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Optimal Guidance of the Isotropic Rocket in the Presence of Wind

Efstathios Bakolas

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Abstract We address the minimum-time guidance problem for the so-called isotropic rocket in the presence of wind under an explicit constraint on the acceleration norm. We consider the guidance problem to a prescribed terminal position and a circular target set with a free terminal velocity in both cases. We employ standard techniques from optimal control theory to characterize the structure of the optimal guidance law as well as the corresponding minimum time-to-go function. It turns out that the complete characterization of the solution to the optimal control problem reduces to the solution of a system of nonlinear equations in triangular form. Numerical simulations, that illustrate the theoretical developments, are presented.

Keywords: Isotropic rocket, optimal control, guidance, Pontryagin's minimum principle

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E. Bakolas

Department of Aerospace Engineering and Engineering Mechanics, The University of Texas at Austin, Austin, Texas 78712-1221

Tel.: +1 512 471 4250

E-mail: bakolas@austin.utexas.edu

1 Introduction

We address the problem of driving, in a minimum time, a controlled object, to which we shall refer as the *isotropic rocket* (ISOR, for short), to either a prescribed position or a circular target set in the presence of wind. It is assumed that the control input of the ISOR is the rate of change of its air velocity, whose norm is bounded by an a priori fixed constant. The problem belongs to the class of optimal control problems, where the control input is "constrained to a hypersphere" according to [1]. The name of the controlled object comes from "the isotropic rocket pursuit game" proposed by Isaacs in [2, 3], in which a *pursuer* with the kinematics of the ISOR tries, in the absence of wind, to *capture*, that is, to get sufficiently close to, a moving target (*evader*) as fast as possible, whereas the latter tries to delay or, if possible, avoid being captured. In the problem considered in this work, the target of the ISOR is not moving, however, the motion of the ISOR is directly affected by a wind field. At a first glance, the two problems look very similar; they can be associated with each other by a simple coordinate transformation as shown in [4]. The key point that will completely differentiate their solutions has to do with the assumption about the information that is available to the ISOR in the two problems regarding, respectively, the velocity of the target and the wind. In the pursuit-evasion problem, which is a zero sum differential game (two player problem), both the ISOR and the moving target are committed to employ the so-called saddle point strategies, which correspond, respectively, to optimal feedback (pursuing) acceleration and (evading) velocity vectors as shown in [2, 3]. By contrast, the wind in the guidance problem addressed herein (one player problem) is an arbitrary function of time, which is a priori known; something that the ISOR can exploit in order to expedite its arrival to the target (if, for example, the wind is favorable).

A similar problem with the one considered in this work, which has received a considerable amount of attention in the literature, is the one of guiding, in a minimum-time, an aerial vehicle to a target in the presence of wind. However, in the majority of the available results in the literature a different kinematic model is utilized, namely the so-called Dubins vehicle (DV, for short) [5, 6]. Both of these kinematic models can be steered by appropriately controlling the rate of change of their air velocities[1], whose norms are assumed, in addition, to be bounded by an a priori given bound. On the one hand, the rate of change of the air velocity of the DV is constrained to be perpendicular to the air velocity of the vehicle; consequently, the air speed of the DV remains constant at all times. On the other hand, the direction of the rate of change of the air velocity of the ISOR can be chosen freely and does not have to be perpendicular to the air velocity. Consequently, in the latter case, there may exist a component of the acceleration vector that is parallel to the air velocity of the ISOR; therefore, the air speed of the ISOR is not constrained to remain constant at all times.

Variations of the minimum-time guidance problem for the DV in the presence of constant or time varying wind can be found in [7–11]. The problem of guiding the DV in the presence of a stochastic wind while minimizing the expected time of arrival at a given target set can be found in [12, 13]. On the other hand, the problem of guiding a controlled object with kinematics similar to those of the ISOR, yet in the absence of wind, has received considerable attention in the Russian literature, mostly by Akulenko and his collaborators in a series of papers [14–18], where various boundary conditions on the position and/or the velocity vectors have been considered. The authors of [19] independently developed results that bear some similarities with the ones presented in [14–18]. The problems treated in the previous references belong to the class of minimum time problems for linear systems whose control input is constrained to attain values in a compact and convex set, which have been studied in detail in [20–23].

The main contribution of this paper is the formulation and solution of the problem of guiding, in a minimum time, the ISOR to either a prescribed terminal position or a circular target set with a free final air velocity in the presence of time-varying wind. To the best of our knowledge, this is the first time that this problem is proposed and addressed in the literature.

The rest of the paper is organized as follows. The kinematic model is presented in Section 2. The optimal guidance problem with a prescribed terminal position is formulated and subsequently analyzed

[1] In this paper, we will use the standard convention that the velocity is a vector and the speed is the norm of the velocity.

in Section 3 using both a general formulation of Pontryagin's Minimum Principle (Section 3.1-3.4) and more specialized techniques for minimum-time problems for linear systems (Section 3.5). The guidance problem with a circular target set is formulated and subsequently addressed in Section 4. The solution to the guidance problem when the wind is constant is presented in Section 5. Numerical simulations are presented in Section 6. Finally, we conclude the paper with a summary of remarks in Section 7.

2 Kinematic Model

The motion of the ISOR is described by the following set of vector equations

$$\begin{aligned} \dot{\mathsf{x}}(t) &= \mathsf{v}(t) + \mathsf{w}(t), & \mathsf{x}(0) &= \bar{\mathsf{x}}, \\ \dot{\mathsf{v}}(t) &= u(t) - \mu \mathsf{v}(t), & \mathsf{v}(0) &= \bar{\mathsf{v}}, \end{aligned} \tag{1}$$

where $\mathsf{x}(t) := [x(t),\ y(t)]^{\mathsf{T}} \in \mathbb{R}^2$ ($\bar{\mathsf{x}} := [\bar{x},\ \bar{y}]^{\mathsf{T}} \in \mathbb{R}^2$) and $\mathsf{v}(t) := [v(t),\ w(t)]^{\mathsf{T}} \in \mathbb{R}^2$ ($\bar{\mathsf{v}} := [\bar{v},\ \bar{w}]^{\mathsf{T}} \in \mathbb{R}^2$) are, respectively, the position and the air velocity vectors of the ISOR at time t (time $t = 0$). We shall also denote by $\mathsf{z}(t)$ (respectively, $\bar{\mathsf{z}}$), where $\mathsf{z}(t) := [\mathsf{x}^{\mathsf{T}}(t),\ \mathsf{v}^{\mathsf{T}}(t)]^{\mathsf{T}} \in \mathbb{R}^4$ (resp., $\bar{\mathsf{z}} := [\bar{\mathsf{x}},\ \bar{\mathsf{v}}]^{\mathsf{T}} \in \mathbb{R}^4$), the composite state vector of the ISOR at time t (resp., time $t = 0$). In addition, u denotes the control input of the ISOR. We assume that $u(\cdot) \in \mathcal{U}$, where $\mathcal{U}$ consists of the piece-wise continuous functions $t \mapsto u(t)$ taking values in the set $U := \{\nu \in \mathbb{R}^2 :\ \|\nu\| \leq \bar{u}\}$, that is,

$$\|u(t)\| \leq \bar{u}, \qquad \text{for all } t \geq 0. \tag{2}$$

Furthermore, μ is a non-negative constant. In particular, when $\mu > 0$, then it will be assumed that an artificial friction force $F(\mathsf{v}) := -\mu \mathsf{v}$, which is parallel to the direction $-\mathsf{v}/\|\mathsf{v}\|$, will be applied to the ISOR. The purpose of the artificial force $F(\mathsf{v})$, which was originally suggested by Isaacs in [3], is to prohibit the ISOR from acquiring a large air speed. In particular, the force F places along with the input constraint (2) the following upper bound on the maximum attainable air speed

$$\|\mathsf{v}(t)\| \leq \frac{\bar{u}}{\mu}, \qquad \text{for all } t \geq 0,$$

provided that $\|\bar{\mathsf{v}}\| \leq \bar{u}/\mu$. Finally, $\mathsf{w}(t) := [\mathsf{w}_x(t),\ \mathsf{w}_y(t)]^\intercal$ denotes the velocity field induced by the wind, which is a piecewise continuous function of time. The kinematic model is illustrated in Fig. 1, where we also highlight its intrinsic differences with the other commonly employed kinematic model, namely the DV, whose motion is described, instead, by the following vector equations [24]

$$\dot{\mathsf{x}}(t) = \mathsf{v}(t) + \mathsf{w}(t), \qquad \mathsf{x}(0) = \bar{\mathsf{x}},$$
$$\dot{\mathsf{v}}(t) = u(t) \times \mathsf{v}(t), \qquad \mathsf{v}(0) = \bar{\mathsf{v}}, \tag{3}$$

where $\times$ denotes the vector product operation. Note that in the latter case we should only consider control inputs u that are perpendicular to v, given that any component of the input that is parallel to v will result in a null vector product. In addition, in order for $\dot{\mathsf{v}}$ to lie on the $x - y$ plane (the plane of motion of the ISOR), the input u should be perpendicular to this plane. Moreover, the rate of change of the air velocity $\dot{\mathsf{v}}$ of the DV, which is equal to the cross product $\mathsf{v} \times u$, will always be perpendicular to the vehicle's air velocity v. Consequently, the DV will always travel with a constant air speed. By contrast, the ISOR has the flexibility of regulating both its air speed and the direction of its motion.

It is interesting to note that the guidance problem for the ISOR in the presence of wind can be equivalently interpreted as an interception problem of a moving target. In particular, consider the ISOR, whose motion, in the absence of wind, is described by the following vector equations

$$\dot{\mathsf{x}}_{\mathrm{R}}(t) = \mathsf{v}_{\mathrm{R}}(t), \qquad \mathsf{x}_{\mathrm{R}}(0) = \bar{\mathsf{x}},$$
$$\dot{\mathsf{v}}_{\mathrm{R}}(t) = u_{\mathrm{R}}(t) - \mu \mathsf{v}_{\mathrm{R}}(t), \qquad \mathsf{v}_{\mathrm{R}}(0) = \bar{\mathsf{v}}_{\mathrm{R}}, \tag{4}$$

where $\mathsf{x}_{\mathrm{R}}(t) \in \mathbb{R}^2$ ($\bar{\mathsf{x}}_{\mathrm{R}} \in \mathbb{R}^2$) and $\mathsf{v}_{\mathrm{R}}(t) \in \mathbb{R}^2$ ($\bar{\mathsf{v}}_{\mathrm{R}} \in \mathbb{R}^2$) are, respectively, the position and the velocity vectors of the ISOR at time t (time $t = 0$) and $u_{\mathrm{R}}(\cdot) \in \mathcal{U}$ is the control input. In addition, consider a moving target, whose motion is described by the following equation

$$\dot{\mathsf{x}}_T(t) = \mathsf{v}_T(t), \qquad \mathsf{x}_T(0) = 0, \tag{5}$$

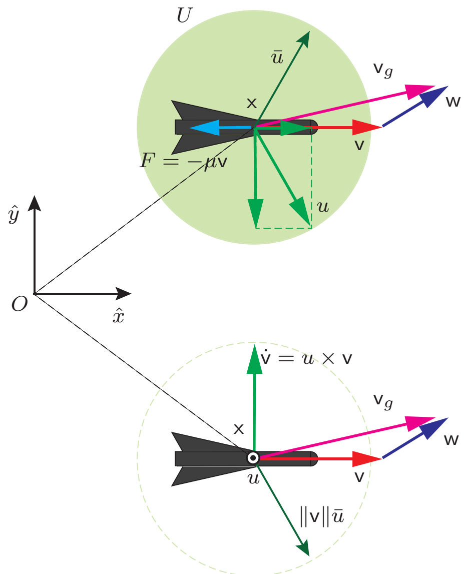


Fig. 1 The kinematics of the ISOR versus those of the DV in the presence of wind. In the first case, the control takes values in U, which is a closed ball of radius $\bar{u}$, whereas in the second case the input u is a vector that is perpendicular to the plane of motion of the ISOR (pointing out of the page) and whose norm is also bounded by $\bar{u}$. In the latter case, $\dot{\mathsf{v}}$ is always perpendicular to v. Consequently, the airspeed of the DV is constant at all times given that no artificial friction force is present in this model. On the other hand, the air speed of the ISOR is not constrained to remain constant.

where $\mathsf{x}_T(t) \in \mathbb{R}^2$ and $\mathsf{v}_T(t) \in \mathbb{R}^2$ are, respectively, the position and velocity vectors of the moving target at time t. Let us consider the following state and input transformations:

$$\mathsf{x}(t) := \mathsf{x}_{\mathrm{R}}(t) - \mathsf{x}_T(t), \qquad \mathsf{v}(t) := \mathsf{v}_{\mathrm{R}}(t), \qquad u(t) = u_{\mathrm{R}}(t).$$

Then (4) and (5) can be written in a compact way as follows

$$\begin{aligned} \dot{\mathsf{x}}(t) &= \mathsf{v}(t) - \mathsf{v}_T(t), & \mathsf{x}(0) &= \bar{\mathsf{x}}_{\mathrm{R}}, \\ \dot{\mathsf{v}}(t) &= u(t) - \mu\mathsf{v}(t), & \mathsf{v}(0) &= \bar{\mathsf{v}}_{\mathrm{R}}. \end{aligned} \tag{6}$$

It follows readily that the problem of finding the guidance law, that will drive, in a minimum time, the ISOR to the origin in the presence of a time-varying wind field $\mathsf{w}(t)$, is equivalent to a minimum time intercept problem. In the latter problem, the objective is to find the control law, that will allow the ISOR, which emanates from $\bar{\mathsf{x}}$ with velocity $\bar{\mathsf{v}}_{\mathrm{R}} = \bar{\mathsf{v}}$, at time $t = 0$, to intercept as fast as possible a moving target, which emanates, in turn, from the origin and whose velocity $\mathsf{v}_T(t) = -\mathsf{w}(t)$. The interpretation of the guidance problem in the presence of wind as an interception problem of a moving target is illustrated in Fig 2.

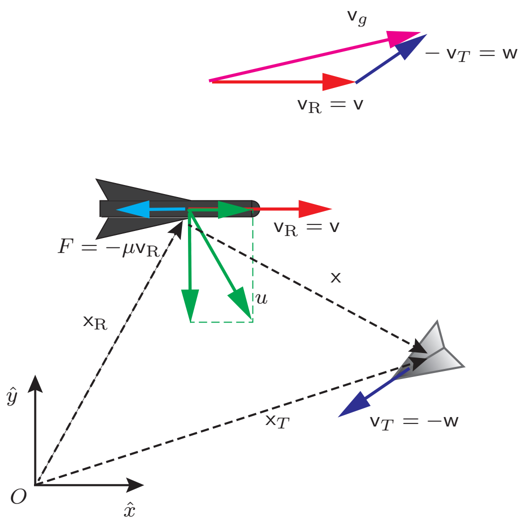


Fig. 2 The interpretation of Problem 3.1, which is a guidance problem to a fixed target, as an interception problem of a moving target. The velocity v_g, where $\mathsf{v}_g := \mathsf{v}_{\mathrm{R}} - \mathsf{v}_T$, corresponds now to the so-called closing velocity.

3 Formulation and Analysis of the Optimal Control Problem with Prescribed Terminal Position

Next we formulate the problem of steering the system described by (1) to a prescribed terminal position with a free terminal air velocity as a deterministic optimal control problem.

Problem 3.1. *Find the control input $u_\star(\cdot) \in \mathcal{U}$ that minimizes the performance index*

$$J(\bar{\mathsf{x}}, \bar{\mathsf{v}}, u(\cdot)) := \int_0^{T_\mathsf{f}} \mathrm{d}t = T_\mathsf{f}, \tag{7}$$

where T_f is the free final time, subject to i) *the dynamic constraint given in* (1) *and* ii) *the following boundary conditions*

$$\mathsf{x}_\star(0) = \bar{\mathsf{x}}, \quad \mathsf{v}_\star(0) = \bar{\mathsf{v}}, \quad \mathsf{x}_\star(T_\mathsf{f}) = 0, \quad \mathsf{v}_\star(T_\mathsf{f}) \ \ \textit{is free.}$$

Remark 3.1. *Problem 3.1 is a standard minimum-time problem, where the control input is a vector constrained to attain values in a closed ball of radius $\bar{\mathsf{u}}$ (a hypersphere). It is known that the class of time-optimal control laws that take values in a hypersphere have, in some cases, a few important advantages over, for example, time-optimal control laws that attain values in a hypercube as is explained in [1]. In particular, as we shall see in the subsequent discussion, the characterization of the time-optimal control law that solves Problem 3.1 can be completely carried out via analytical techniques. One key reason for the analytical tractability of the problem addressed herein has to do with the fact that, in general, the computation of time-optimal control laws that attain values in a hypersphere does not require the characterization of the so-called switch curves, which appear in the solution of minimum-time problems where the input value set is a hypercube.*

3.1 Analysis of the Minimum-Time Problem Based on Pontryagin's Minimum Principle

In this section, we shall completely characterize the solution to Problem 3.1 using standard techniques from optimal control theory [1, 25]. To this aim, we will first utilize Pontryagin's Minimum Principle [26], which furnishes a necessary condition for a control input to be optimal. It turns out that, for the problem considered in this work, Pontryagin's Minimum Principle suffices to naturally lead to the complete characterization of the time-optimal control law. This is a consequence of the well known fact that for minimum-time problems for many systems with linear or affine dynamics and convex input value sets, such as the one considered in this work, Pontryagin's Minimum Principle essentially furnishes a condition that is both necessary and sufficient for optimality as is highlighted, for example, in [23].

In particular, let $t \mapsto \mathsf{z}_\star(t)$, where

$$\mathsf{z}_\star(t) := \begin{bmatrix} \mathsf{x}_\star(t) \\ \mathsf{v}_\star(t) \end{bmatrix} \in \mathbb{R}^4, \tag{8}$$

be an optimal (state) trajectory generated with the application of the control input $u_\star(\cdot) \in \mathcal{U}$ that solves Problem 3.1, for $t \in [0, T_\mathsf{f}]$. Then, there exists a scalar $\lambda_0^\star \in \{0, 1\}$ and an absolutely continuous function

$t \mapsto \lambda^\star_{\mathsf{z}}(t)$, known as the costate, where

$$\lambda^\star_{\mathsf{z}}(t) := \begin{bmatrix} \lambda^\star_{\mathsf{x}}(t) \\ \\ \lambda^\star_{\mathsf{v}}(t) \end{bmatrix} \in \mathbb{R}^4, \tag{9}$$

such that

(i) $\|\lambda^\star_{\mathsf{x}}(t)\| + \|\lambda^\star_{\mathsf{v}}(t)\| + |\lambda^\star_0| \neq 0$, for all $t \in [0, T_{\mathsf{f}}]$[2].

(ii) For all $t \in [0, T_{\mathsf{f}}]$, $\lambda^\star_{\mathsf{x}}$ and $\lambda^\star_{\mathsf{v}}$ satisfy the following differential equations

$$\dot{\lambda}^\star_{\mathsf{x}}(t) = -\nabla^{\mathsf{T}}_{\mathsf{x}} \mathcal{H}(t, \mathsf{z}_\star(t), \lambda^\star_{\mathsf{z}}(t), u_\star(t)) = 0, \tag{10}$$

$$\dot{\lambda}^\star_{\mathsf{v}}(t) = -\nabla^{\mathsf{T}}_{\mathsf{v}} \mathcal{H}(t, \mathsf{z}_\star(t), \lambda^\star_{\mathsf{z}}(t), u_\star(t)) = -\lambda^\star_{\mathsf{x}}(t) + \mu \lambda^\star_{\mathsf{v}}(t), \tag{11}$$

where $\lambda^\star_{\mathsf{x}}(T_{\mathsf{f}})$ is free, $\lambda^\star_{\mathsf{v}}(T_{\mathsf{f}}) = 0$, and $\mathcal{H}$ is the Hamiltonian, which is defined by the following equation

$$\mathcal{H}(t, \mathsf{z}, \lambda_{\mathsf{z}}, u) := \lambda_0 + \langle \lambda_{\mathsf{x}}, \mathsf{v} + \mathsf{w}(t) \rangle + \langle \lambda_{\mathsf{v}}, u - \mu \mathsf{v} \rangle. \tag{12}$$

(iii) The Hamiltonian satisfies the following transversality condition at time $t = T_{\mathsf{f}}$

$$\mathcal{H}(T_{\mathsf{f}}, \mathsf{z}_\star(T_{\mathsf{f}}), \lambda^\star_{\mathsf{z}}(T_{\mathsf{f}}), u_\star(T_{\mathsf{f}})) = 0. \tag{13}$$

(iv) Furthermore, the optimal control $u^\star$ necessarily minimizes the Hamiltonian evaluated along the optimal state and costate trajectories $t \mapsto \mathsf{z}_\star(t)$ and $t \mapsto \lambda^\star_{\mathsf{z}}(t)$, respectively, that is,

$$\mathcal{H}(t, \mathsf{z}_\star(t), \lambda^\star_{\mathsf{z}}(t), u_\star(t)) = \min_{\nu \in U} \mathcal{H}(t, \mathsf{z}_\star(t), \lambda^\star_{\mathsf{z}}(t), \nu), \quad \text{for all } t \in [0, T_{\mathsf{f}}]. \tag{14}$$

Remark 3.2. *Note that the equations of motion of the ISOR along an optimal trajectory $t \mapsto \mathsf{z}_\star(t)$ can be written as follows*

$$\dot{\mathsf{x}}_\star(t) = \nabla^{\mathsf{T}}_{\lambda_{\mathsf{x}}} \mathcal{H}(t, \mathsf{z}_\star(t), \lambda^\star_{\mathsf{z}}(t), u_\star(t)) \tag{15}$$

$$\dot{\mathsf{v}}_\star(t) = \nabla^{\mathsf{T}}_{\lambda_{\mathsf{v}}} \mathcal{H}(t, \mathsf{z}_\star(t), \lambda^\star_{\mathsf{z}}(t), u_\star(t)). \tag{16}$$

[2] We shall refrain from using the expressions "for almost all $t \in [0, T_{\mathsf{f}}]$" or "a.e. on $[0, T_{\mathsf{f}}]$" throughout the manuscript to avoid the unnecessary distraction that this may cause to the reader.

Equations (15)-(16) *along with* (10)-(11) *form the system of canonical equations that determine the evolution of the Hamiltonian system, whose trajectories correspond to the solutions to the minimum-time problem. In particular, the state* $\mathsf{z}_\star$ *and the costate* $\lambda_{\mathsf{z}}^\star$ *correspond, respectively, to the generalized position vector and the generalized momentum or impulse of the Hamiltonian system (see, for example, [27,28]). Note that when* $\lambda_0^\star = 0$, *the minimization of the performance index is not reflected on the trajectories of the Hamiltonian system.*

Note that (14) implies that

$$u_\star(t) = \begin{cases} -\bar{u}\dfrac{\lambda_{\mathsf{v}}^\star(t)}{\|\lambda_{\mathsf{v}}^\star(t)\|}, & \text{if} \quad \lambda_{\mathsf{v}}^\star(t) \neq 0 \\ \nu \in U, & \text{otherwise.} \end{cases} \tag{17}$$

3.2 The case $\mu > 0$

Next, we consider the case when $\mu > 0$. We first show that $\lambda_{\mathsf{v}}^\star(t) \neq 0$ for every $t \in [0, T_{\mathsf{f}}[$; note that we already know from the boundary conditions of (11) that $\lambda_{\mathsf{v}}^\star(t)$ vanishes at $t = T_{\mathsf{f}}$.

Proposition 3.1. *The costates* $\lambda_{\mathsf{x}}^\star$ *and* $\lambda_{\mathsf{v}}^\star$ *satisfy the following equations*

$$\lambda_{\mathsf{x}}^\star(t) = \bar{\lambda}_{\mathsf{x}}^\star, \qquad \lambda_{\mathsf{v}}^\star(t) = \frac{(1 - \exp(-\mu(T_{\mathsf{f}} - t)))}{\mu}\,\bar{\lambda}_{\mathsf{x}}^\star, \tag{18}$$

for all $t \in [0, T_{\mathsf{f}}]$, *where* $\bar{\lambda}_{\mathsf{x}}^\star$ *is a non-zero vector in* $\mathbb{R}^2$. *Furthermore,* $\lambda_{\mathsf{v}}^\star(t) \neq 0$, *for all* $t \in [0, T_{\mathsf{f}}[$.

Proof. Equation (10) reduces to $\lambda_{\mathsf{x}}^\star(t) \equiv \bar{\lambda}_{\mathsf{x}}^\star$, where $\bar{\lambda}_{\mathsf{x}}^\star = \lambda_{\mathsf{x}}^\star(0) = \lambda_{\mathsf{x}}^\star(T_{\mathsf{f}})$, whereas the equation for $\lambda_{\mathsf{v}}^\star$ given in (18) follows readily from (11). Now, let us assume on the contrary that $\bar{\lambda}_{\mathsf{x}}^\star = 0$. Then, Equation (18) implies that $\lambda_{\mathsf{v}}^\star(t) = 0$, for all $t \in [0, T_{\mathsf{f}}]$, which implies, in the light of Equation (13), that $\lambda_0^\star = 0$. Hence $|\lambda_0^\star| + \|\lambda_{\mathsf{x}}^\star(t)\| + \|\lambda_{\mathsf{v}}^\star(t)\| = 0$, for all $t \in [0, T_{\mathsf{f}}]$, which violates Pontryagin's Minimum Principle. Therefore, we conclude that $\bar{\lambda}_{\mathsf{x}}^\star \neq 0$, which implies, in turn, that the left hand side of the second equation in (18) is nonzero for all $t \in [0, T_{\mathsf{f}}[$, and the proof is complete. □

Now we can explicitly characterize the time-optimal control law $u_\star$, that solves Problem 3.1.

Proposition 3.2. *The time-optimal control that solves Problem 3.1 is given by*

$$u_\star(t) = -\bar{u}\frac{\bar{\lambda}^\star_{\mathsf{x}}}{\|\bar{\lambda}^\star_{\mathsf{x}}\|}, \qquad \text{for all } t \in [0, T_{\mathsf{f}}]. \tag{19}$$

Proof. In the light of (18), Equation (17) gives

$$u_\star(t) = -\bar{u}\frac{\lambda^\star_{\mathsf{v}}(t)}{\|\lambda^\star_{\mathsf{v}}(t)\|} = -\bar{u}\frac{(1-\exp(-\mu(T_{\mathsf{f}}-t)))\bar{\lambda}^\star_{\mathsf{x}}}{\|(1-\exp(-\mu(T_{\mathsf{f}}-t)))\bar{\lambda}^\star_{\mathsf{x}}\|} = -\bar{u}\frac{\bar{\lambda}^\star_{\mathsf{x}}}{\|\bar{\lambda}^\star_{\mathsf{x}}\|}, \quad \text{for all } \ t \in [0, T_{\mathsf{f}}[, \tag{20}$$

where we have used the fact that the function $t \mapsto \vartheta(t)$, where $\vartheta(t) := 1 - \exp(-\mu(T_{\mathsf{f}} - t))$, is positive for all $t \in [0, T_{\mathsf{f}}[$, and that $\bar{\lambda}^\star_{\mathsf{x}} \neq 0$ in view of Proposition 3.1. Note that $u_\star(t)$ is indefinite at $t = T_{\mathsf{f}}$; we can take $u_\star(T_{\mathsf{f}}) = \nu$, where $\nu \in U$ can be chosen arbitrarily[3]. In particular, we take $\nu = -\bar{u}\bar{\lambda}^\star_{\mathsf{x}}/\|\bar{\lambda}^\star_{\mathsf{x}}\|$ and the result follows readily. □

Proposition 3.2 simply states that the optimal control $u_\star(t)$ is constant for all $t \in [0, T_{\mathsf{f}}]$; we write $u_\star(t) \equiv \alpha$, where $\alpha := -\bar{u}\bar{\lambda}^\star_{\mathsf{x}}/\|\bar{\lambda}^\star_{\mathsf{x}}\|$. Therefore, Problem 3.1 reduces to the characterization of the constant vector α (it suffices to characterize the direction of α, given that its norm is constant and equal to $\bar{u}$) along with the free terminal time T_{f}.

3.3 Reduction of the Optimal Control Problem to a System of Nonlinear Equations in Triangular Form

Next, we characterize the equations for α and T_{f}. To this aim, we integrate (1) from time $t = 0$ to time t for $u(t) = \alpha$ to obtain the following system of (vector) equations

$$\mathsf{x}_\star(t) = \bar{\mathsf{x}} + \int_0^t (\mathsf{v}_\star(\sigma) + w(\sigma))\,\mathrm{d}\sigma, \tag{21}$$

$$\mathsf{v}_\star(t) = \exp(-\mu t)\bar{\mathsf{v}} + \frac{(1-\exp(-\mu t))}{\mu}\alpha, \tag{22}$$

which implies, in turn, that

$$\mathsf{x}_\star(t) = \bar{\mathsf{x}} + \frac{(1-\exp(-\mu t))}{\mu}\bar{\mathsf{v}} + \varpi(t) + \frac{(\mu t + \exp(-\mu t) - 1)}{\mu^2}\alpha, \tag{23}$$

[3] We are allowed to do so given that the value of the input at an isolated time instant is irrelevant to the evolution of a dynamical system

where $\varpi(t) := \int_0^t w(\sigma)\mathrm{d}\sigma$. Therefore, the terminal condition $\mathsf{x}_\star(T_\mathsf{f}) = 0$ gives the following (vector) equation

$$\alpha = \frac{\mu}{(\mu T_\mathsf{f} + \exp(-\mu T_\mathsf{f}) - 1)} \left((\exp(-\mu T_\mathsf{f}) - 1)\bar{\mathsf{v}} - \mu\bar{\mathsf{x}} - \mu\varpi(T_\mathsf{f}) \right), \tag{24}$$

where the function $t \mapsto \kappa(t)$, $\kappa(t) := \mu t + \exp(-\mu t) - 1$, is non-zero for all $t > 0$. The two unknowns in Equation (24) are the final time T_f and the direction of the vector α. Next, we take the norms of the two vectors on both sides of (24), which gives, in the light of the fact that $\|\alpha\| = \bar{u}$, the following (scalar) nonlinear equation

$$|\mu T_\mathsf{f} + \exp(-\mu T_\mathsf{f}) - 1|\bar{u} = \mu\| \left(\exp(-\mu T_\mathsf{f}) - 1\right) \bar{\mathsf{v}} - \mu\bar{\mathsf{x}} - \mu\varpi(T_\mathsf{f})\|. \tag{25}$$

Consequently, if Problem 3.1 admits a solution, then Equation (25) has at least one positive root; in particular, the minimum time T_f is the smallest positive root of Equation (25).

Note that Equations (24)-(25) form a system of nonlinear equations in triangular form. In particular, one has to solve Equation (25), which is independent of the vector α, to find the time T_f and subsequently compute α from (24) with back substitution. So essentially, the complete characterization of the solution to Problem 3.1 requires the solution of a single (scalar) nonlinear equation, namely Equation (25).

Finally, we examine the existence of abnormal candidate solutions to Problem 3.1, that is, the existence of quadruples $(\lambda_0^\star, \mathsf{z}_\star(\cdot), \lambda_\mathsf{z}^\star(\cdot), u_\star(\cdot))$ that satisfy Pontryagin's Minimum Principle when $\lambda_0^\star = 0$. We call such a quadruple an *abnormal extremal* of the optimal control problem.

Proposition 3.3. *Problem 3.1 admits an abnormal candidate solution if, and only if,*

$$\langle \alpha, \mathsf{v}_\star(T_\mathsf{f}) + \mathsf{w}(T_\mathsf{f}) \rangle = 0, \tag{26}$$

where $\alpha := -\bar{u}\bar{\lambda}_\mathsf{x}^\star / \|\bar{\lambda}_\mathsf{x}^\star\|$.

Proof. The quadruple $(\lambda_0^\star, \mathsf{z}_\star(\cdot), \lambda_\mathsf{z}^\star(\cdot), u_\star(\cdot))$ corresponds to an abnormal candidate solution to Problem 3.1 if, and only if, $\lambda_0^\star = 0$, which along with the boundary conditions

$$\lambda_\mathsf{v}^\star(T_\mathsf{f}) = 0, \qquad \mathcal{H}(T_\mathsf{f}, \mathsf{z}_\star(T_\mathsf{f}), \lambda_\mathsf{z}^\star(T_\mathsf{f}), \alpha) = 0$$

implies that $\langle \bar{\lambda}_{\mathsf{x}}^{\star}, \mathsf{v}_{\star}(T_{\mathsf{f}}) + \mathsf{w}(T_{\mathsf{f}})\rangle = 0$, where we have used the fact that $\lambda_{\mathsf{x}}^{\star}(T_{\mathsf{f}}) = \bar{\lambda}_{\mathsf{x}}^{\star}$. In the light of (19), we have that $\langle \bar{\lambda}_{\mathsf{x}}^{\star}, \mathsf{v}_{\star}(T_{\mathsf{f}}) + \mathsf{w}(T_{\mathsf{f}})\rangle = 0$ if, and only if, $\langle \alpha, \mathsf{v}_{\star}(T_{\mathsf{f}}) + \mathsf{w}(T_{\mathsf{f}})\rangle = 0$ and the proof is complete. □

Remark 3.3. *After having characterized T_{f} and subsequently α and $\mathsf{v}_{\star}(T_{\mathsf{f}})$ for a given initial condition $\bar{\mathsf{z}}$, one can easily conclude whether Problem 3.1 admits an abnormal solution or not by simply checking whether* (26) *is satisfied. The importance of identifying the existence of abnormal extremals has to do with the fact that the latter are intrinsically associated with the appearance of singularities in the optimal synthesis of an optimal control problem. For example, their existence in the optimal synthesis of minimum-time problems can be related to the appearance of discontinuities of the minimum time T_{f}, when the latter is considered as a function of the initial state of the system (the reader is referred to [29] for more details).*

3.4 The case $\mu = 0$

Next, we briefly discuss the solution to Problem 3.1 when $\mu = 0$.

Proposition 3.4. *The costates $\lambda_{\mathsf{x}}^{\star}$ and $\lambda_{\mathsf{v}}^{\star}$ satisfy the following equations*

$$\lambda_{\mathsf{x}}^{\star}(t) = \bar{\lambda}_{\mathsf{x}}^{\star}, \qquad \lambda_{\mathsf{v}}^{\star}(t) = (T_{\mathsf{f}} - t)\,\bar{\lambda}_{\mathsf{x}}^{\star}, \tag{27}$$

for all $t \in [0, T_{\mathsf{f}}]$, where $\bar{\lambda}_{\mathsf{x}}^{\star}$ is a non-zero vector in $\mathbb{R}^2$. Furthermore, $\lambda_{\mathsf{v}}^{\star}(t) \neq 0$, for all $t \in [0, T_{\mathsf{f}}[$, and the optimal control that solves Problem 3.1 is given by Equation (19)*. Finally, Problem 3.1 admits an abnormal candidate solution if, and only if,* (26) *is satisfied.*

Proposition 3.4 implies that the optimal control is a constant vector $\alpha \in \mathbb{R}^2$ as in the case when $\mu > 0$ (of course, the value of α will be, in general, different for $\mu = 0$ and $\mu > 0$). By carrying out a similar analysis as in Section 3.2, we can show that T_{f} is the smallest positive root of the following nonlinear (scalar) equation

$$-\frac{\bar{u}^2}{4}T_{\mathsf{f}}^4 + \|\bar{\mathsf{v}}\|^2 T_{\mathsf{f}}^2 + 2\langle \bar{\mathsf{v}}, \bar{\mathsf{x}} + \varpi(T_{\mathsf{f}})\rangle T_{\mathsf{f}} + 2\langle \bar{\mathsf{x}}, \varpi(T_{\mathsf{f}})\rangle + \|\varpi(T_{\mathsf{f}})\|^2 + \|\bar{\mathsf{x}}\|^2 = 0. \tag{28}$$

Furthermore, by considering the terminal condition $\mathsf{x}_\star(T_\mathsf{f}) = 0$, it follows that the optimal control $u_\star(t) \equiv \alpha$ is determined by the following vector equation

$$\alpha = -\frac{2}{T_\mathsf{f}^2}(\bar{\mathsf{v}}T_\mathsf{f} + \varpi(T_\mathsf{f}) + \bar{\mathsf{x}}). \tag{29}$$

3.5 An Alternative Formulation and Solution of the Minimum-Time Problem

As we have already mentioned, the solution of the minimum-time problem that was previously discussed was based on a general formulation of Pontryagin's Minimum Principle. Next, we briefly discuss an alternative formulation of Problem 3.1 that will allow us to associate it with a well studied class of optimal control problems. To this aim, let us consider the following state transformation

$$\chi(t) := \mathsf{x}(t) - \int_0^t \mathsf{w}(\tau)\mathrm{d}\tau. \tag{30}$$

Then, the equations of motion of the ISOR can be written as follows

$$\begin{aligned} \dot{\chi}(t) &= \mathsf{v}(t), & \chi(0) &= \bar{\mathsf{x}}, \\ \dot{\mathsf{v}}(t) &= u(t) - \mu\mathsf{v}(t), & \mathsf{v}(0) &= \bar{\mathsf{v}}. \end{aligned} \tag{31}$$

Note that (31) can be written more compactly as follows

$$\dot{\zeta} = \mathbf{A}\zeta + \mathbf{B}u(t), \qquad \zeta(0) = \bar{\mathsf{z}}, \tag{32}$$

where

$$\zeta(t) := \begin{bmatrix} \chi(t) \\ \mathsf{v}(t) \end{bmatrix}, \quad \mathbf{A} := \begin{bmatrix} \mathbf{0}_2 & \mathbf{I}_2 \\ \mathbf{0}_2 & -\mu\mathbf{I}_2 \end{bmatrix}, \quad \mathbf{B} := \begin{bmatrix} \mathbf{0}_2 \\ \mathbf{I}_2 \end{bmatrix}.$$

The boundary terminal condition $\mathsf{x}(T_\mathsf{f}) = 0$, where T_f is the (free) final time, expressed now in terms of $\chi(T_\mathsf{f})$, is given by

$$\chi(T_\mathsf{f}) = \mathsf{x}(T_\mathsf{f}) - \int_0^{T_\mathsf{f}} \mathsf{w}(\tau)\mathrm{d}\tau = -\int_0^{T_\mathsf{f}} \mathsf{w}(\tau)\mathrm{d}\tau. \tag{33}$$

Therefore, Problem 3.1 can now be formulated as follows.

Problem 3.2. *Find the control input $u_\star(\cdot) \in \mathcal{U}$ that minimizes the time required for the system described by (32), emanating from the state $\bar{\mathsf{z}}$, at time $t = 0$, to reach a target that moves on the manifold $\Gamma(t) := \{\zeta = [\chi^\mathsf{T},\ \mathsf{v}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^4 : \zeta = \gamma(t;\mathsf{v})\}$, where, for each $\mathsf{v} \in \mathbb{R}^2$, the mapping $t \mapsto \gamma(t;\mathsf{v})$, where*

$$\gamma(t;\mathsf{v}) := \begin{bmatrix} -\int_0^t \mathsf{w}(\tau)\mathrm{d}\tau \\ \\ \mathsf{v} \end{bmatrix},$$

corresponds to a continuous curve in $\mathbb{R}^4$.

Therefore, we are dealing with a minimum-time problem for a linear time invariant system whose target is moving along a family of continuous curves. This particular class of problems has been extensively studied by Krasovskii and LaSalle and their collaborators (the reader may refer to, for example, [20–22,30]).

Next, we show that one can arrive at the same results presented in Section 3.1, by making use of the techniques described in [22] and the formulation of the minimum-time problem given in Problem 3.2. In particular, let us consider the matrix valued function $t \mapsto \mathbf{Y}(t)$, where

$$\mathbf{Y}(t) := \exp(-\mathbf{A}t)\mathbf{B}. \tag{34}$$

It is easy to show that

$$\mathbf{Y}(t) = \begin{bmatrix} \dfrac{(1-\exp(\mu t))}{\mu}\mathbf{I}_2 \\ \\ \exp(\mu t)\mathbf{I}_2 \end{bmatrix} \quad \text{when} \quad \mu > 0, \quad \text{and} \quad \mathbf{Y}(t) = \begin{bmatrix} -t\mathbf{I}_2 \\ \\ \mathbf{I}_2 \end{bmatrix} \quad \text{when} \quad \mu = 0. \tag{35}$$

Let us consider the following state transformation

$$\mathsf{y}(t) := \int_0^t \mathbf{Y}(\tau)u(\tau)\mathrm{d}\tau. \tag{36}$$

or, equivalently,

$$\dot{\mathsf{y}}(t) = \mathbf{Y}(t)u(t), \qquad \mathsf{y}(0) = 0, \tag{37}$$

Let us also consider a continuous curve $t \mapsto \xi(t;\mathsf{v})$ in $\mathbb{R}^4$, where $\mathsf{v} \in \mathbb{R}^2$ and

$$\xi(t;\mathsf{v}) := \exp(-\mathbf{A}t)\gamma(t;\mathsf{v}) - \bar{\mathsf{z}}. \tag{38}$$

Problem 3.2 is then equivalent to the problem of characterizing a control $u_\star(\cdot) \in \mathcal{U}$ that will steer the system described by (37) to a target that moves on the manifold $\Xi(t) := \{\mathsf{y} \in \mathbb{R}^4 : \mathsf{y} = \xi(t; \mathsf{v})\}$ as fast as possible. Using the approach described in [22], one can easily show that the control that solves Problem 3.2 satisfies the following equation

$$u_\star(t) = \begin{cases} \bar{u}\dfrac{\mathbf{Y}^\intercal(t)\eta}{\|\mathbf{Y}^\intercal(t)\eta\|}, & \text{if} \quad \mathbf{Y}^\intercal(t)\eta \neq 0 \\ \nu \in U, & \text{otherwise,} \end{cases} \tag{39}$$

where $\eta = [\eta_1, \eta_2, \eta_3, \eta_4]^\intercal$ is a constant vector in $\mathbb{R}^4$ that is outward normal to the reachable set $\mathfrak{R}(T_\mathsf{f})$, where

$$\mathfrak{R}(t) := \left\{ \mathsf{y} = \int_0^t \mathbf{Y}(\tau)u(\tau)\mathrm{d}\tau \in \mathbb{R}^4 : \ u(\cdot) \in \mathcal{U} \right\},$$

for all $t \in [0, T_\mathsf{f}]$. Note that if Problem 3.2, which is equivalent to Problem 3.1, admits a solution, then, at time $t = T_\mathsf{f}$, the point $\xi(T_\mathsf{f}; \mathsf{v})$ will belong to the boundary of $\mathfrak{R}(T_\mathsf{f})$, for some (free) $\mathsf{v} \in \mathbb{R}^2$ and the hyperplane $\pi(\xi(T_\mathsf{f}; \mathsf{v}), \eta)$, which passes through the point $\xi(T_\mathsf{f}; \mathsf{v})$ and whose normal is the vector η, will be a *support plane* of the reachable set $\mathfrak{R}(T_\mathsf{f})$. Equivalently, the vector formed by the first two components of the vector $\xi(T_\mathsf{f}; \mathsf{v}) \in \mathbb{R}^4$, for any $\mathsf{v} \in \mathbb{R}^2$, which we denote by $\xi_\mathsf{x}(T_\mathsf{f}) \in \mathbb{R}^2$, will belong to the boundary of the set $\mathfrak{R}_\mathsf{x}(T_\mathsf{f})$, where the latter set consists of all the vectors in $\mathbb{R}^2$ that correspond, in turn, to the first two components of each vector in $\mathfrak{R}(T_\mathsf{f})$. Therefore, it suffices to consider outward normal vectors η of the following form: $\eta = [\eta_1, \eta_2, 0, 0]^\intercal \in \mathbb{R}^4$, or more compactly, $\eta = [\eta_\mathsf{x}^\intercal, 0]^\intercal$, where $\eta_\mathsf{x} := [\eta_1, \eta_2]^\intercal \in \mathbb{R}^2$, which are normal to a support plane of $\mathfrak{R}_\mathsf{x}(T_\mathsf{f})$ passing through $\xi_\mathsf{x}(T_\mathsf{f}) \in \mathbb{R}^2$. Then, it follows readily that

$$\mathbf{Y}^\intercal(t)\eta = \begin{cases} \dfrac{(1 - \exp(\mu t))}{\mu}\eta_\mathsf{x}, & \text{when} \quad \mu > 0 \\ -t\eta_\mathsf{x}, & \text{when} \quad \mu = 0. \end{cases} \tag{40}$$

Therefore,

$$\frac{\mathbf{Y}^\intercal(t)\eta}{\|\mathbf{Y}^\intercal(t)\eta\|} = -\frac{\eta_\mathsf{x}}{\|\eta_\mathsf{x}\|}, \tag{41}$$

for $\mu \geq 0$ and for all $t \in]0, T_\mathsf{f}]$, where we have used the fact that the functions $t \mapsto \vartheta_\mu(t)$ and $t \mapsto \vartheta_0(t)$, where $\vartheta_\mu(t) := 1 - \exp(\mu t)$, for $\mu > 0$, and $\vartheta_0(t) := -t$, are negative for all $t \in]0, T_\mathsf{f}]$. Therefore, the equivalent result to the one given in Propositions 3.2 and 3.4 would be, in the light of (41), that the optimal control satisfies the following equation [22]

$$u_\star(t) = -\bar{u}\frac{\eta_{\mathsf{x}}}{\|\eta_{\mathsf{x}}\|}, \tag{42}$$

for all $t \in [0, T_{\mathsf{f}}]$. Therefore, the optimal control $u_\star$ is identically equal to a constant vector, call it α, where $\alpha := -\bar{u}\eta_{\mathsf{x}}/\|\eta_{\mathsf{x}}\|$, whose magnitude is equal to $\bar{u}$.

We thus conclude that the solution to Problem 3.1, which was obtained by means of a general formulation of Pontryagin's Minimum Principle, and the solution to its equivalent formulation given in Problem 3.2, which was obtained by means of the approach that was just described, are identical, as expected.

4 The Guidance Problem to a Prescribed Circular Target Set in Minimum Time

Instead of requiring the ISOR to exactly reach a prescribed terminal position, it is reasonable, from a practical point of view, to consider the problem of guiding the ISOR to a target set. In particular, we consider the target set $\mathcal{T}_\rho := \{[\mathsf{x}^\mathsf{T},\ \mathsf{v}^\mathsf{T}]^\mathsf{T} \in \mathbb{R}^4 :\ \|\mathsf{x}\| = \rho\}$, where ρ is a positive constant. In the latter case, the optimal control problem is formulated as follows.

Problem 4.1. *Let $\rho > 0$ be given and let $\bar{\mathsf{x}} \in \{\mathsf{x} \in \mathbb{R}^2 :\ \|\mathsf{x}\| > \rho\}$. Find the control input $u_\star(\cdot) \in \mathcal{U}$ that minimizes the performance index*

$$J(\bar{\mathsf{x}}, \bar{\mathsf{v}}, u(\cdot)) := \int_0^{T_{\mathsf{f}}} \mathrm{d}t = T_{\mathsf{f}}, \tag{43}$$

where T_{f} is the free final time, subject to i) *the dynamic constraint given in Equation* (1) *and* ii) *the following boundary conditions*

$$\mathsf{x}_\star(0) = \bar{\mathsf{x}}, \quad \mathsf{v}_\star(0) = \bar{\mathsf{v}}, \quad \|\mathsf{x}_\star(T_{\mathsf{f}})\| = \rho, \quad \mathsf{v}_\star(T_{\mathsf{f}}) \ \textit{ is free.}$$

Proposition 4.1. *The time-optimal control that solves Problem 4.1, when $\mu \geq 0$, is given by*

$$u_\star(t) = -\frac{\bar{u}}{\rho}\mathsf{x}_\star(T_{\mathsf{f}}), \qquad \text{for all } t \in [0, T_{\mathsf{f}}]. \tag{44}$$

Finally, Problem 4.1 admits an abnormal candidate solution if, and only if,

$$\langle \mathsf{x}_\star(T_\mathsf{f}), \mathsf{v}_\star(T_\mathsf{f}) + \mathsf{w}(T_\mathsf{f})\rangle = 0. \tag{45}$$

Proof. Let $g(\mathsf{x}) = \frac{1}{2}(\|\mathsf{x}\|^2 - \rho^2)$. The costate equations that correspond to Problem 4.1 are given by (10)-(11), as in Problem 3.1, and the transversality conditions are given by the following equations

$$\lambda_\mathsf{x}^\star(T_\mathsf{f}) = c\nabla_\mathsf{x}^\mathsf{T} g(\mathsf{x}(T_\mathsf{f})) = c\mathsf{x}_\star(T_\mathsf{f}), \quad \lambda_\mathsf{v}^\star(T_\mathsf{f}) = 0, \quad \mathcal{H}(T_\mathsf{f}, \mathsf{z}_\star(T_\mathsf{f}), \lambda_\mathsf{z}^\star(T_\mathsf{f}), u_\star(T_\mathsf{f})) = 0,$$

where $c > 0$. It follows that

$$\lambda_\mathsf{x}^\star(t) \equiv c\,\mathsf{x}_\star(T_\mathsf{f}), \qquad \lambda_\mathsf{v}^\star(t) = \begin{cases} \dfrac{c(1-\exp(-\mu(T_\mathsf{f}-t)))}{\mu}\,\mathsf{x}_\star(T_\mathsf{f}), & \text{when } \mu > 0, \\[2ex] c\,(T_\mathsf{f}-t)\mathsf{x}_\star(T_\mathsf{f}), & \text{when } \mu = 0, \end{cases}$$

for all $t \in [0, T_\mathsf{f}]$, and Equation (44) follows readily. Finally, Equation (45) follows by using similar arguments as in the proof of Proposition 3.3. □

Proposition 4.1 implies that the implementation of the optimal control law requires in this case knowledge of the terminal state $\mathsf{x}_\star(T_\mathsf{f}) \in \mathcal{T}_\rho$. Next, we briefly discuss how to compute $\mathsf{x}_\star(T_\mathsf{f})$ and thus the corresponding time-optimal control $u_\star$ when $\mu > 0$ and $\mu = 0$.

4.1 The case $\mu > 0$

Before specifying $\mathsf{x}_\star(T_\mathsf{f})$, we need to compute the minimum time T_f required to steer the ISOR to the target set $\mathcal{T}_\rho$. By integrating (21) from $t = 0$ to $t = T_\mathsf{f}$ for $u(t) = -\bar{u}/\rho\,\mathsf{x}_\star(T_\mathsf{f})$, it follows that

$$\mathsf{x}_\star(T_\mathsf{f}) = \bar{\mathsf{x}} + \delta(T_\mathsf{f})\bar{\mathsf{v}} + \varepsilon(T_\mathsf{f})\mathsf{x}_\star(T_\mathsf{f}) + \varpi(T_\mathsf{f}), \tag{46}$$

where $\delta(T_\mathsf{f}) := (1-\exp(-\mu T_\mathsf{f}))/\mu$ and $\varepsilon(T_\mathsf{f}) := \bar{u}\,(\delta(T_\mathsf{f}) - T_\mathsf{f})\,/\rho\mu$. By taking the norm of the vectors on both sides of Equation (46) and using the fact that $\|\mathsf{x}_\star(T_\mathsf{f})\| = \rho$, it follows that

$$|1 - \varepsilon(T_\mathsf{f})|\rho = \|\bar{\mathsf{x}} + \delta(T_\mathsf{f})\bar{\mathsf{v}} + \varpi(T_\mathsf{f})\|. \tag{47}$$

Thus, the minimum time T_{f} is the minimum positive root of Equation (47) (provided that such a root exists). After having characterized T_{f}, we can obtain $\mathsf{x}_\star(T_{\mathsf{f}})$ from (46) and subsequently the vector $\alpha \in \mathbb{R}^2$, where $\alpha := -u/\rho \mathsf{x}_\star(T_{\mathsf{f}})$, with back substitution of T_{f} and $\mathsf{x}_\star(T_{\mathsf{f}})$ to (44). In particular,

$$\mathsf{x}_\star(T_{\mathsf{f}}) = \frac{1}{1-\varepsilon(T_{\mathsf{f}})}(\bar{\mathsf{x}} + \delta(T_{\mathsf{f}})\bar{\mathsf{v}} + \varpi(T_{\mathsf{f}})), \qquad \alpha = -\frac{\bar{u}}{\rho(1-\varepsilon(T_{\mathsf{f}}))}(\bar{\mathsf{x}} + \delta(T_{\mathsf{f}})\bar{\mathsf{v}} + \varpi(T_{\mathsf{f}})), \tag{48}$$

where $1 - \varepsilon(t) \neq 0$, for all $t > 0$. An important observation is that (44), (46) and (47) form a system of nonlinear equations in triangular form, which can be easily solved numerically. In particular, the whole problem essentially reduces to the characterization of the minimum positive root of Equation (47).

4.2 The case when $\mu = 0$

Using similar arguments as in the case when $\mu > 0$, we can show that T_{f} is the smallest positive root of the following nonlinear equation

$$\rho + \frac{\bar{u}}{2}T_{\mathsf{f}}^2 = \|\bar{\mathsf{x}} + T_{\mathsf{f}}\bar{\mathsf{v}} + \varpi(T_{\mathsf{f}})\|. \tag{49}$$

After having computed T_{f}, we can easily characterize $\mathsf{x}_\star(T_{\mathsf{f}})$ and subsequently α, where $\alpha := -u/\rho \mathsf{x}_\star(T_{\mathsf{f}})$, by the following equations

$$\mathsf{x}_\star(T_{\mathsf{f}}) = \frac{2\rho}{2\rho + \bar{u}T_{\mathsf{f}}^2}(\bar{\mathsf{x}} + T_{\mathsf{f}}\bar{\mathsf{v}} + \varpi(T_{\mathsf{f}})), \qquad \alpha = -\frac{2\bar{u}}{2\rho + \bar{u}T_{\mathsf{f}}^2}(\bar{\mathsf{x}} + T_{\mathsf{f}}\bar{\mathsf{v}} + \varpi(T_{\mathsf{f}})). \tag{50}$$

5 The Constant Wind Case

One particular case of special interest is when the wind field is constant, that is, $\mathsf{w}(t) \equiv \bar{\mathsf{w}}$ for all $t \geq 0$, where $\bar{\mathsf{w}} \in \mathbb{R}^2$. The importance of this special case stems from the fact that, at a constant altitude, the mean wind field varies slowly with time whereas its spatial variations are insignificant when one considers relatively small travel distances.

5.1 The case $\mu > 0$

First, we simply set $\varpi(t) = T_{\mathsf{f}}\bar{\mathsf{w}}$ in (25) and (47) for Problem 3.1 and Problem 4.1, respectively. Then, if either Problem 3.1 or Problem 4.1 admits a solution, T_{f} is the smallest positive (real) root of the corresponding nonlinear equation. Subsequently, one can characterize α from (24) and (48), respectively.

5.2 The case $\mu = 0$

By plugging $\varpi(T_{\mathsf{f}}) = T_{\mathsf{f}}\bar{\mathsf{w}}$ in (28), it follows that the minimum time of Problem 3.1 satisfies the following polynomial equation

$$-\frac{\bar{u}^2}{4}T_{\mathsf{f}}^4 + \|\bar{\mathsf{v}} + \bar{\mathsf{w}}\|^2 T_{\mathsf{f}}^2 + 2\langle \bar{\mathsf{x}}, \bar{\mathsf{v}} + \bar{\mathsf{w}}\rangle T_{\mathsf{f}} + \|\bar{\mathsf{x}}\|^2 = 0. \tag{51}$$

Note that (51) is a depressed quartic equation, which can be solved analytically using, for example, the Ferrari solution technique. We can then compute α from (24). Similarly, by plugging $\varpi(T_{\mathsf{f}}) = T_{\mathsf{f}}\bar{\mathsf{w}}$ in (49), we can show that the minimum time of Problem 4.1 satisfies the following equation

$$-\frac{\bar{u}^2}{4}T_{\mathsf{f}}^4 + (\|\bar{\mathsf{v}} + \bar{\mathsf{w}}\|^2 - \rho\bar{u})T_{\mathsf{f}}^2 + 2\langle \bar{\mathsf{x}}, \bar{\mathsf{v}} + \bar{\mathsf{w}}\rangle T_{\mathsf{f}} + (\|\bar{\mathsf{x}}\|^2 - \rho^2) = 0, \tag{52}$$

which is also a depressed quartic equation. After computing T_{f}, we can compute $\mathsf{x}_\star(T_{\mathsf{f}})$ and α from (50).

6 Numerical Simulations

In this section, we present numerical simulations to illustrate the previous theoretical developments. For our simulation purposes, we consider a constant wind field $\bar{\mathsf{w}} = 0.5[\cos\frac{\pi}{6},\ \sin\frac{\pi}{6}]^\mathsf{T}$ and $\bar{u} = 2$. Now let $\bar{\mathsf{x}} \mapsto T_{\mathsf{f}}(\bar{\mathsf{x}}; \bar{\mathsf{v}})$ denote the optimal value function of Problem 3.1 (respectively, Problem 4.1) as a function of the ISOR's initial position $\bar{\mathsf{x}}$, for a fixed initial air velocity $\bar{\mathsf{v}}$. In particular, T_{f} is the minimum time required for the ISOR driven by the time-optimal control law that solves Problem 3.1 (resp., Problem 4.1) to reach the origin $\mathsf{x} = 0$ (respectively, a ball of radius ρ centered at $\mathsf{x} = 0$) with free terminal air velocity, assuming that, at time $t = 0$, it commences at an arbitrary point $\bar{\mathsf{x}} \in \mathbb{R}^2$ with a fixed air velocity $\bar{\mathsf{v}} \in \mathbb{R}^2$. Given $k > 0$ and $\bar{\mathsf{v}} \in \mathbb{R}^2$, we will refer to the set $\ell_k(\bar{\mathsf{v}}) := \{\mathsf{x} \in \mathbb{R}^2 : \ T_{\mathsf{f}}(\bar{\mathsf{x}}; \bar{\mathsf{v}}) = k\}$ as the k-th level set of the optimal value function $\bar{\mathsf{x}} \mapsto T_{\mathsf{f}}(\bar{\mathsf{x}}; \bar{\mathsf{v}})$ of Problem 3.1 (respectively, Problem 4.1). Figures 3 and 4 illustrate the k-level sets $\ell_k(\bar{\mathsf{v}})$ of $T_{\mathsf{f}}(\cdot; \bar{\mathsf{v}}(\bar{\theta}))$ for several positive constants k and $\bar{\mathsf{v}}(\theta) = 1.2[\cos\bar{\theta}, \ \sin\bar{\theta}]^{\mathsf{T}}$, where $\bar{\theta} \in \{\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}\}$ (the initial air velocity $\bar{\mathsf{v}}$ is denoted by the red arrow in both Figures 3 and 4), for $\mu = 0$ (Fig. 3) and $\mu = 0.5$ (Fig. 4). For the characterization of the level sets $\ell_k(\bar{\mathsf{v}})$ that are illustrated in Figs. 3 and 4, we have numerically solved Equations (25), for $\varpi(t) = t\mathsf{w}$, and (51), respectively, for different initial positions $\bar{\mathsf{x}}$ of the ISOR, where the latter correspond to the nodes of a uniform grid on a square domain. We observe that, in both cases, the optimal value function is symmetric with respect to the initial inertial (or ground) velocity $\bar{\mathsf{v}}_g$, where $\bar{\mathsf{v}}_g := \bar{\mathsf{v}} + \bar{\mathsf{w}}$ (denoted by the blue arrow in both Figures 3 and 4). As expected, T_{f} attains, in general, higher values in the case when $\mu > 0$ (the virtual force $F = -\mu\mathsf{v}$ slows down the ISOR) than when $\mu = 0$.

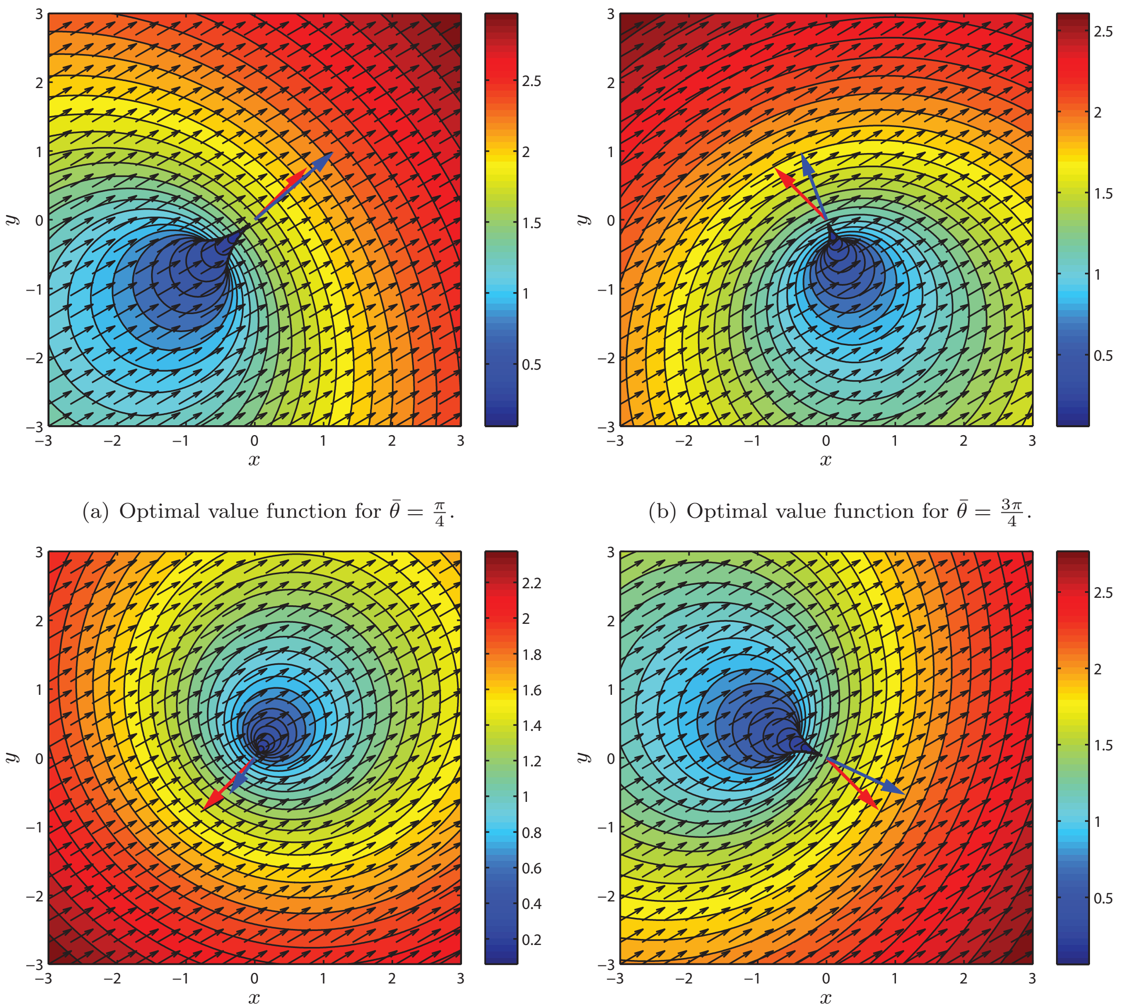
Another important observation is that the value function undergoes discontinuous jumps. The manifolds where these discontinuous jumps take place (thick black solid lines in the vicinity of the origin) are the counterparts of the so-called *barriers* that appear in the solution of the ISOR differential game [3]. The physical interpretation of these discontinuous jumps is the following: even if the ISOR is sufficiently close to the target, it is possible that will be able to reach it only after a significant amount of time unless the ISOR's inertial velocity is pointing towards the target, that is, its inertial velocity is parallel to the so-called line-of-sight (LOS) direction. Now, let $\mathfrak{A}_{t\leq\tau}(0; \bar{\mathsf{v}})$ denote the *accessibility set* of the ISOR to the origin, which is defined as the union of all the initial positions $\bar{\mathsf{x}}$ from which the ISOR, which has a fixed initial air velocity $\bar{\mathsf{v}}$, will reach the origin in the $x - y$ plane at time $t = \tau$ or faster using an admissible control input $u(\cdot) \in \mathcal{U}$. We observe that, due to these discontinuities of the value function, the origin is not an interior point of the accessibility set $\mathfrak{A}_{t\leq\tau}(0; \bar{\mathsf{v}})$, for all $\tau > 0$; the system does not have the so-called local accessibility property. Actually, the origin becomes an interior point of the accessibility set $\mathfrak{A}_{t\leq\tau}(0; \bar{\mathsf{v}})$, for all $\tau \geq \tau^\star$, where $\tau^\star$ is some strictly positive constant.

Furthermore, Fig. 5 illustrates the vector fields that correspond to the $\dot{\mathsf{v}}$ equation of the ISOR, that is, the second equation from (1), for Problem 3.1 and for $\mu = 0$ and $\bar{\theta} = \frac{\pi}{4}$ (Fig. 5(a)) and $\bar{\theta} = \frac{3\pi}{4}$ (Fig. 5(b)). Finally, Fig. 6 illustrates the level sets of the optimal value function $T_{\mathsf{f}}(\cdot; \bar{\mathsf{v}}(\bar{\theta}))$ of Problem 4.1 for $\rho = 0.1$, $\mu = 0$ and $\bar{\theta} = \frac{\pi}{4}$ (Fig. 6(a)) and $\bar{\theta} = \frac{3\pi}{4}$ (Fig. 6(b)). For the characterization of the latter level sets, we have now solved numerically Equation (52) for different initial positions $\bar{\mathsf{x}}$, where the latter correspond again to the nodes of a uniform grid on a square domain.

7 Conclusions

In this paper, we have addressed the problem of guiding, in a minimum time, the so-called isotropic rocket to either a prescribed terminal position or a target set with free terminal air velocity in the presence of wind that varies uniformly with time. We have completely characterized the solution to the guidance problem by using standard techniques from optimal control theory, and in particular, a general formulation of Pontryagin's Minimum Principle as well as more specialized techniques for minimum-time problems for linear systems. It turns out that the optimal guidance law is constant and its implementation simply requires the computation of the solution of a system of nonlinear equations in triangular form. Moreover, we have shown that in the case of constant wind the problem admits an analytic solution.

Future work involves the treatment of the guidance problem with both prescribed terminal positions and velocities. It can be shown that in the latter case, the optimal guidance law is, in general, time-varying. Consequently, some of the analytical tractability of the problem considered in this work is expected to be lost. Another possibility is to consider the case when no accurate information about the wind is available to the rocket. In the latter case, the wind may be modeled as the velocity of a second player, which is adversarial to the rocket, and in particular, its objective is to delay or avoid, if possible, its capture by the rocket. The latter case will naturally lead to the formulation of a differential game, where capture occurs when both the velocity and the positions of the two players coincide [31]. This particular class of



(a) Optimal value function for $\bar{\theta} = \frac{\pi}{4}$.

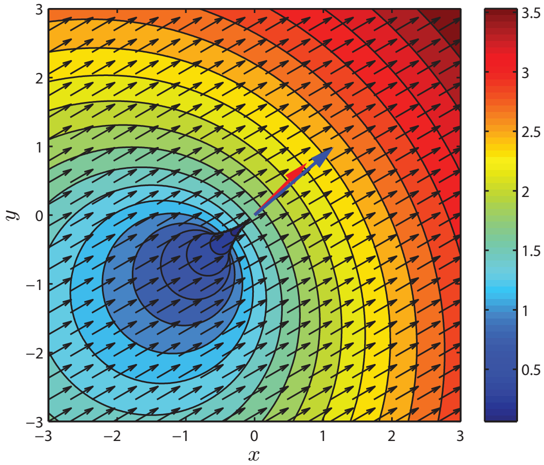
(b) Optimal value function for $\bar{\theta} = \frac{3\pi}{4}$.

(c) Optimal value function for $\bar{\theta} = \frac{5\pi}{4}$.

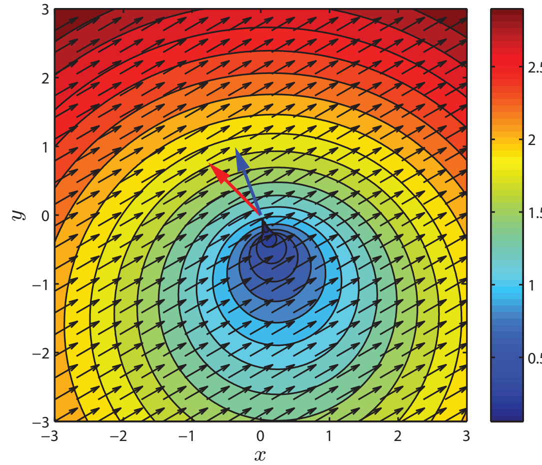
(d) Optimal value function for $\bar{\theta} = \frac{7\pi}{4}$.

Fig. 3 Level sets of the optimal value function $T_{\mathsf{f}}(\cdot;\bar{\mathsf{v}})$ of Problem 3.1 for $\mu = 0$ and different initial air velocities $\bar{\mathsf{v}}$.

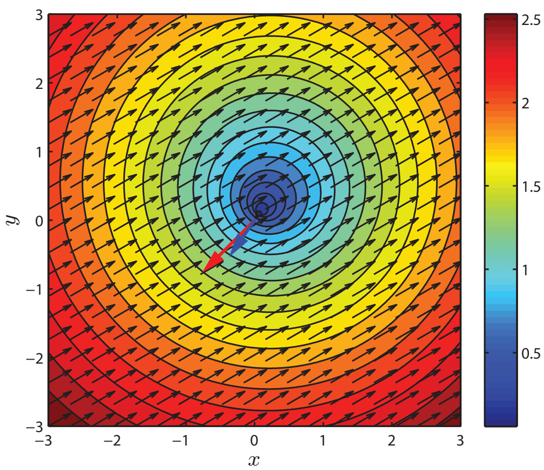
problems can be addressed using techniques from the theory of differential games involving players with linear dynamics and compact and convex input value sets [30,32]. Note that in the previous case, the wind velocity is modeled as a deterministic noise signal; its exact evolution is unknown but its magnitude will satisfy known bounds at all times. Consequently, the characterization of the control law of the rocket will be based on a worst case design approach. Another possibility is to model the wind by a continuous stochastic



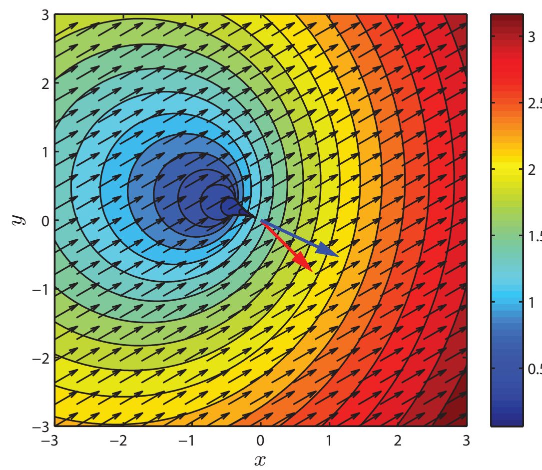
(a) Optimal value function T_{f}, for $\bar{\theta} = \frac{\pi}{4}$.



(b) Optimal value function T_{f} for $\bar{\theta} = \frac{3\pi}{4}$.



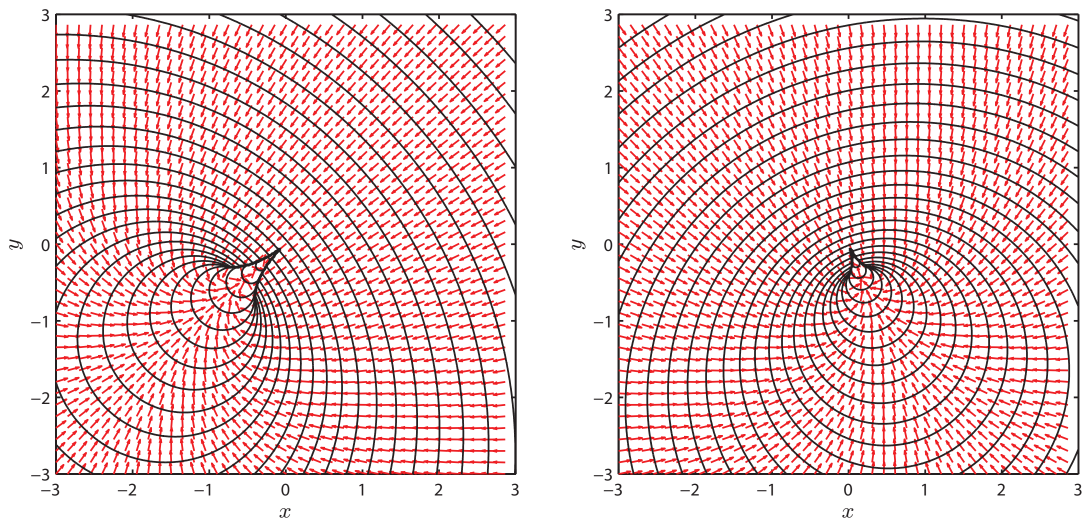
(c) Optimal value function T_{f}, for $\bar{\theta} = \frac{5\pi}{4}$.



(d) Optimal value function T_{f}, for $\bar{\theta} = \frac{7\pi}{4}$.

Fig. 4 Level sets of the optimal value function $T_{\mathsf{f}}(\cdot;\bar{\mathsf{v}})$ of Problem 3.1 for $\mu = 0.5$ and different initial air velocities $\bar{\mathsf{v}}$.

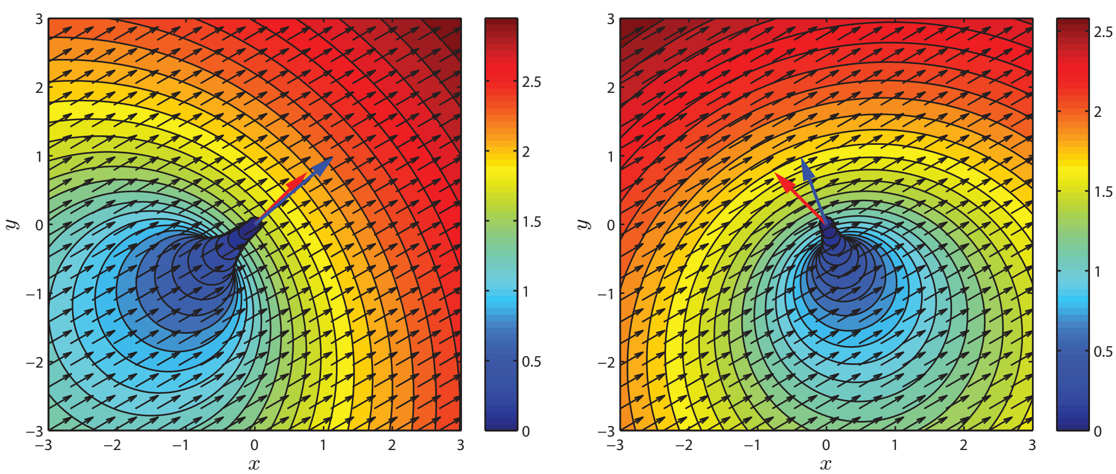
process as in [12,13]. In the latter case, it is assumed that the statistics of the stochastic process are known a priori. By utilizing this information, one can then design less conservative control laws than the ones of the differential game approach. One particular problem that we wish to study is the one of determining a control law that minimizes the expected minimum time of arrival of the rocket to a given target set in the presence of stochastic wind (stochastic optimal control problem).



(a) Vector field for $\dot{\mathsf{v}}$ equation for $\bar{\theta} = \frac{\pi}{4}$.

(b) Vector field for $\dot{\mathsf{v}}$ equation for $\bar{\theta} = \frac{3\pi}{4}$.

Fig. 5 Vector field for $\dot{\mathsf{v}}$ equation for Problem 3.1 for $\mu = 0$ and different initial air velocities $\bar{\mathsf{v}}$.



(a) Optimal value function T_{f}, for $\bar{\theta} = \frac{\pi}{4}$.

(b) Optimal value function T_{f}, for $\bar{\theta} = \frac{3\pi}{4}$.

Fig. 6 Level sets of the optimal value function of Problem 4.1 for $\mu = 0$, $\rho = 0.1$ and different initial air velocities $\bar{\mathsf{v}}$.

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