Mechanics of Solids

The study of the mechanics of solids is concerned with determining the stresses (loads/forces) and strains (displacements/deformation) in an object or structure. Such information can then be used to determine if and/or how the component will fail.

Generally there are 3 components to the solution of mechanics of solids problems. (*Note that this course is also called mechanics of materials or strength of materials.)

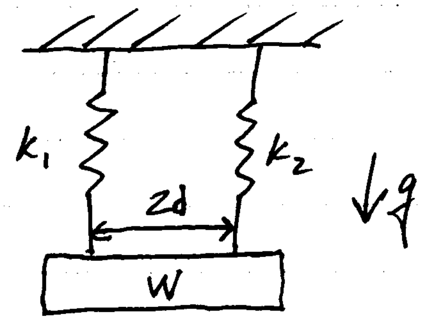
1) Equilibrium - You all have taken Statics, so you know what equilibrium is all about. Equilibrium tells us how forces are distributed on and in an object. In this course we will also introduce the concept of stress, which is closely related to force. Considerations of equilibrium will allow us to understand how stresses are distributed in an object. As in statics, equilibrium analyses are always aided by correct free-body diagrams.

   In short, equilibrium tells us that to be in equilibrium an object, and in fact each part of the object, must have no net force and no net moment acting on it.

2) <u>Kinematics</u> - Kinematics relates the displacements of points within an object to the deformation of and strain within the object. These relationships are dictated by simple geometry. There are mechanics problems, specifically statically determinate problems, where considerations of kinematics are not necessary for the solution.

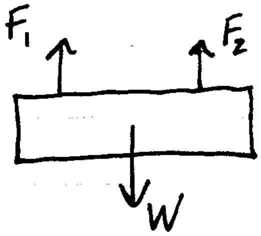
3) <u>Material Behavior / Constitutive behavior</u>
The constitutive behavior of the material is the ultimate link between forces/stresses, which must obey the equilibrium equations, and the displacements/strains that are governed by kinematics. The simplest equation for constitutive behavior that you have likely encountered is that for a linear spring, $F = k\delta$. Here, the linear spring constant k tells us the relationship between the force carried by the spring and its deflection. Again, in statically determinate problems this information is not always necessary. Another aspect of the material behavior is concerned with the failure of the material. Once we know the stresses within an object, a failure criterion will tell us if the material can support that stress without failing.

Example:



** Assume that the deflections of the springs are similar such that the geometric configuration is essentially as shown. What are the forces in the springs?

FBD:

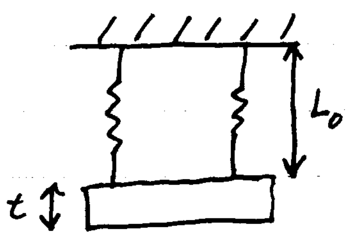


Equilibrium: $\sum F_y = F_1 + F_2 - W = 0$

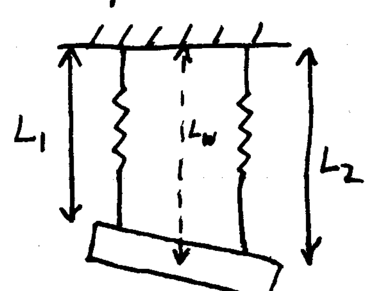
$$\sum M_z^w = F_2 d - F_1 d = 0 \rightarrow F_1 = F_2 = \frac{W}{2}$$

That's it. You see this was a statically determinate. We did not need to use the spring constants (constitutive law) or the spring deflections (kinematics).

But what if we want to know how far the weight descends and its slope with respect to the ground?



Prior to descent



After descent

** If the deflections of the springs are similar then the slope of the weight is small and the distance between the springs remains approximately at 2d.

Constitutive behavior $\rightarrow F_1 = k_1(L_1 - L_0) \rightarrow L_1 = L_0 + \frac{W}{2k_1}$

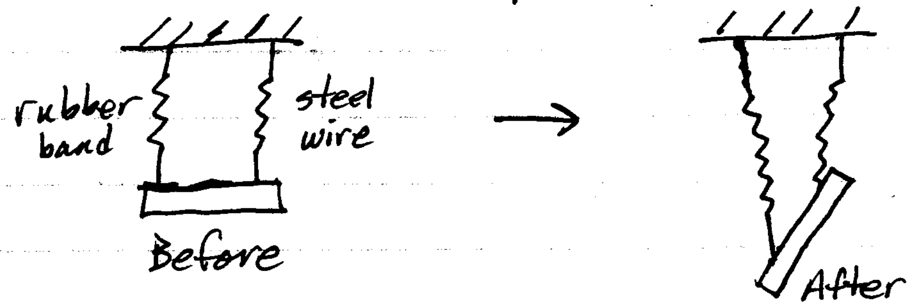
$$F_2 = k_2(L_2 - L_0) \rightarrow L_2 = L_0 + \frac{W}{2k_2}$$

if L_1 & L_2 are similar then:

$$L_W = \frac{L_1 + L_2}{2} + \frac{t}{2} = \underbrace{L_0 + \frac{t}{2}}_{\text{prior to descent}} + \frac{k_1 + k_2}{4k_1k_2} W$$

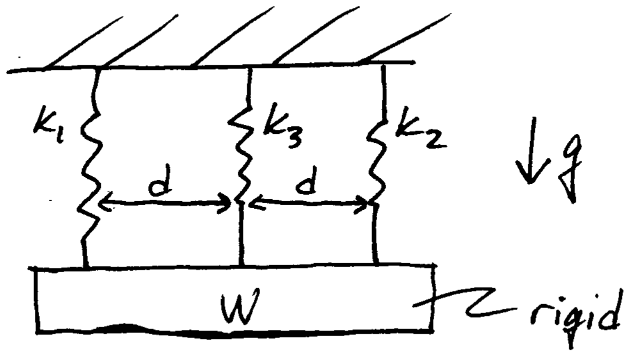
$$\text{slope} = \frac{L_1 - L_2}{2d} = \frac{k_2 - k_1}{4k_1k_2}\frac{W}{d}$$

** Note that if the slope is large then the simplifying assumption that we used for the kinematics would not be valid and a more thorough (and difficult) analysis would be required.

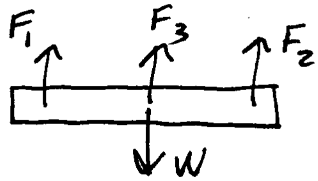


Such a large change in the geometry due to deformation would also affect the equilibrium equations due to the directional changes in the forces and the changes in lengths of the moment arms. Luckily, most structural materials are very stiff and so usually deflections and changes in slopes are very small.

This was a rather simple statically determinate problem. How would you handle this?



FBD :



$$\Sigma F_y = F_1 + F_2 + F_3 - W = 0$$

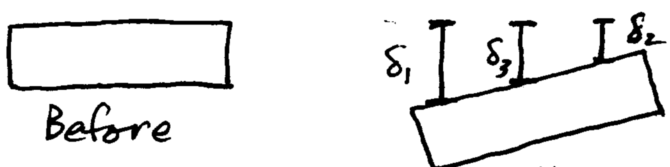
(3 forces, 2 Equilibrium equations)

$$\Sigma M_z^W = F_2 d - F_1 d = 0$$

$$\rightarrow F_1 = F_2 \quad (A)$$

$$\rightarrow F_3 = W - 2F_1 \quad (B)$$

That is as far as we can go with equilibrium. Now what? Well, since the weight is rigid, and if we assume small slope changes (which we used in our equilibrium analysis) then we can construct a geometric relationship between the deflections of the springs.



Before

After

$$\delta_3 = \frac{1}{2}(\delta_1 + \delta_2) \quad (C)$$

(3 deflections, 1 kinematic constraint) → 6 unknowns & 3 equations

Our last 3 equations come from the constitutive response of the springs.

$$\overset{D}{F_1 = k_1\delta_1}\ , \quad \overset{E}{F_2 = k_2\delta_2}\ , \quad \overset{F}{F_3 = k_3\delta_3}$$

D & E in A $\rightarrow$ $k_1\delta_1 = k_2\delta_2$

F, B and D $\rightarrow$ $W - 2k_1\delta_1 = k_3\delta_3 \overset{C}{=} \frac{1}{2}k_3(\delta_1+\delta_2)$

$$\rightarrow W - 2k_1\delta_1 = \frac{1}{2}k_3\left(\delta_1 + \frac{k_1}{k_2}\delta_1\right)$$

$$W = \delta_1\left[\frac{1}{2}k_3\left(1+\frac{k_1}{k_2}\right) + 2k_1\right]$$

$$\delta_1 = \frac{W}{k_1\left[2 + \frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)\right]} \rightarrow \boxed{F_1 = \frac{W}{2+\frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)}}$$

$$\delta_2 = \frac{k_1}{k_2}\delta_1 = \frac{W}{k_2\left[2+\frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)\right]} \rightarrow \boxed{F_2 = \frac{W}{2+\frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)}} \checkmark$$

$$\delta_3 = \frac{1}{2}(\delta_1+\delta_2) = \frac{W(k_1+k_2)}{2k_1k_2\left[2+\frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)\right]} = \frac{W}{4\left(\frac{k_1k_2}{k_1+k_2}\right)+k_3}$$

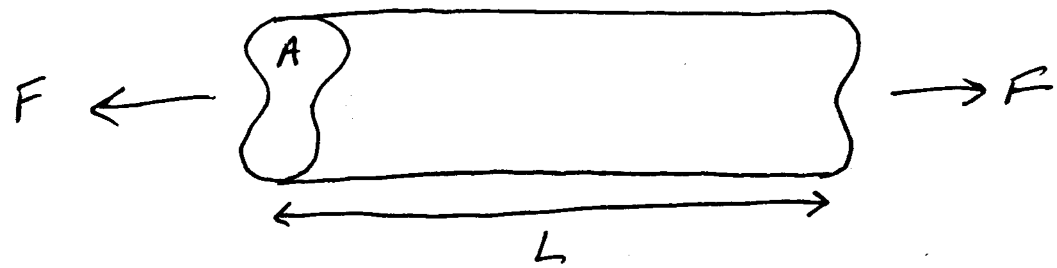
$$\rightarrow F_3 = \frac{Wk_3}{k_3 + 4\left(\frac{k_1k_2}{k_1+k_2}\right)} = \boxed{\frac{\frac{1}{2}W\left(\frac{k_1+k_2}{k_1k_2}\right)k_3}{\frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)+2} = F_3}$$

Check : $F_1+F_2+F_3 = \dfrac{2W + \frac{1}{2}W\left(\frac{k_1+k_2}{k_1k_2}\right)k_3}{2+\frac{1}{2}k_3\left(\frac{k_1+k_2}{k_1k_2}\right)} = W \checkmark$

Check $k_3=0 \rightarrow F_1 = F_2 = \frac{W}{2}$ $\checkmark$ (same as previous problem)

In our previous example we analyzed the forces and deformations of a set of springs. In that analysis we utilized the spring constant k to describe the constitutive behavior of the spring. k is a structural property of the spring relating the force carried by the spring to its deflection. Every structure is a combination of its geometry and its material. The structural response will depend upon both of these features.

The first structure that we will analyze is a bar loaded by an axial force.



A = cross-sectional area, the bar is said to be "prismatic" because this area is extruded along the axial direction without rotation.

L = length of the bar

A and L are clearly the geometric properties of this structure.

Now, we should have some intuition that the response of the bar will depend on what material it is made from. We expect that a steel bar will be stiffer than a bar made of rubber. Keep this in mind as we introduce the stiffness of the bar.

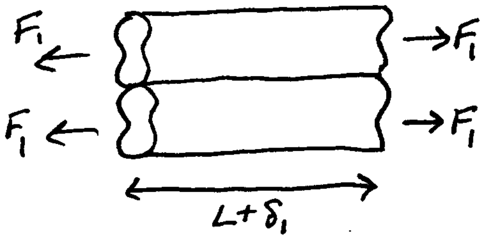
$$F = k\delta$$

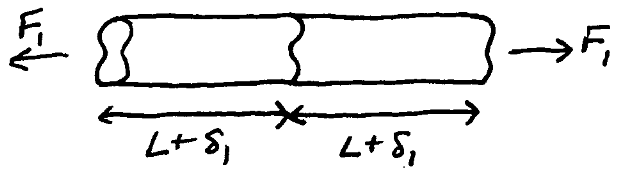
axial force — stiffness — deflection (i.e. change in length)

Now let's consider how k depends on the geometry and material of our structure.

First, take a single bar and stretch it by δ_1. Then, the force in this bar is $F_1 = k_1\delta_1$. Now take 2 of these bars side by side and stretch them by δ_1. We expect the sum of the forces on the bars to be $2F_1$. So if we consider the two bars together as a single structure we have $F_2 = 2F_1 = 2k_1\delta_1 = k_2\delta_1$. So the stiffness of the two bars side by side is twice that of a single bar. What changed? The material is the same, the length is the same, but the area has doubled. So it is a small logical step to claim k is proportional to A.

Next, let's consider the effects of L. Instead of side by side, let's attach the bars end to end.





Side by side $\rightarrow k_2 = 2k_1$ End to end $\rightarrow k_2 = \frac{1}{2}k_1$

For the end to end configuration the same force F_1 is transmitted through each bar, and each bar deflects by δ_1. So the total deflection is $2\delta_1$.

$$F_2 = F_1$$
$$k_2\delta_2 = k_1\delta_1$$
$$k_2(2\delta_1) = k_1\delta_1 \rightarrow k_2 = \frac{1}{2}k_1$$

So, in this case the only feature that has changed is the length of the bar. In this case we conclude that k is inversely proportional to L.

Finally, we have already argued that there must exist some intrinsic material stiffness. Let's call this material property E and for now let's assume k is proportional to E.

This all gives us : $k = \frac{EA}{L}$ for a bar

$$F = \frac{EA}{L}\delta$$

So, to get a better understanding of E let's rearrange this equation.

$$F = \frac{EA}{L}\delta \longrightarrow \underbrace{\left(\frac{F}{A}\right)}_{\sigma} = E\underbrace{\left(\frac{\delta}{L}\right)}_{\varepsilon}$$

$\sigma \equiv$ axial stress in the bar. It is equal to the axial force carried by the bar divided by the cross-sectional area A. Note that the force F is perpendicular/normal to A, and so σ is also called a normal stress.

$\varepsilon \equiv$ axial strain in the bar. It is equal to the deflection of the bar divided by its initial length. For reasons we will not discuss now this is also called engineering strain.

Note that both structures drawn at the top of page 9 have the same stress and the same strain. The material property E that relates stress to strain is called Young's modulus or the elastic modulus. The "law" $\sigma = E\varepsilon$ is called Hooke's law after Robert Hooke (a contemporary of Isaac Newton).

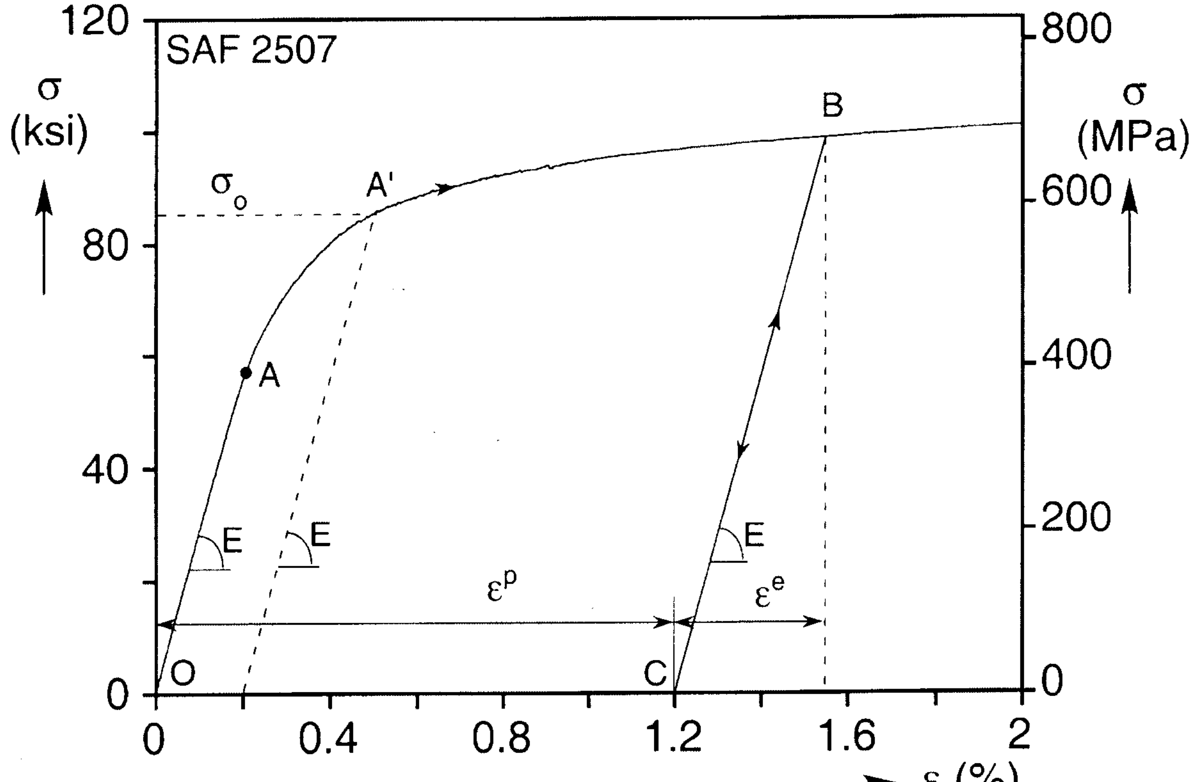
The question we have yet to answer is, do materials obey Hooke's law?

You might imagine that since it has been elevated to a "law" and named that the answer is yes. In fact this is the case for very, very many structural materials of interest. There are usually special circumstances (which we will not discuss in detail here) when Hooke's law does not apply over some small range of stress in a material. The reason that I have placed "law" in quotes is because Hooke's law should not be thought of as a fundamental law of nature like Newton's Laws. It is a phenomenological relationship between stress and strain that is at least approximately true over some range of stress.

Page 12 contains two plots of the same $\sigma - \varepsilon$ curve on different strain scales for a stainless steel SAF 2507. From this curve we will identify the following features.

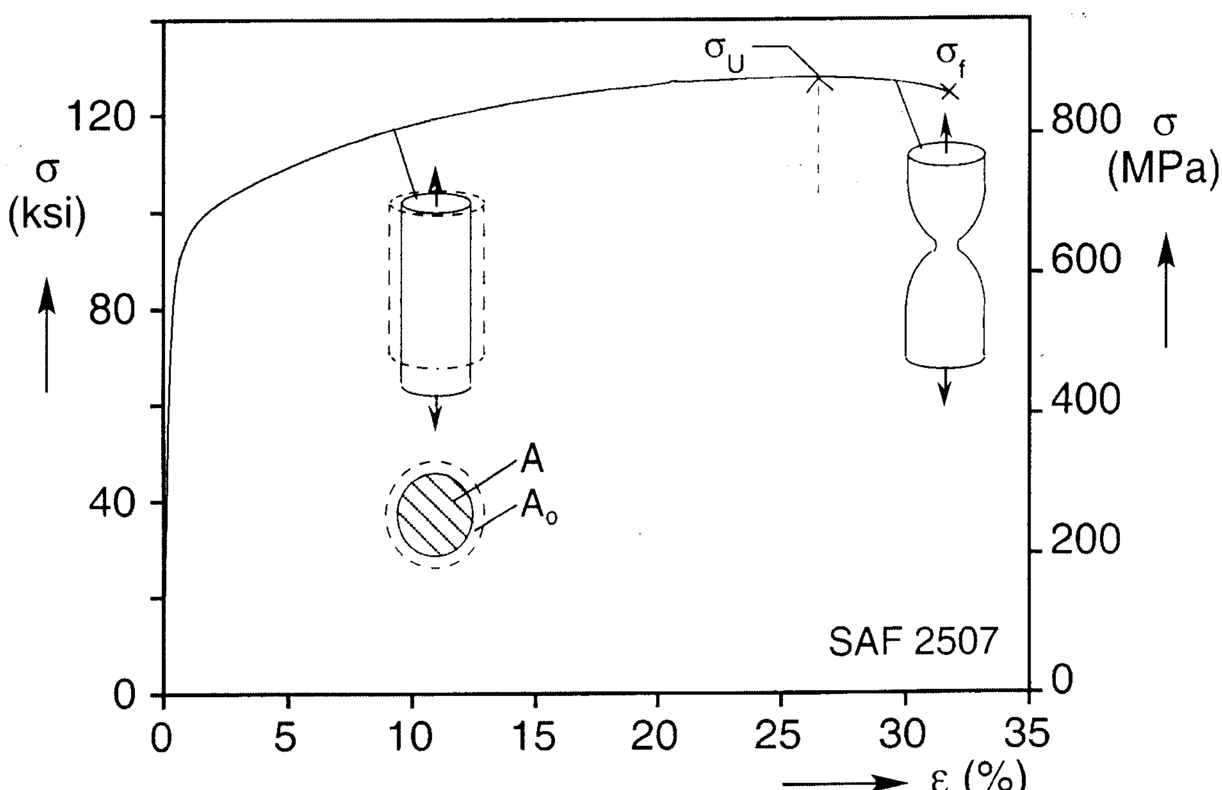
Young's modulus — E
Proportional limit
0.2% offset yield strength
Plastic strain
Elastic strain
Strain hardening
Ultimate strength
Reduction in area
Necking

12



SAF 2507
σ (ksi)
σ (MPa)
120
80
40
0
800
600
400
200
0
σ_o
A'
B
A
E
E
E
ε^p
ε^e
O
C
0
0.4
0.8
1.2
1.6
2
ε (%)

(a)



σ_U
σ_f
σ (ksi)
σ (MPa)
120
80
40
0
800
600
400
200
0
A
A_o
SAF 2507
0
5
10
15
20
25
30
35
ε (%)

(b)

- **Elastic modulus / Young's modulus : E** $\rightarrow$ $\sigma = E\varepsilon$

Notice that there is an initial linear part of the stress-strain curve where it appears that the relationship $\sigma = E\varepsilon$ is valid. Let's find E. Point A is approximately at $\sigma = 60$ ksi and $\varepsilon = 0.2\%$. First, 1 ksi = 1000 psi = 1000 lb/in^2 (psi = pounds per square inch). Second, it is common to quote strain as a percentage, as in a percent change in length of the bar. So $\varepsilon = 0.002 = 0.2\%$. Another common terminology is the use of microstrain ($\mu\varepsilon$). Micro = 10^{-6} so $\varepsilon = 0.002$ is the same as 2000 $\mu\varepsilon$.

$$E = \frac{\sigma}{\varepsilon} = \frac{60{,}000 \text{ psi}}{0.002} = 30{,}000{,}000 \text{ psi} = 30 \text{ Msi}$$

valid only in the linear region — mega pounds per square inch

The conversion ~~factor~~ from ksi to MPa is 7. (actually 6.89)

1 ksi = 7 MPa or 1000 psi = 7×10^6 Pa

1 psi = 7000 Pa (6890 Pa)

So $E = 210$ GPa in SI units.

gigapascals

Other materials

Aluminum	–	10 Msi	70 GPa
Copper	–	18 Msi	120 GPa
Wood	–	1.8 Msi	12 GPa
Diamond	–	180 Msi	1200 GPa

Proportional limit - The maximum stress to which the equation $\sigma = E\varepsilon$ remains valid. This is a very ~~..~~ inexact quantity since it depends upon how closely you zoom in on the elastic region. In figure (a) it appears to be about 60 ksi, but on (b) it looks to be about 80 ksi.

x% Offset Yield Strength - This is a more precise number than the proportional limit but perhaps less useful. To determine this quantity construct a straight line passing through the point $\varepsilon = x\%$ $\sigma = 0$ that is parallel to the line $\sigma = E\varepsilon$ (i.e. with slope $\frac{d\sigma}{d\varepsilon} = E$). The intersection of this line with the $\sigma - \varepsilon$ curve is the x% yield strength. On Figure (a) the 0.2% yield strength is about 85 ksi. Yield strength has significantly more variability in a given material than Young's modulus. Young's modulus depends primarily on the types of atoms in the material. For example, pure iron and steel (iron plus a little carbon) both have $E \approx 30$ Msi. However the yield strength depends on carbon content, other alloying elements, temperature history, and deformation history. Yield strength can be a factor of 3-5 different in ~~..~~ materials with the exact same chemistry.

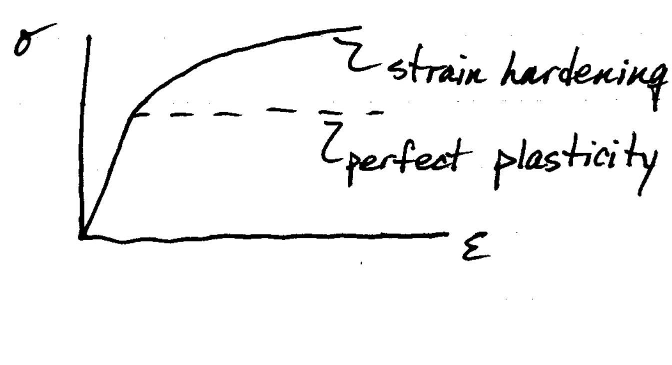
<u>Plastic strain</u> – Beyond the proportional limit the stress-strain response is nonlinear and metals begin to accumulate plastic strain and permanent deformation, i.e. when unloaded the material does not return to zero strain. During monotonic loading the plastic strain can be determined from the equation:

$$\sigma = E\underbrace{(\varepsilon - \varepsilon^{p})}_{= \varepsilon^{e} \equiv \text{ the elastic strain}}$$

Notice that when metals unload, they unload along a straight line with slope $\frac{d\sigma}{d\varepsilon} = E$.

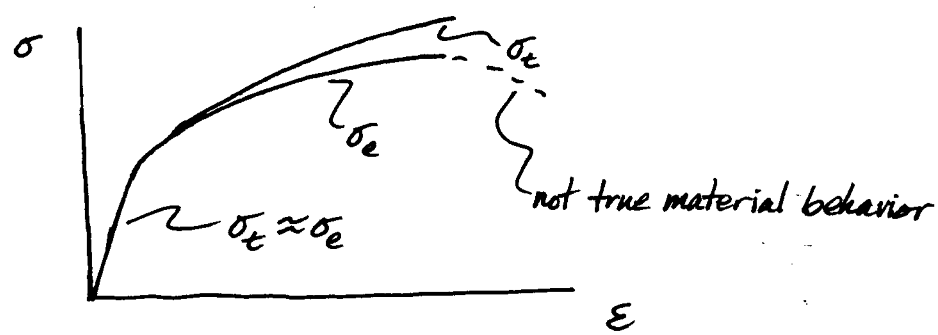
<u>Elastic strain</u> – $\varepsilon^{e} = \varepsilon - \varepsilon^{p}$

<u>Strain hardening</u> – After reaching the proportional limit the stress continues to increase with increasing strain. This is called strain hardening, which is in contrast to idealized "perfect plasticity" where the stress remains constant with continued deformation.

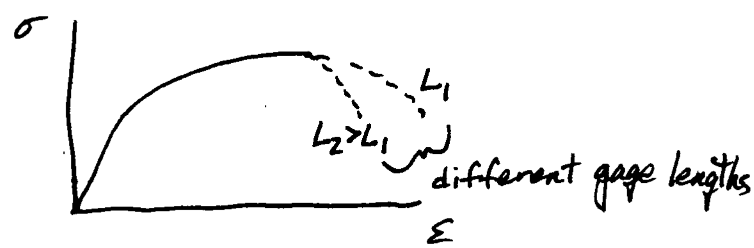


<u>Ultimate strength</u> - The peak stress on the σ-ε curve.

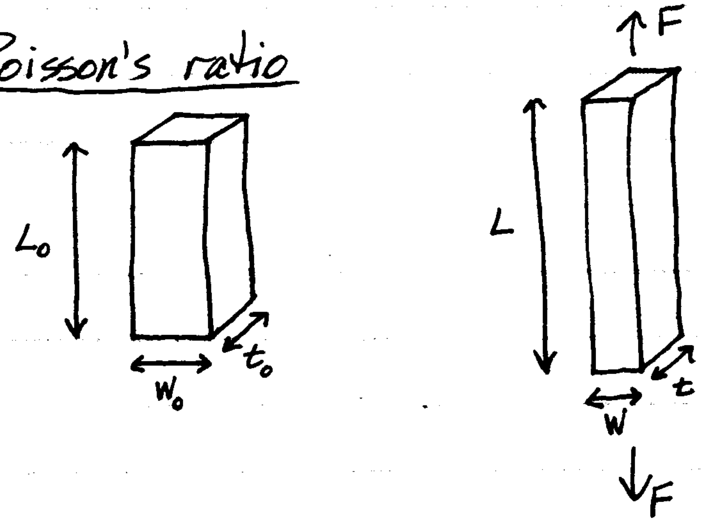
<u>Reduction in area</u> - As a material is stretched the lateral dimensions (of most materials) of the specimen decrease. This effect is more pronounced after significant plastic straining and leads to different stress definitions. Engineering stress $= \frac{F}{A_o}$ and true stress $= \frac{F}{A}$.



<u>Necking</u> - Up to the ultimate strength the deformation of a tensile bar remains uniform. After this point the deformation localizes into a small region called a neck. During the deformation of the neck the material outside of the neck unloads. The $\sigma = \frac{F}{A_o}$ versus $\varepsilon = \frac{\delta}{L_o}$ behavior after the ultimate strength depends on the geometric properties of the bar such as its length and shape of cross-section. This means that the part of the curve after σ_{ult} is not true material behavior.



Poisson's ratio



$$\sigma = \frac{F}{A_o} = \frac{F}{W_o t_o} \quad , \quad \varepsilon = \frac{\delta}{L_o} = \frac{L - L_o}{L_o}$$

$$\sigma = E\varepsilon \rightarrow \frac{F}{A_o} = E\frac{\delta}{L_o} \rightarrow F = \underbrace{\frac{EA_o}{L_o}}_{K}\delta$$

For an isotropic, homogeneous material the two lateral or transverse strains will be equal. This is only true because the loading is uniaxial.

For uniaxial loading $\varepsilon_t = \frac{W - W_o}{W_o} = \frac{t - t_o}{t_o}$

The ratio of $-\varepsilon_t$ to $\varepsilon_a = \varepsilon$ is the Poisson's ratio.

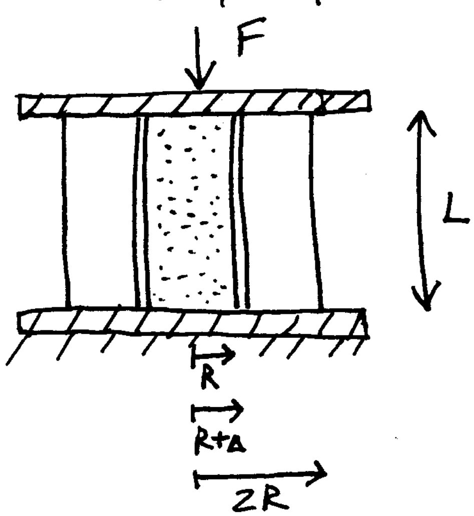
$$\boxed{\nu = \frac{-\varepsilon_t}{\varepsilon_a}}$$

$\varepsilon_a = \frac{1}{E}\sigma_a$ axial strain and stress

$$\boxed{\varepsilon_t = -\nu\varepsilon_a = -\frac{\nu}{E}\sigma_a}$$

Steel	$\nu = 0.27 - 0.3$
Aluminum	$\nu = 0.33$
Concrete	$\nu = 0.1 - 0.2$
Glass	$\nu = 0.17 - 0.27$
Polyethylene	$\nu = 0.4$
Rubber	$\nu = 0.45 - 0.5$

__Example__ : A polyethylene [redacted] cylinder of radius R and length L is placed inside an aluminum tube with inner radius $R+\Delta$ and outer radius $2R$ ($\Delta \ll R$). The assembly is placed between two rigid plates and compressed. Determine the load on the plates when the polyethylene touches the aluminum.



Determine the lateral deflections of each component.

Polyethylene : $\varepsilon_t^{Poly} = -\nu_p \varepsilon_a$

$$\frac{\delta_p}{R} = -\nu_p \varepsilon_a \rightarrow \delta_p = -\nu_p \varepsilon_a R$$

Aluminum: $\varepsilon_t^{Al} = -\nu_A \varepsilon_a$

$$\frac{\delta_A}{R+\Delta} = -\nu_A \varepsilon_a \rightarrow \delta_A = -\nu_A \varepsilon_a (R+\Delta)$$

If the cylinder is centered within the tube then the two will touch when $\delta_p - \delta_A = \Delta$

$$\rightarrow -\nu_p R \varepsilon_a + \nu_A (R+\Delta) \varepsilon_a = \Delta$$

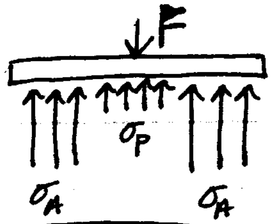
$$\varepsilon_a = \frac{\Delta}{\nu_A(R+\Delta) - \nu_p R} \approx \frac{1}{\nu_A - \nu_p}\frac{\Delta}{R}, \quad \Delta \ll R$$

$$\rightarrow \sigma_A = E_A \varepsilon_a = \frac{E_A}{\nu_A - \nu_p}\frac{\Delta}{R}$$

$$F_A = \sigma_A A_A = \frac{E_A}{\nu_A - \nu_p}\frac{\Delta}{R}\underbrace{3\pi R^2}_{\text{again using } \Delta \ll R} = \frac{3\pi E_A R \Delta}{\nu_A - \nu_p}$$

$$\sigma_p = E_p \varepsilon_a = \frac{E_p}{\nu_A - \nu_p}\frac{\Delta}{R}$$

$$F_p = \sigma_p A_p = \frac{E_p}{\nu_A - \nu_p}\frac{\Delta}{R}\pi R^2 = \frac{\pi E_p R \Delta}{\nu_A - \nu_p}$$

Finally from 

$\rightarrow F = F_A + F_p$

$$\boxed{F = -\frac{(3E_A + E_p)\pi R \Delta}{\nu_A - \nu_p}}$$

** Sign change is introduced b/c I drew the σ's in compression. This is a bad practice.