, which is, in general, a continuous function of time, by reducing the minimum time problem to a system of coupled nonlinear equations. Interestingly, the latter system of nonlinear equations can be brought, in some special cases, in triangular form, which can be easily solved numerically. In our future work, we intend to examine the problem when



(a) Level sets of the minimum time function for $\omega = 1$.



(b) Level sets of the minimum time function for $\omega = 10$.



(c) Level sets of the minimum time function for $\omega = 100$.

Fig. 1. Level sets of the minimum time t_f as a function of the particle initial position for a prescribed initial velocity in the presence of a time-varying flow field. We observe that the minimum time function undergoes discontinuous jumps along the manifolds that correspond to the thick black curve segments.

the flow field is not perfectly known a priori; for example, besides the known component of the drift, there is also an uncertain component, which can be modeled by means of either a continuous stochastic process or a deterministic noise signal (worst-case approach) leading, respectively, to the formulation of a stochastic optimal control and a differential game problem.

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