Finite-Horizon Separation-Based Covariance Control for Discrete-Time Stochastic Linear Systems

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***Abstract*— In this paper, we address a finite-horizon stochastic optimal control problem with covariance assignment and input energy constraints for discrete-time stochastic linear systems with partial state information. In our approach, we consider separation-based control policies that correspond to sequences of control laws that are affine functions of either the complete history of the output estimation errors, that is, the differences between the actual output measurements and their corresponding estimated outputs produced by a discrete-time Kalman filter, or a truncation of the same history. This particular feedback parametrization allows us to associate the stochastic optimal control problem with a tractable semi-definite (convex) program. We argue that the proposed procedure for the reduction of the stochastic optimal control problem to a convex program has significant advantages in terms of improved scalability and tractability over the approaches proposed in the relevant literature.**

I. INTRODUCTION

We consider a finite-horizon stochastic optimal control problem for discrete-time stochastic linear systems with partial state information subject to a constraint on the terminal state covariance (covariance assignment constraints) and another constraint on the expected value of the ℓ_2-norm of the utilized control sequence (input energy constraint). To streamline the analysis of the problem and simplify the computation of its solution, we will only consider (admissible) control policies that correspond to sequences of feedback control laws that can be expressed as affine combinations of either the complete history of output residuals or a truncation of the latter history. In this context, the term "output residual" is used to describe the difference between the measured output of a system and its estimated output as computed by a discrete-time Kalman filtering algorithm. The proposed feedback control parametrization is based on the famous principle of separation between estimation and control [1].

Literature Review: Finite-horizon and infinite-horizon stochastic control problems with terminal constraints on the state covariance for stochastic linear systems in both the discrete-time and continuous-time frameworks have received a lot of attention in the literature [2]–[8]. Recently, a series of recent papers on this topic [9]–[11] addressed similar finite-horizon stochastic control problems for continuous-time linear, Gaussian (stochastic) systems. Similar problems with those in the previous references but in the discrete-time framework have been studied recently for both the cases of complete and partial state information in [12], [13] and [14], [15], respectively. In particular, in our previous work in [14], [15], we have leveraged certain tools and ideas from [16] to develop systematic approaches for the reduction of stochastic optimal control problems with covariance assignment constraints for discrete-time stochastic linear systems with partial state information to (deterministic) tractable convex programs. The methods proposed in [14], [15] utilize a special family of control policies in which the feedback control law at each stage is an affine function of the history of all the output measurements up to the current stage without explicitly using state estimators. Consequently, the dimension of the resulting convex programs can be significantly large especially for multi-stage problems. It should be mentioned at this point that, despite the fact that stochastic optimal control problems subject to different types of constraints (especially, in the infinite-horizon case) can be addressed, in principle, by means of stochastic model predictive control (SMPC) solution techniques [17]–[21], to the best of the author's knowledge, problems with covariance assignment constraints have never been studied within the framework of SMPC in the literature.

Main Contribution: The main contribution of this work is the presentation of a systematic approach for the reduction of stochastic optimal control problems subject to covariance assignment constraints for discrete-time linear systems with partial state information to convex programs that are more tractable than those proposed in the relevant literature. To this aim, we propose a particular feedback control parametrization according to which the admissible control policies are sequences of control laws that are affine functions of the history of the output residuals of the discrete-time system whose state is estimated by a recursive Kalman filter algorithm. One of the key advantages of this parametrization is that it allows one to reduce the size of the convex program that corresponds to the stochastic optimal problem by restricting the feedback control laws at each stage to depend on a truncated history of the most recent output residuals in lieu of the whole history. The flexibility in determining the size of the convex program by truncating accordingly the history of past and present state information is a feature of the proposed approach that is missing from the approaches proposed in [14], [15]. This is because in these references, the decision variables of the convex program are determined by the original parameters of the control policy after the application of a certain bilinear transformation [16] to them, which is agnostic to the length of the truncated sequence.

Structure of the paper: The constrained stochastic optimal

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control problem is formulated in Section II. A systematic procedure for the reduction of the latter problem to a tractable convex program is presented in Section III. Section IV concludes the paper with a number of remarks and ideas for future research.

II. Problem Formulation

A. Notation

We denote by $\mathbb{R}^n$ and $\mathbb{R}^{m\times n}$ the set of real n-dimensional (column) vectors and real $m \times n$ matrices, respectively. We write $\mathbb{Z}^+$ and $\mathbb{Z}^{++}$ to denote the set of non-negative integers and strictly positive integers, respectively. Given τ_1, $\tau_2 \in \mathbb{Z}^+$ with $\tau_1 \leq \tau_2$, we define the *discrete interval* from τ_1 to τ_2 as follows: $[\tau_1, \tau_2]_d = [\tau_1, \tau_2] \cap \mathbb{Z}^+$. We denote by $\mathbb{E}[\cdot]$ the expectation operator. In addition, we denote by $\mathrm{vec}(X_{0:N})$ the vector that is formed by concatenating the (column) vectors that comprise $X_{0:N}$, that is, $\mathrm{vec}(X_{0:N}) := [x(0)^\mathrm{T}, \ldots,\ x(N)^\mathrm{T}]^\mathrm{T}$. If $\mathbf{A} \in \mathbb{R}^{n\times n}$, then we denote its trace by $\mathrm{trace}(\mathbf{A})$ and by $\mathbf{A}^{-1}$ its inverse (provided that the latter is well defined). We write $\mathbf{0}$ and $\mathbf{I}$ to denote the zero matrix and the identity matrix. The space of real symmetric $n \times n$ matrices will be denoted by $\mathbb{S}_n$. Furthermore, we will denote the convex cone of $n \times n$ (symmetric) positive semi-definite and (symmetric) positive definite matrices by $\mathbb{S}_n^+$ and $\mathbb{S}_n^{++}$, respectively. Given a matrix $\mathbf{A} \in \mathbb{S}_n^{++}$ (resp. $\mathbf{A} \in \mathbb{S}_n^{+}$), we will also write $\mathbf{A} \succ \mathbf{0}$ (resp., $\mathbf{A} \succeq \mathbf{0}$). Finally, we write $\mathrm{bdiag}(\mathbf{A}_1,\ \ldots, \mathbf{A}_\ell)$ to denote the block diagonal matrix formed by the matrices $\mathbf{A}_i$, $i \in \{1, \ldots, \ell\}$, which have compatible dimensions.

B. Vectorization of the State Space Model of a Discrete-Time Stochastic Linear System

We consider a discrete-time stochastic linear system that is described by the following equations:

$$x(t+1) = \mathbf{A}x(t) + \mathbf{B}u(t) + w(t), \tag{1a}$$
$$y(t) = \mathbf{C}x(t) + v(t), \tag{1b}$$

for $t \in [0, N-1]_d$, where $X_{0:N} := \{x(t) \in \mathbb{R}^n :\ t \in [0, N]_d\}$ is the state (random) process, $U_{0:N-1} := \{u(t) \in \mathbb{R}^m :\ t \in [0, N-1]_d\}$ is the input process acting together with the noise process $W_{0:N-1} := \{w(t) \in \mathbb{R}^n :\ t \in [0, N-1]_d\}$ upon the system, $Y_{0:N-1} := \{y(t) \in \mathbb{R}^p :\ t \in [0, N-1]_d\}$ is the output process, and finally, $V_{0:N-1} := \{v(t) \in \mathbb{R}^p :\ t \in [0, N-1]_d\}$ is the measurement noise process. In addition, $W_{0:N-1}$ and $V_{0:N-1}$ are sequences of independent and identically distributed normal random variables with

$$\mathbb{E}[w(t)] = \mathbf{0}, \qquad \mathbb{E}\left[w(t)w(\tau)^\mathrm{T}\right] = \delta(t,\tau)\mathbf{W}, \tag{2a}$$
$$\mathbb{E}[v(t)] = \mathbf{0}, \qquad \mathbb{E}\left[v(t)v(\tau)^\mathrm{T}\right] = \delta(t,\tau)\mathbf{V}, \tag{2b}$$

for all $t, \tau \in [0, N-1]_d$, where $\mathbf{W} \in \mathbb{S}_n^{++}$, $\mathbf{V} \in \mathbb{S}_p^{++}$, and $\delta(t,\tau) := 1$, when $t = \tau$, and $\delta(t,\tau) := 0$, otherwise. In addition, $W_{0:N-1}$ and $V_{0:N-1}$ are independent, which implies that

$$\mathbb{E}\left[w(t)v(\tau)^\mathrm{T}\right] = \mathbf{0},\quad \mathbb{E}\left[v(t)w(\tau)^\mathrm{T}\right] = \mathbf{0}, \tag{3}$$

for all $(t,\tau) \in [0, N-1]_d \times [0, N-1]_d$. Similarly, x_0 is independent of both $W_{0:N-1}$ and $V_{0:N-1}$, that is,

$$\mathbb{E}\left[x_0 w(t)^\mathrm{T}\right] = \mathbf{0},\quad \mathbb{E}\left[w(t)x_0^\mathrm{T}\right] = \mathbf{0}, \tag{4a}$$
$$\mathbb{E}\left[x_0 v(t)^\mathrm{T}\right] = \mathbf{0},\quad \mathbb{E}\left[v(t)x_0^\mathrm{T}\right] = \mathbf{0}, \tag{4b}$$

for all $t \in [0, N-1]_d$.

We can write equations (1a)-(1b) compactly as follows:

$$\boldsymbol{x} = \mathbf{X}_0 x_0 + \mathbf{X}_u \boldsymbol{u} + \mathbf{X}_w \boldsymbol{w}, \tag{5a}$$
$$\boldsymbol{y} = \mathbf{Y}_0 x_0 + \mathbf{Y}_u \boldsymbol{u} + \mathbf{Y}_w \boldsymbol{w} + \boldsymbol{v}, \tag{5b}$$

where $\boldsymbol{x} := \mathrm{vec}(X_{0:N}) \in \mathbb{R}^{(N+1)n}$, $\boldsymbol{u} := \mathrm{vec}(U_{0:N-1}) \in \mathbb{R}^{Nm}$, $\boldsymbol{w} := \mathrm{vec}(W_{0:N-1}) \in \mathbb{R}^{Nn}$, $\boldsymbol{y} := \mathrm{vec}(Y_{0:N-1}) \in \mathbb{R}^{Np}$ and $\boldsymbol{v} := \mathrm{vec}(V_{0:N-1}) \in \mathbb{R}^{Np}$. In addition, $\mathbf{X}_0 \in \mathbb{R}^{(N+1)n\times n}$, $\mathbf{X}_u \in \mathbb{R}^{(N+1)n\times Nm}$, and $\mathbf{X}_w \in \mathbb{R}^{(N+1)n\times Nn}$. In particular,

$$\mathbf{X}_0 := \left[\mathbf{I},\ \mathbf{A}^\mathrm{T}, \ldots,\ (\mathbf{A}^{N-1})^\mathrm{T},\ (\mathbf{A}^N)^\mathrm{T}\right]^\mathrm{T},$$

whereas $\mathbf{X}_u$ and $\mathbf{X}_w$ are block lower triangular matrices; in particular, $\mathbf{X}_u = \mathbf{M}(\mathbf{B}; \mathbf{A}, N)$ and $\mathbf{X}_w = \mathbf{M}(\mathbf{I}; \mathbf{A}, N)$ with

$$\mathbf{M}(\mathbf{N}; \mathbf{A}, N) := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{N} & \mathbf{0} & \ldots & \mathbf{0} \\ \mathbf{A}\mathbf{N} & \mathbf{N} & \ldots & \mathbf{0} \\ \vdots & \vdots & \ldots & \vdots \\ \mathbf{A}^{N-1}\mathbf{N} & \mathbf{A}^{N-2}\mathbf{N} & \ldots & \mathbf{N} \end{bmatrix}. \tag{6}$$

Furthermore, $\mathbf{Y}_0 := \mathbf{Y}_x \mathbf{X}_0$, $\mathbf{Y}_u := \mathbf{Y}_x \mathbf{X}_u$, $\mathbf{Y}_w := \mathbf{Y}_x \mathbf{X}_w$ where $\mathbf{Y}_x \in \mathbb{R}^{Np\times(N+1)n}$ is a block lower triangular matrix with

$$\mathbf{Y}_x := [\mathrm{bdiag}(\mathbf{C}, \ldots, \mathbf{C}), \mathbf{0}]. \tag{7}$$

Note that for the derivation of (5b), we have used the fact that $\boldsymbol{y} = \mathbf{Y}_x \boldsymbol{x} + \boldsymbol{v}$, which follows from (1b) and itself implies, in view of (5a), that

$$\boldsymbol{y} = \mathbf{Y}_x \mathbf{X}_0 x_0 + \mathbf{Y}_x \mathbf{X}_u \boldsymbol{u} + \mathbf{Y}_x \mathbf{X}_w \boldsymbol{w} + \boldsymbol{v},$$

from which (5b) follows readily.

The following basic assumption will be useful in the subsequent discussion.

Assumption 1: The pair $(\mathbf{A},\ \mathbf{B})$ is controllable, that is,

$$\mathrm{rank}(\begin{bmatrix}\mathbf{B} & \ldots & \mathbf{A}^{n-1}\mathbf{B}\end{bmatrix}) = n. \tag{8}$$

In addition, the pair $(\mathbf{C},\ \mathbf{A})$ is observable, that is,

$$\mathrm{rank}(\begin{bmatrix}\mathbf{C}^\mathrm{T} & \ldots & \mathbf{C}^\mathrm{T}(\mathbf{A}^{n-1})^\mathrm{T}\end{bmatrix}^\mathrm{T}) = n. \tag{9}$$

Remark 1 The controllability assumption given in (8) will ensure that the expected value of the state can be steered to any vector in $\mathbb{R}^n$ with the application of an appropriate control sequence, at least, in the absence of input constraints. In addition, the observability assumption is made to ensure that the output measurements will always contain the amount of information needed to extract good state estimates from them.

C. State Estimator Dynamics

We assume that a recursive state estimator provides estimates of the current state of the control system based on its output measurements. The state of the estimator and its corresponding output, which are denoted by $\hat{x}(\cdot)$ and $\hat{y}(\cdot)$, respectively, satisfy the following equations:

$$\hat{x}(0) = \hat{x}(0|-1) + \mathbf{\Lambda}^{\circ}(0)\big(y(0) - \mathbf{C}\hat{x}(0|-1)\big), \quad (10a)$$
$$\hat{x}(t+1) = \mathbf{A}\hat{x}(t) + \mathbf{B}u(t) + \mathbf{\Lambda}^{\circ}(t)\big(y(t+1) - \mathbf{CA}\hat{x}(t) - \mathbf{CB}u(t)\big), \quad (10b)$$

for $t \in [0, N-1]_d$, and

$$\hat{y}(0) = \mathbf{C}\hat{x}(0), \quad \hat{y}(t) = \mathbf{C}\hat{x}(t), \quad (11)$$

for $t \in [1, N-1]_d$, where $\hat{x}(0|-1) = \mathbb{E}(x_0) = \mu_0$ and $\mathbf{\Lambda}^{\circ}(t)$ denotes the optimal estimation gain (or "Kalman" gain) matrix at time t, which is determined by the following recursive scheme [22], [23]:

$$\mathbf{P}(0|-1) = \mathbb{E}[(x_0 - \mu_0)(x_0 - \mu_0)^{\mathrm{T}}] = \mathbf{\Sigma}_0, \quad (12a)$$
$$\mathbf{P}(t|t-1) = \mathbf{AP}(t-1|t-1)\mathbf{A}^{\mathrm{T}} + \mathbf{W}, \quad (12b)$$
$$\mathbf{\Lambda}^{\circ}(t) = \mathbf{P}(t|t-1)\mathbf{C}^{\mathrm{T}}\big[\mathbf{CP}(t|t-1)\mathbf{C}^{\mathrm{T}} + \mathbf{V}\big]^{-1}, \quad (12c)$$
$$\mathbf{P}(t|t) = \big[\mathbf{I} - \mathbf{\Lambda}^{\circ}(t)\mathbf{C}\big]\mathbf{P}(t|t-1), \quad (12d)$$

for $t \in [0, N-1]_d$. Now, let us denote by $e(t)$ the state estimation error, where $e(t) := x(t) - \hat{x}(t)$, for $t \in [0, N]_d$. In addition, we denote by $\psi(t)$ the output estimation error, which is also known as the output residual, where $\psi(t) := y(t) - \hat{y}(t)$, for $t \in [0, N-1]_d$. In light of (1a)-(1b) and (10a)-(11), we have that

$$e(t+1) = \mathbf{A}_e(t)e(t) + \mathbf{B}_e(t)w(t) + \mathbf{D}_e(t)v(t), \quad (13a)$$
$$\psi(t) = \mathbf{C}e(t) + v(t), \quad (13b)$$

for $t \in [0, N-1]_d$, with

$$\mathbf{A}_e(t) := \mathbf{A} - \mathbf{\Lambda}^{\circ}(t)\mathbf{CA},$$
$$\mathbf{B}_e(t) := \mathbf{I} - \mathbf{\Lambda}^{\circ}(t)\mathbf{C},$$
$$\mathbf{D}_e(t) := -\mathbf{\Lambda}^{\circ}(t).$$

In addition, $e(0) = e_0$, with $e_0 \sim \mathcal{N}(0, \tilde{\mathbf{\Sigma}}_0)$ where the error covariance matrix $\tilde{\mathbf{\Sigma}}_0$ satisfies the following equation [23]:

$$\tilde{\mathbf{\Sigma}}_0 := \mathbf{\Sigma}_0 - \mathbf{\Sigma}_0\mathbf{C}^{\mathrm{T}}(\mathbf{C}\mathbf{\Sigma}_0\mathbf{C}^{\mathrm{T}} + \mathbf{V})^{-1}\mathbf{C}\mathbf{\Sigma}_0. \quad (14)$$

A well-known property of the Kalman filter is that the state estimation error is orthogonal to the state estimate, which implies that

$$\mathbb{E}\big[\hat{x}(t)e(t)^{\mathrm{T}}\big] = \mathbf{0}, \quad \mathbb{E}\big[e(t)\hat{x}(t)^{\mathrm{T}}\big] = \mathbf{0}, \quad (15)$$

for all $t \in [0, N]_d$.

Now, let $E_{0:N} := \{e(t) \in \mathbb{R}^n : t \in [0, N]_d\}$ and $\Psi_{0:N-1} := \{\psi(t) \in \mathbb{R}^p : t \in [0, N-1]_d\}$ and let $\boldsymbol{e} := \mathrm{vec}(E_{0:N})$ and $\boldsymbol{\psi} := \mathrm{vec}(\Psi_{0:N-1})$. Then, we can write equations (13a)–(13b) compactly as follows:

$$\boldsymbol{e} = \mathbf{E}_0 e_0 + \mathbf{E}_w \boldsymbol{w} + \mathbf{E}_v \boldsymbol{v}, \quad (16a)$$
$$\boldsymbol{\psi} = \mathbf{\Psi}_0 e_0 + \mathbf{\Psi}_w \boldsymbol{w} + \mathbf{\Psi}_v \boldsymbol{v}, \quad (16b)$$

where $\mathbf{E}_0 \in \mathbb{R}^{(N+1)n \times n}$, $\mathbf{E}_w \in \mathbb{R}^{(N+1)n \times Nn}$ and $\mathbf{E}_v \in \mathbb{R}^{(N+1)n \times Np}$. In particular, $\mathbf{E}_0$ is defined as follows:

$$\mathbf{E}_0 := \begin{bmatrix}\mathbf{I} & \mathbf{\Phi}_e(1,0)^{\mathrm{T}} & \dots & \mathbf{\Phi}_e(N,0)^{\mathrm{T}}\end{bmatrix}^{\mathrm{T}},$$

where $\mathbf{\Phi}_e(t,\tau) := \mathbf{A}_e(t-1)\dots\mathbf{A}_e(\tau)$, for all $(t,\tau) \in [1,N]_d \times [0, N-1]_d$ with $t > \tau$ (note that $\mathbf{\Phi}_e(t,\tau) := \mathbf{A}_e(t-1)$, when $t = \tau+1$) and $\mathbf{\Phi}_e(t,t) = \mathbf{I}$, for all $t \in [1,N]_d$. In addition, $\mathbf{E}_w$ and $\mathbf{E}_v$ are block lower triangular matrices, and in particular, $\mathbf{E}_w := \boldsymbol{\mathcal{M}}(\mathbf{B}_e(\cdot); \mathbf{\Phi}_e(\cdot,\cdot), N)$ and $\mathbf{E}_v := \boldsymbol{\mathcal{M}}(\mathbf{D}_e(\cdot); \mathbf{\Phi}_e(\cdot,\cdot), N)$, where

$$\boldsymbol{\mathcal{M}}(\boldsymbol{\mathcal{P}}(\cdot); \mathbf{\Phi}_e(\cdot,\cdot), N) := \begin{bmatrix} \mathbf{0} & \mathbf{0} & \dots & \mathbf{0} \\ \boldsymbol{\mathcal{P}}(0) & \mathbf{0} & \dots & \mathbf{0} \\ \mathbf{\Phi}_e(2,1)\boldsymbol{\mathcal{P}}(0) & \boldsymbol{\mathcal{P}}(1) & \dots & \mathbf{0} \\ \vdots & \vdots & \dots & \vdots \\ \mathbf{\Phi}_e(N,1)\boldsymbol{\mathcal{P}}(0) & \mathbf{\Phi}_e(N,2)\boldsymbol{\mathcal{P}}(1) & \dots & \boldsymbol{\mathcal{P}}(N-1) \end{bmatrix}.$$

Furthermore, we have that

$$\mathbf{\Psi}_0 := \mathbf{Y}_x\mathbf{E}_0, \;\; \mathbf{\Psi}_w := \mathbf{Y}_x\mathbf{E}_w, \;\; \mathbf{\Psi}_v := \mathbf{Y}_x\mathbf{E}_v + \mathbf{I}.$$

For the derivation of (16b), we have used the fact that

$$\boldsymbol{\psi} = \mathbf{Y}_x\boldsymbol{e} + \boldsymbol{v},$$

which follows from (13b) and itself implies, in view of (16a), that

$$\boldsymbol{\psi} = \mathbf{Y}_x\mathbf{E}_0 e_0 + \mathbf{Y}_x\mathbf{E}_w\boldsymbol{w} + (\mathbf{Y}_x\mathbf{E}_v + \mathbf{I})\boldsymbol{v},$$

from which the result follows readily.

D. Formulation of the Stochastic Optimal Control Problem with Covariance Assignment Constraints

Our objective is to find a control policy that minimizes the expected value of a finite sum of convex quadratic functions (costs per stage) of the state $x(t)$ of the stochastic linear system (1a)-(1b), subject to an inequality constraint on the squared ℓ_2-norm of the input (random) sequence $U_{0:N-1}$. We will assume that the set of admissible control policies, which is denoted by $\mathcal{P}$, consists of all control policies $\pi := \{\mu(\cdot;t);\; t \in [0, N-1]_d\}$, where at each stage t, the control law $\mu(\cdot;t)$ is a causal (non-anticipative), measurable function of the elements of the output process $Y_{0:t}$ and in particular, an *affine* combination of the elements of the latter process (or more precisely, the complete filtration of the sigma field generated by the elements of $Y_{0:t}$). In particular, for each $t \in [0, N-1]_d$, the control law $\mu(\cdot;t)$ will map a given (random) finite-length sequence $Y_{0:t}$ to a (random) m-dimensional input vector $u(t)$. We write $\pi = \{\mu(Y_{0:t};t) : t \in [0, N-1]_d\}$. Next, we give the precise formulation of the stochastic optimal control problem with *incomplete* and imperfect state information subject to covariance assignment constraints.

Problem 1: Let N, $q \in \mathbb{Z}^{++}$ and $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$ be given. In addition, let $\{\mathbf{Q}(t) \in \mathbb{S}_n^{+} : t \in [0, N-1]_d\}$ and $\{\mathbf{R}(t) \in \mathbb{S}_m^{+} : t \in [0, N-1]_d\}$ be given sequences of positive semi-definite matrices and let also ℓ be a given positive number. Then, find an optimal control policy $\pi^{\circ} :=$

$\{\mu^\circ(Y_{0:0};0),\ldots,\mu^\circ(Y_{0:N-1};N-1)\} \in \mathcal{P}$ that minimizes the performance index

$$J(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} x(t)^{\mathrm{T}}\mathbf{Q}(t)x(t)\Big] \tag{17}$$

over all admissible feedback control policies $\pi = \{\mu(Y_{0:0};0),\ldots,\mu(Y_{0:N-1};N-1)\} \in \mathcal{P}$ subject to (i) the difference equation (1a)-(1b), (ii), the following input constraint:

$$C(\pi) \leq 0, \quad C(\pi) := \mathbb{E}\Big[\sum_{t=0}^{N-1} u(t)^{\mathrm{T}}\mathbf{R}(t)u(t)\Big] - \ell, \tag{18}$$

and (iii) the following terminal constraints in terms of the mean and the covariance of the (random) state vector $x(t)$ at $t = N$:

$$h(x(N)) = \mathbf{0}, \quad \mathbf{H}(x(N)) \succeq \mathbf{0},$$

where

$$h(x(N)) := \mathbb{E}\left[x(N)\right], \tag{19a}$$

$$\mathbf{H}(x(N)) := \mathbf{\Sigma}_{\mathsf{f}} - \mathbb{E}\left[x(N)x(N)^{\mathrm{T}}\right]. \tag{19b}$$

Remark 2 Note that instead of the positive semi-definite constraint $\mathbf{H}(x(N)) \succeq \mathbf{0}$, where $\mathbf{H}(x(N))$ is given in (19b), one should in principle enforce the following matrix equality constraint: $\mathbf{H}(x(N)) = \mathbf{0}$, or equivalently, $\mathbb{E}\left[x(N)x(N)^{\mathrm{T}}\right] = \mathbf{\Sigma}_{\mathsf{f}}$. Note that the latter matrix equality constraint together with the vector equality constraint $h(x(N)) = \mathbb{E}\left[x(N)\right] = \mathbf{0}$ imply that the terminal state covariance should be equal to a prescribed positive definite matrix (strict covariance assignment constraint). As we have shown in our previous work in [12], the matrix equality constraint $\mathbf{H}(x(N)) = \mathbf{0}$ is non-convex, whereas the positive semi-definite constraint $\mathbf{H}(x(N)) \succeq \mathbf{0}$ corresponds to a convex relaxation of the latter. These remarks will become more clear later on, when we discuss the process of converting Problem 1 into a tractable finite-dimensional optimization problem.

III. Reduction of the Stochastic Optimal Control Problem to a Tractable Convex Program

A. Set of Admissible Control Policies

Finding the solution to Problem 1 can be a very complex task. In our previous work [14], [15], we have proposed solution techniques in which the proposed feedback control policy was taken to be a sequence of control laws that were affine functions of the present and all past output measurements. It turns out that the computation of the latter feedback policy can incur a significant cost when the number of stages, N, is large given that the control law at each stage depends on the present and all past measurements. Herein, we restrict our search to a subset of $\mathcal{P}$ that consists of policies $\pi = \{\mu(\cdot;t) : \ t \in [0,N-1]_d\}$, where

$$\mu(\Psi_{0:t};t) = \bar{u}(t) + \sum_{\tau=0}^{t} \mathbf{F}(t,\tau)\psi(t), \tag{20}$$

for $t \in [0,N-1]_d$, where $\mathbf{F}(t,\tau) \in \mathbb{R}^{m\times p}$ for all $(t,\tau) \in [0,N-1]_d \times [0,N-1]_d$ with $t \geq \tau$, and $\bar{U}_{0:N-1} := \{\bar{u}(t) \in \mathbb{R}^m : \ t \in [0,N-1]_d\}$ is a finite-length sequence of (open-loop) reference input signals. We will denote this subset of $\mathcal{P}$ as $\widehat{\mathcal{P}}$. Note that the fact that $\widehat{\mathcal{P}}$ is a subset of $\mathcal{P}$ is a direct consequence of the fact that the state estimate $\hat{x}(t)$ is an affine function of the present and past output measurements, that is, $\hat{x}(t)$ is an affine function of the elements of $Y_{0:t}$ [23].

Note that there is nothing that prevents us from setting $\mathbf{F}(t,\tau) = \mathbf{0}$ for all $\tau \in [0, t-\sigma-1]_d$, for some $\sigma \in [0, t-1]_d$. In this case $\mu(\cdot;t)$ will be an affine function of the elements of the truncated output process $\Psi_{t-\sigma:t}$, that is,

$$\mu(\Psi_{t-\sigma:t};t) = \bar{u}(t) + \sum_{\tau=t-\sigma}^{t} \mathbf{F}(t,\tau)\psi(t). \tag{21}$$

For instance, if $\sigma = 1$, then $\mu(\cdot;t)$ will depend only on the current and the most recent output residuals, $\psi(t)$ and $\psi(t-1)$, respectively, whereas if $\sigma = 0$, then $\mu(\cdot;t)$ will depend only on the current output residual, $\psi(t)$. In the subsequent discussion, we will present the most general cases in which the control law depends on the whole history of output estimation errors and satisfies equation (20). The analysis for the case when the control law depends only on a truncated version of the history of output estimation errors can be done in a similar (and obvious) way after the necessary modifications have been carried out.

In order to find the closed-loop dynamics of the discrete-time linear system given in (1a)-(1b), we will have to set $u(t) = \mu(\Psi_{0:t};t)$, where $\mu(\Psi_{0:t};t)$ is defined in (20). Then,

$$\boldsymbol{u} := \bar{\boldsymbol{u}} + \mathbf{K}\boldsymbol{\psi}, \tag{22}$$

where $\bar{\boldsymbol{u}} := \mathrm{vec}(\bar{U}_{0:N-1})$ and $\mathbf{K} \in \mathbb{R}^{Nm\times Np}$ is an $N \times N$ block lower triangular matrix with blocks $\mathbf{K}_{i,j} \in \mathbb{R}^{m\times p}$. In particular, $\mathbf{K}_{i,j} := \mathbf{F}(i-1, j-1)$, if $i \geq j$, and $\mathbf{K}_{i,j} := \mathbf{0}$, if $i < j$. In view of (16b), equation (22) gives

$$\boldsymbol{u} = \bar{\boldsymbol{u}} + \mathbf{U}_0(\mathbf{K})e_0 + \mathbf{U}_w(\mathbf{K})\boldsymbol{w} + \mathbf{U}_v(\mathbf{K})\boldsymbol{v}, \tag{23}$$

with $\mathbf{U}_0(\mathbf{K}) := \mathbf{K}\mathbf{\Psi}_0$, $\mathbf{U}_w(\mathbf{K}) := \mathbf{K}\mathbf{\Psi}_w$, and $\mathbf{U}_v(\mathbf{K}) := \mathbf{K}\mathbf{\Psi}_v$. Equation (23) induces an one-to-one mapping that associates a control policy $\pi \in \widehat{\mathcal{P}}$ with the decision variables $(\bar{\boldsymbol{u}}, \mathbf{K})$. In particular, given $(\bar{\boldsymbol{u}}, \mathbf{K})$, the corresponding control policy $\pi = \{\mu(\mathbf{E}_{t+1}\boldsymbol{\psi};t) : \ t \in [0,N-1]_d\}$, with

$$\mu(\mathbf{E}_{t+1}\boldsymbol{\psi};t) := \mathbf{E}_{t+1}(\bar{\boldsymbol{u}} + \mathbf{U}_0(\mathbf{K})e_0 + \mathbf{U}_w(\mathbf{K})\boldsymbol{w} + \mathbf{U}_v(\mathbf{K})\boldsymbol{v}),$$

where $\mathbf{E}_{t+1} \in \mathbb{R}^{1\times Nm}$, for $t \in [0,N-1]_d$, is a block row vector with N blocks $(\mathbf{E}_{t+1})_{1,i} \in \mathbb{R}^m$, for $i \in [1,N]_d$. In particular, $(\mathbf{E}_{t+1})_{1,i} = \mathbf{I}$, for $i = t+1$, and $(\mathbf{E}_{t+1})_{1,i} = \mathbf{0}$, otherwise. We denote the latter mapping as ϖ and we write $\pi = \varpi(\bar{\boldsymbol{u}}, \mathbf{K})$. The inverse mapping, ϖ^{-1}, can be defined similarly; we write $(\bar{\boldsymbol{u}}, \mathbf{K}) = \varpi^{-1}(\pi)$.

B. Closed-loop dynamics

In view of (5a) and (23), the closed loop dynamics of the control system can be written compactly as follows:

$$\begin{aligned}\boldsymbol{x} = \mathbf{X}_0 x_0 + \mathbf{X}_u(\mathbf{U}_0(\mathbf{K})e_0 + \mathbf{U}_w(\mathbf{K})\boldsymbol{w} + \mathbf{U}_v(\mathbf{K})\boldsymbol{v}) \\ + \mathbf{X}_u\bar{\boldsymbol{u}} + \mathbf{X}_w\boldsymbol{w},\end{aligned} \tag{24}$$

or equivalently,

$$\boldsymbol{x} = \boldsymbol{\mathcal{G}}_{x_0} x_0 + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} + \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) e_0 + \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{w} + \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{v}, \tag{25}$$

where $\boldsymbol{\mathcal{G}}_{x_0} := \mathbf{X}_0$, $\boldsymbol{\mathcal{G}}_{\bar{u}} := \mathbf{X}_u$, $\boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) := \mathbf{X}_u \mathbf{K} \mathbf{\Psi}_0$, $\boldsymbol{\mathcal{G}}_w(\mathbf{K}) := \mathbf{X}_w + \mathbf{X}_u \mathbf{K} \mathbf{\Psi}_w$, and $\boldsymbol{\mathcal{G}}_v(\mathbf{K}) := \mathbf{X}_u \mathbf{K} \mathbf{\Psi}_v$.

C. Expressions of the cost and constraint functions in terms of the decision variables $\bar{\boldsymbol{u}}$ and $\mathbf{K}$

The cost function can be written as follows:

$$J(\pi) = \mathbb{E}\big[\boldsymbol{x}^{\mathrm{T}} \boldsymbol{\mathcal{Q}} \boldsymbol{x}\big] = \mathbb{E}\big[\mathrm{trace}(\boldsymbol{x}\boldsymbol{x}^{\mathrm{T}} \boldsymbol{\mathcal{Q}})\big], \tag{26}$$

where $\boldsymbol{\mathcal{Q}} := \mathrm{bdiag}(\mathbf{Q}(0), \dots, \mathbf{Q}(N-1), \mathbf{0}) \in \mathbb{S}^{+}_{(N+1)n}$. In view of (25), Eq. (26) can be written as follows:

$$\begin{aligned} J(\pi) = \mathbb{E}\big[\mathrm{trace}\big((&\boldsymbol{\mathcal{G}}_{x_0} x_0 + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} \\ &+ \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) e_0 + \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{w} + \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{v}) \\ &\times (\boldsymbol{\mathcal{G}}_{x_0} x_0 + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} \\ &+ \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) e_0 + \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{w} + \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{v})^{\mathrm{T}} \boldsymbol{\mathcal{Q}}\big)\big] \\ =: \mathcal{J}(\mathbf{K}). \end{aligned} \tag{27}$$

In view of (2a)– (4b) and (15), Eq. (27) implies that

$$\begin{aligned} \mathcal{J}(\mathbf{K}) = \mathrm{trace}\big((&\boldsymbol{\mathcal{G}}_{x_0}(\mathbf{\Sigma}_0 + \mu_0 \mu_0^{\mathrm{T}}) \boldsymbol{\mathcal{G}}_{x_0}^{\mathrm{T}} \\ &+ 2\boldsymbol{\mathcal{G}}_{x_0} \mu_0 \bar{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{\bar{u}}^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} \bar{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{\bar{u}}^{\mathrm{T}} \\ &+ 2\boldsymbol{\mathcal{G}}_{x_0} \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K})^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K})^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{G}}_w(\mathbf{K})^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{G}}_v(\mathbf{K})^{\mathrm{T}}) \boldsymbol{\mathcal{Q}}\big), \end{aligned} \tag{28}$$

where $\boldsymbol{\mathcal{W}} = \mathrm{bdiag}(\mathbf{W}, \dots, \mathbf{W})$ and $\boldsymbol{\mathcal{V}} = \mathrm{bdiag}(\mathbf{V}, \dots, \mathbf{V})$. In the previous derivation, we have used the fact that, in the light of (15), we have that

$$\mathbb{E}\big[x_0 e_0^{\mathrm{T}}\big] = \mathbb{E}\big[(\hat{x}_0 + e_0) e_0^{\mathrm{T}}\big] = \tilde{\mathbf{\Sigma}}_0. \tag{29}$$

The input constraint function $C(\pi)$ can be written compactly as follows:

$$C(\pi) = \mathbb{E}\big[\boldsymbol{u}^{\mathrm{T}} \boldsymbol{\mathcal{R}} \boldsymbol{u}\big] - \ell = \mathbb{E}\big[\mathrm{trace}(\boldsymbol{u}\boldsymbol{u}^{\mathrm{T}} \boldsymbol{\mathcal{R}})\big] - \ell, \tag{30}$$

where $\boldsymbol{\mathcal{R}} := \mathrm{bdiag}(\mathbf{R}(0), \dots, \mathbf{R}(N-1)) \in \mathbb{S}^{+}_{Nm}$. In view of (23), Eq. (30) can be written as follows:

$$\begin{aligned} C(\pi) = \mathbb{E}\big[\mathrm{trace}\big((&\bar{\boldsymbol{u}} + \mathbf{U}_0(\mathbf{K}) e_0 + \mathbf{U}_w(\mathbf{K}) \boldsymbol{w} + \mathbf{U}_v(\mathbf{K}) \boldsymbol{v}) \\ &\times (\bar{\boldsymbol{u}} + \mathbf{U}_0(\mathbf{K}) e_0 + \mathbf{U}_w(\mathbf{K}) \boldsymbol{w} + \mathbf{U}_v(\mathbf{K}) \boldsymbol{v})^{\mathrm{T}} \boldsymbol{\mathcal{R}}\big)\big] - \ell \\ =: \mathcal{C}(\mathbf{F}). \end{aligned} \tag{31}$$

In light of (2a)–(4b) and (15), Eq. (31) implies that

$$\begin{aligned} \mathcal{C}(\mathbf{K}) = \mathrm{trace}\big(&\bar{\boldsymbol{u}} \bar{\boldsymbol{u}}^{\mathrm{T}} + \mathbf{U}_0(\mathbf{K}) \tilde{\mathbf{\Sigma}}_0 \mathbf{U}_0(\mathbf{K})^{\mathrm{T}} \\ &+ \mathbf{U}_w(\mathbf{K}) \boldsymbol{\mathcal{W}} \mathbf{U}_w(\mathbf{K})^{\mathrm{T}} \\ &+ \mathbf{U}_v(\mathbf{K}) \boldsymbol{\mathcal{V}} \mathbf{U}_v(\mathbf{K})^{\mathrm{T}}) \boldsymbol{\mathcal{R}}\big) - \ell. \end{aligned} \tag{32}$$

Next, we express the terminal constraint $h(x(N)) = \mathbf{0}$ in terms of the decision variables $(\bar{\boldsymbol{u}}, \mathbf{K})$. To this aim, we observe that in view of (25), equation (19a) becomes

$$\begin{aligned} h(x(N)) = \mathbb{E}\big[\mathbf{E}_{N+1}\big(&\boldsymbol{\mathcal{G}}_{x_0} x_0 + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} + \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) e_0 \\ &+ \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{w} + \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{v}\big)\big] =: \tilde{h}(\bar{\boldsymbol{u}}), \end{aligned} \tag{33}$$

where $\mathbf{E}_{N+1} := [\mathbf{0},\ \mathbf{0}, \dots, \mathbf{0},\ \mathbf{I}]$. In view of (2a)–(2b) and the fact that $\mathbb{E}[x_0] = \mu_0$ and $\mathbb{E}[e_0] = \mathbf{0}$, it follows that

$$\tilde{h}(\bar{\boldsymbol{u}}) = \mathbf{E}_{N+1}(\boldsymbol{\mathcal{G}}_{x_0} \mu_0 + \boldsymbol{\mathcal{G}}_{\bar{\boldsymbol{u}}} \bar{\boldsymbol{u}}). \tag{34}$$

From (6) and (34), it follows that the terminal constraint $\tilde{h}(\bar{\boldsymbol{u}}) = \mathbf{0}$ can be written equivalently as follows:

$$\big[\mathbf{A}^{N-1}\mathbf{B} \quad \dots \quad \mathbf{B}\big] \bar{\boldsymbol{u}} = \boldsymbol{\chi}, \quad \boldsymbol{\chi} := -\mathbf{E}_{N+1} \boldsymbol{\mathcal{G}}_{x_0} \mu_0. \tag{35}$$

Proposition 1: Suppose that $N \geq n$. If Assumption 1 holds true, then the linear constraint $\tilde{h}(\bar{\boldsymbol{u}}) = \mathbf{0}$, where $\tilde{h}(\bar{\boldsymbol{u}})$ is defined in (34), will always be feasible.

Proof: If (8) holds true, then the system of (algebraic) linear equations that is given in (35) will always admit a solution. This is because the vector $\boldsymbol{\chi} := -\mathbf{E}_{N+1} \boldsymbol{\mathcal{G}}_{x_0} \boldsymbol{\mu}_0$ will always belong to the range of $\big[\mathbf{B} \quad \dots \quad \mathbf{A}^{n-1}\mathbf{B}\big]$ given that $N \geq n$, by hypothesis. ■

Next, we will express the positive semi-definite constraint, $\mathbf{H}(x(N)) \succeq \mathbf{0}$, in terms of the elements of the decision variables $(\bar{\boldsymbol{u}}, \mathbf{K})$. To this aim, we note that in view of (2a)–(3), (15), (29), equation (19b) gives

$$\begin{aligned} \mathbf{H}(x(N)) = \mathbf{\Sigma}_{\mathsf{f}} - \mathbf{E}_{N+1}\big(&\boldsymbol{\mathcal{G}}_{x_0}(\mathbf{\Sigma}_0 + \mu_0 \mu_0^{\mathrm{T}}) \boldsymbol{\mathcal{G}}_{x_0}^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_{x_0} \mu_0 \bar{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{\bar{u}}^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} \mu_0^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{x_0}^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}} \bar{\boldsymbol{u}}^{\mathrm{T}} \boldsymbol{\mathcal{G}}_{\bar{u}}^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{x_0}^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_{x_0} \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K})^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K})^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{G}}_w(\mathbf{K})^{\mathrm{T}} \\ &+ \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{G}}_v(\mathbf{K})^{\mathrm{T}}\big) \mathbf{E}_{N+1}^{\mathrm{T}} =: \boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K}). \end{aligned} \tag{36}$$

In particular,

$$\begin{aligned} \boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K}) &= \widehat{\mathbf{\Sigma}}_{\mathsf{f}} - \boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}}) \boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}})^{\mathrm{T}} - \boldsymbol{\mathcal{H}}_2(\mathbf{K}) \boldsymbol{\mathcal{H}}_2(\mathbf{K})^{\mathrm{T}} \\ &\quad - \boldsymbol{\mathcal{H}}_3(\mathbf{K}) \\ &= \widehat{\mathbf{\Sigma}}_{\mathsf{f}} - \mathbf{\Lambda}(\bar{\boldsymbol{u}}, \mathbf{K}) \mathbf{\Lambda}(\bar{\boldsymbol{u}}, \mathbf{K})^{\mathrm{T}} - \boldsymbol{\mathcal{H}}_3(\mathbf{K}), \end{aligned} \tag{37}$$

where

$$\begin{aligned} \widehat{\mathbf{\Sigma}}_{\mathsf{f}} &:= \mathbf{\Sigma}_{\mathsf{f}} - \mathbf{E}_{N+1} \boldsymbol{\mathcal{G}}_{x_0} \mathbf{\Sigma}_0 \boldsymbol{\mathcal{G}}_{x_0}^{\mathrm{T}}, \\ \boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}}) &:= \mathbf{E}_{N+1}\big(\boldsymbol{\mathcal{G}}_{x_0} \mu_0 + \boldsymbol{\mathcal{G}}_{\bar{u}} \bar{\boldsymbol{u}}\big), \\ \boldsymbol{\mathcal{H}}_2(\mathbf{K}) &:= \mathbf{E}_{N+1}\big(\boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K})^{\mathrm{T}} \\ &\qquad + \boldsymbol{\mathcal{G}}_w(\mathbf{K}) \boldsymbol{\mathcal{W}} \boldsymbol{\mathcal{G}}_w(\mathbf{K})^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_v(\mathbf{K}) \boldsymbol{\mathcal{V}} \boldsymbol{\mathcal{G}}_v(\mathbf{K})^{\mathrm{T}}\big)^{1/2}, \\ \boldsymbol{\mathcal{H}}_3(\mathbf{K}) &:= \mathbf{E}_{N+1}\big(\boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K}) \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{x_0}^{\mathrm{T}} + \boldsymbol{\mathcal{G}}_{x_0} \tilde{\mathbf{\Sigma}}_0 \boldsymbol{\mathcal{G}}_{e_0}(\mathbf{K})^{\mathrm{T}}\big) \mathbf{E}_{N+1}^{\mathrm{T}}, \end{aligned}$$

and $\mathbf{\Lambda}(\bar{\boldsymbol{u}}, \mathbf{K}) := \big[\boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}}) \quad \boldsymbol{\mathcal{H}}_2(\mathbf{K})\big]$.

An important observation at this point is that $\boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}})$ and $\boldsymbol{\mathcal{H}}_2(\mathbf{K})$ are affine functions of $\bar{\boldsymbol{u}}$ and $\mathbf{K}$, respectively, and consequently, $\mathbf{\Lambda}(\bar{\boldsymbol{u}}, \mathbf{K})$ is an affine (joint) function of $(\bar{\boldsymbol{u}}, \mathbf{K})$.

Proposition 2: Let $\mathbf{\Sigma}_0$, $\mathbf{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$. The constraints $\boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}$ and $\boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}$ are equivalent in the sense that the set $S_{\boldsymbol{\mathcal{H}}} := \{(\bar{\boldsymbol{u}}, \mathbf{K}) \in \mathbb{R}^{Nm} \times \mathbb{R}^{Nm \times Np} :\ \boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}\}$ and the set $S_{\boldsymbol{\mathcal{M}}} := \{(\bar{\boldsymbol{u}}, \mathbf{K}) \in \mathbb{R}^{Nm} \times \mathbb{R}^{Nm \times Np} :\ \boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}\}$, where

$$\begin{aligned} \boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K}) &:= \begin{bmatrix} \mathbf{I} & [\boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}}) \ \ \boldsymbol{\mathcal{H}}_2(\mathbf{K})] \\ [\boldsymbol{\mathcal{H}}_1(\bar{\boldsymbol{u}}) \ \ \boldsymbol{\mathcal{H}}_2(\mathbf{K})]^{\mathrm{T}} & \widehat{\mathbf{\Sigma}}_{\mathsf{f}} - \boldsymbol{\mathcal{H}}_3(\mathbf{K}) \end{bmatrix} \\ &= \begin{bmatrix} \mathbf{I} & \mathbf{\Lambda}(\bar{\boldsymbol{u}}, \mathbf{K}) \\ \mathbf{\Lambda}(\bar{\boldsymbol{u}}, \mathbf{K})^{\mathrm{T}} & \widehat{\mathbf{\Sigma}}_{\mathsf{f}} - \boldsymbol{\mathcal{H}}_3(\mathbf{K}) \end{bmatrix} \end{aligned} \tag{38}$$

are equal. In addition, the positive semi-definite constraint $\boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}$ can be written as an LMI (convex) constraint in terms of the elements of $(\bar{\boldsymbol{u}}, \mathbf{K})$.

Proof: Because the matrix $\boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K})$ is the Schur complement of $\mathbf{I}$ in the matrix $\boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K})$, which is defined in (38), it follows that the constraint $\boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}$ is equivalent to the following constraint $\boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}$. Note that the latter positive semi-definite constraint can be expressed as an LMI constraint in terms of the elements of $(\bar{\boldsymbol{u}}, \mathbf{K})$ [24]. ■

Problem 2: Given $\ell > 0$ and $\boldsymbol{\Sigma}_0$, $\boldsymbol{\Sigma}_{\mathsf{f}} \in \mathbb{S}_n^{++}$, find the matrix $\mathbf{K}^\star \in \mathbb{R}^{Nm \times Np}$ that minimizes $\mathcal{J}(\mathbf{K})$ subject to $\mathcal{C}(\mathbf{K}) \leq 0$, $\tilde{h}(\bar{\boldsymbol{u}}) = \mathbf{0}$, and $\mathbf{H}(\bar{\boldsymbol{u}}, \mathbf{K}) \succeq \mathbf{0}$ where $\mathcal{J}(\mathbf{K})$, $\mathcal{C}(\mathbf{K})$, $\tilde{\tilde{h}}(\bar{\boldsymbol{u}})$, and $\boldsymbol{\mathcal{M}}(\bar{\boldsymbol{u}}, \mathbf{K})$ are defined in (28), (32), (34) and (38), respectively.

Proposition 3: Under the assumption that the set of control policies is restricted to the subset $\widehat{\mathcal{P}}$ of $\mathcal{P}$, Problem 1 and Problem 2 are equivalent in the sense that if $\pi^\circ \in \widehat{\mathcal{P}}$ solves Problem 1, then $(\bar{\boldsymbol{u}}^\circ, \mathbf{K}^\circ) = \varpi(\pi^\circ)$ solves Problem 2, and vice versa.

Proof: The proof follows readily after noting that $J(\pi) = \mathcal{J}(\mathbf{K})$, $C(\pi) = \mathcal{C}(\mathbf{K})$, $\tilde{h}(\bar{\boldsymbol{u}}) = h(x(N))$, and $\mathbf{H}(x(N)) = \boldsymbol{\mathcal{H}}(\bar{\boldsymbol{u}}, \mathbf{K})$ provided that $\pi = \varpi(\bar{\boldsymbol{u}}, \mathbf{K})$ together with Proposition 2. ■

Remark 3 An important observation at this point is that with the proposed reduction of the stochastic optimal control problem (Problem 1) to a convex program (Problem 2), we can decrease the dimension of the latter convex program by truncating the history of output residuals that the control laws at each stage will depend to. Note, on the other hand that the longer the history of the output residuals, the better performance can be achieved, especially in the presence of stringent constraints. Therefore, one should choose the length of this history in such a way that strikes a balance between performance and computational tractability.

IV. Conclusion

In this work, we have proposed a systematic approach for the reduction of a stochastic optimal control problem with partial state information subject to covariance assignment and input energy constraints into a tractable convex program. In contrast with our previous work on similar problems, in this work we have proposed separation-based control policies which are sequences of feedback control laws that are affine mappings of either the complete history of output estimation errors, which are computed with the aid of a discrete-time Kalman filter algorithm, or a truncation of the latter history. In our approach, the size of the resulting convex problem depends on the length of the truncated history of the output estimation errors; something that allows us to design control policies that strike a balance between good performance and computational scalability. In our future work, we will consider the nonlinear stochastic optimal control problem with state covariance assignment constraints.

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