EM 306 - Statics

This course will focus on the concepts of force and moment (a.k.a. torque), their mathematical description as vectors, and their application to the analysis of the equilibrium of statically determinate structures.

Statics is the first semester course in a series on engineering mechanics which also includes dynamics and strength of materials. The material covered in statics builds on concepts introduced in your introductory physics class on mechanics.

Mechanics is concerned with both the study of motion (kinematics) and the study of how forces affect motion (kinetics). The study of motion, or kinematics, is concerned with both rigid body motions as described by displacements, velocities, and accelerations and with deformations as described by strain. You will study velocity and acceleration in dynamics and strain in strength of materials.

In statics the structures we will be analyzing will be at rest, i.e. static, and so there is no need to consider velocity or acceleration since these quantities will be zero. Also, we will be analyzing statically determinate structures, so the analysis of deformation will not be necessary.

In statics, when there are no accelerations, kinetics simplifies to the study of equilibrium.

All of these issues are best understood and analyzed against the back-drop of Newton's Laws.

<u>Newton's Laws of Motion</u>

I) A body at rest will remain at rest unless acted upon by a force. Similarly, a body in motion will remain in motion with a constant velocity (speed and direction) unless acted upon by a force.

II) A body acted upon by a force will accelerate in the direction of the force. The magnitude of the acceleration is proportional to the force and inversely proportional to the mass of the body.

III) For every action there is an equal and opposite reaction. In other words, the force that body A imparts on body B is equal in magnitude but opposite in direction to the force that body B imparts on body A.

In the study of statics we will be concerned with (I) and (III). The mathematical statements associated with (I) are

$$\Sigma \vec{F} = 0 \quad \text{and} \quad \Sigma \underbrace{\vec{r} \times \vec{F}}_{\vec{M}} = 0$$

We will get to these details soon, but first we must cover some fundamentals.

<u>Units and Dimensions</u>

The dimensions you will encounter in the study of mechanics include mass, force, length, and time.

SI units : mass $\Rightarrow$ kg , length $\Rightarrow$ m, time $\Rightarrow$ s
Force is then a derived unit from Newton's second law.

$$\vec{F} = m\vec{a} \rightarrow F \Rightarrow kg\frac{m}{s^2} \equiv N \text{ (newton)}$$

US units : force $\Rightarrow$ lb , length $\Rightarrow$ ft , time $\Rightarrow$ s
Mass is then a derived unit from Newton's 2nd

$$\vec{F} = m\vec{a} \rightarrow \text{lb} = M\frac{\text{ft}}{\text{s}^2} \rightarrow M \Rightarrow \frac{\text{lb s}^2}{\text{ft}} \equiv \text{slug}$$

Calculus expressions : Differentiation is like division (but be careful of how you interpret higher order derivatives. Integration is like multiplication.

$$a_x = \frac{d^2x}{dt^2} = \frac{d}{dt}\left(\frac{dx}{dt}\right) \Rightarrow \frac{1}{T}\left(\frac{L}{T}\right) \Rightarrow \frac{L}{T^2}$$

$$m = \int_V \rho\, dV \Rightarrow M \Rightarrow \rho L^3 \rightarrow \rho \Rightarrow \frac{M}{L^3}$$

The most important consideration about dimensions and units is that they <u>must</u> be consistent throughout an equation. It makes absolutely no sense to try to add a length to a mass or to equate a force to a time.

<u>Example</u>) $KE = \frac{1}{2}mv^2 + \frac{1}{2}mk^2\omega^2$

$\omega \Rightarrow$ radians/s , Determine the dimensions for KE and k.

$$KE \Rightarrow \frac{1}{2}mv^2 \Rightarrow M\frac{L^2}{T^2} \Rightarrow \underbrace{\left(\frac{ML}{T^2}\right)}_{F}L \Rightarrow FL$$

$$\frac{1}{2}mk^2\omega^2 \Rightarrow \frac{ML^2}{T^2} \Rightarrow Mk^2\frac{1}{T^2} \rightarrow k \Rightarrow L$$

Vector Algebra

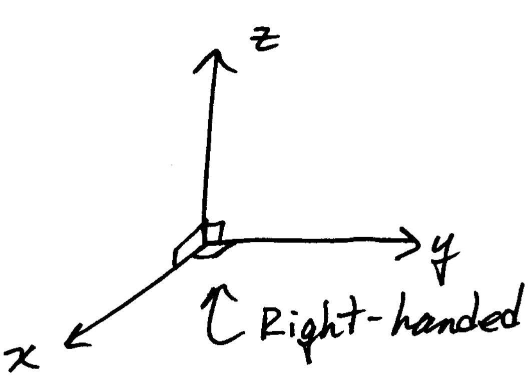
Scalars: A scalar quantity has magnitude only.
e.g. temperature, voltage, time, mass, charge

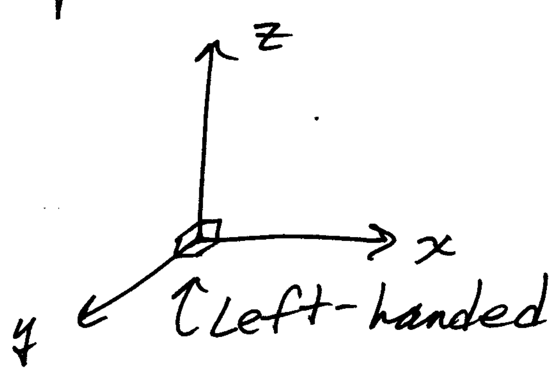
Vectors: A vector quantity has both magnitude and direction. e.g. velocity, acceleration, force, moment, electric field

In order to specify a vector both its magnitude <u>and</u> direction must be given.

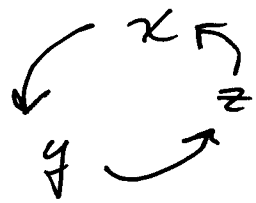
Graphically we can draw a directed line segment with the length proportional to the magnitude of the quantity being depicted.

In a more quantitative manner we can give the <u>components</u> of a vector in 3 mutally exclusive directions. Furthermore, the simplest way to choose these 3 directions is to make them perpendicular to one another, i.e. orthogonal. Finally, we will also always choose these 3 directions to be right-handed.

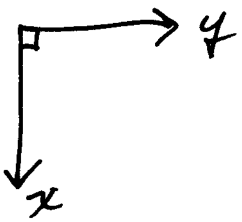




Using your right hand, make a hitch-hiking thumb with your fingers curling from x to y, then your thumb is in the z-direction. Curl your fingers from y to z and your thumb will be in the x-direction. Finally, curl from z to x and your thumb should be in the y-direction.



So if I choose x and y as follows

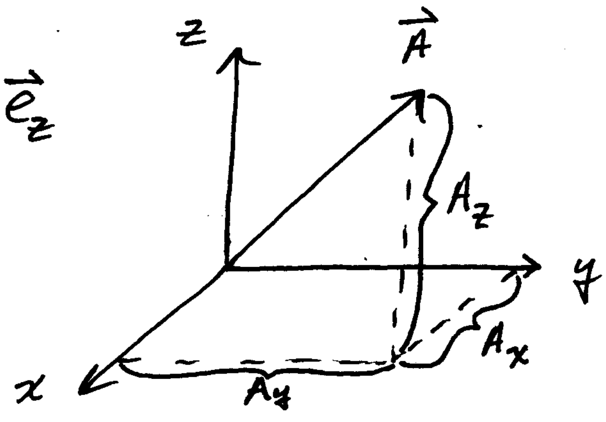


then the z-direction will be out of the page for a right-handed system.

- Back to the specification of a vector:

Let $\vec{e}_x$, $\vec{e}_y$ and $\vec{e}_z$ be unit vectors in the x, y, and z directions. By unit vector we mean that its magnitude is 1. Then any vector $\vec{A}$ can be written as:

$$\vec{A} = A_x \vec{e}_x + A_y \vec{e}_y + A_z \vec{e}_z$$



A_x, A_y, and A_z are the components of the vector $\vec{A}$ in the x, y and z directions.

It is also common to use the notation $\vec{e}_x = \vec{i}$, $\vec{e}_y = \vec{j}$, $\vec{e}_z = \vec{k}$ such that $\vec{A}$ is written as

$$\vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k}$$

$\vec{A}$ can also be written as $\vec{A} = A \vec{e}_A$ where A is the magnitude of $\vec{A}$ and $\vec{e}_A$ is a unit vector in the same direction as $\vec{A}$.

If we know the scalar components of $\vec{A}$, how can we determine its magnitude A?

Before we answer this, let's look at scalar-vector multiplication and dot products.

<u>Scalar-Vector Multiplication</u>

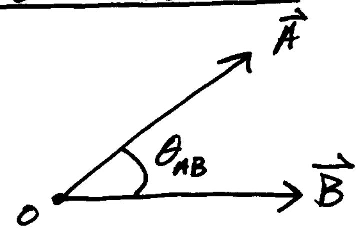
$$\vec{B} = c\vec{A} \quad , \quad \vec{A} = A_x \vec{i} + A_y \vec{j} + A_z \vec{k} = A\vec{e}_A$$

then

$$\vec{B} = \underbrace{cA_x}_{B_x} \vec{i} + \underbrace{cA_y}_{B_y} \vec{j} + \underbrace{cA_z}_{B_z} \vec{k} = \underbrace{cA}_{B} \vec{e}_A$$

So, scalar-vector multiplication produces a new vector in the same direction as the original vector with magnitude scaled by the scalar factor.

Dot Product (Scalar Product)



Two vectors $\vec{A}$ and $\vec{B}$ can always be used to define a plane (except in the special case when they are parallel). Just as 3 points can be used to define a plane (except when they are co-linear). Let one of these points lie at the tails of the vectors, i.e. point O, another at the tip of $\vec{A}$, and the third at the tip of $\vec{B}$. Then, the angle <u>in this plane</u> between $\vec{A}$ and $\vec{B}$ will be called θ_{AB}. The dot product is then defined as:

$$\vec{A} \cdot \vec{B} = AB\cos\theta_{AB}$$

where A is the magnitude of $\vec{A}$ and B is the magnitude of $\vec{B}$.

Note that the dot product is commutative, i.e.

$$\vec{B} \cdot \vec{A} = BA\cos\theta_{AB} = AB\cos\theta_{AB} = \vec{A} \cdot \vec{B}$$

Let's now analyze the dot product assuming that we know the components of $\vec{A}$ and $\vec{B}$.

$$\vec{A}\cdot\vec{B} = (A_x\vec{i} + A_y\vec{j} + A_z\vec{k})\cdot(B_x\vec{i} + B_y\vec{j} + B_z\vec{k})$$

$$= A_xB_x\,\vec{i}\cdot\vec{i} + A_xB_y\,\vec{i}\cdot\vec{j} + A_xB_z\,\vec{i}\cdot\vec{k}$$
$$+ A_yB_x\,\vec{j}\cdot\vec{i} + A_yB_y\,\vec{j}\cdot\vec{j} + A_yB_z\,\vec{j}\cdot\vec{k}$$
$$+ A_zB_x\,\vec{k}\cdot\vec{i} + A_zB_y\,\vec{k}\cdot\vec{j} + A_zB_z\,\vec{k}\cdot\vec{k}$$

but $\vec{i}\cdot\vec{i} = 1$ because $\theta = 0$ and $|\vec{i}| = 1$
similarly $\vec{j}\cdot\vec{j} = 1$ and $\vec{k}\cdot\vec{k} = 1$

also $\vec{i}\cdot\vec{j} = 0$ because $\theta = 90°$
similarly $\vec{i}\cdot\vec{k} = 0$, $\vec{j}\cdot\vec{k} = 0$, $\vec{j}\cdot\vec{i} = 0$, $\vec{k}\cdot\vec{i} = 0$, $\vec{k}\cdot\vec{j} = 0$

$$\therefore \boxed{\vec{A}\cdot\vec{B} = A_xB_x + A_yB_y + A_zB_z}$$

Now we can use the dot product to find the magnitude of a vector if we know its components.

Recall: $\vec{A} = A_x\vec{i} + A_y\vec{j} + A_z\vec{k} = A\,\vec{e}_A$

$$\vec{A}\cdot\vec{A} = A_x^2 + A_y^2 + A_z^2 = A^2\,\underbrace{\vec{e}_A\cdot\vec{e}_A}_{1}$$

$$\therefore \boxed{A = \sqrt{A_x^2 + A_y^2 + A_z^2}}$$

When we <u>normalize</u> a vector we are simply finding a unit vector in the direction of the original vector. The previous result can be applied for this purpose if we know the components.

$$A\vec{e}_A = A_x\vec{i} + A_y\vec{j} + A_z\vec{k}$$

$$\rightarrow \vec{e}_A = \frac{A_x\vec{i} + A_y\vec{j} + A_z\vec{k}}{A} = \frac{A_x\vec{i} + A_y\vec{j} + A_z\vec{k}}{\sqrt{A_x^2 + A_y^2 + A_z^2}}$$

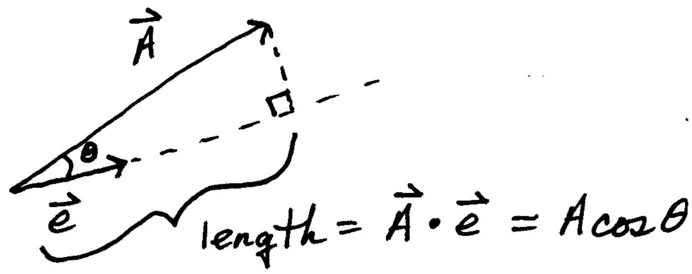
The dot product is also useful for determining the direction cosines of a vector, i.e. the angle that the vector makes with a given coordinate direction.

For example: $\vec{A}\cdot\vec{i} = A\cdot 1\cdot\cos\theta_x = A_x$

$$\rightarrow \cos\theta_x = A_x/A$$

Similarly
$$\cos\theta_y = A_y/A$$
$$\cos\theta_z = A_z/A$$

* Finally, the physical interpretation of taking the dot product of a general vector $\vec{A}$ with a unit vector $\vec{e}$ is that $\vec{A}\cdot\vec{e}$ gives the amount of $\vec{A}$ in the $\vec{e}$ direction.



Cross Product (Vector Product)

Define $\vec{A}$, $\vec{B}$ and θ_{AB} as on page 8. Then define the vector $\vec{C} = \vec{A} \times \vec{B}$ such that the vector $\vec{C}$ is perpendicular to both $\vec{A}$ and $\vec{B}$ (and therefore perpendicular to the plane containing $\vec{A}$ & $\vec{B}$), and the magnitude of $\vec{C}$ is

$$C = AB \sin \theta_{AB}$$

or $\vec{C} = AB \sin \theta_{AB} \, \vec{e}_{\perp}$ where $\vec{e}_{\perp}$ is a unit vector perpendicular to $\vec{A}$ and $\vec{B}$ and its positive sense will be determined using the right-hand rule by sweeping $\vec{A}$ into $\vec{B}$.

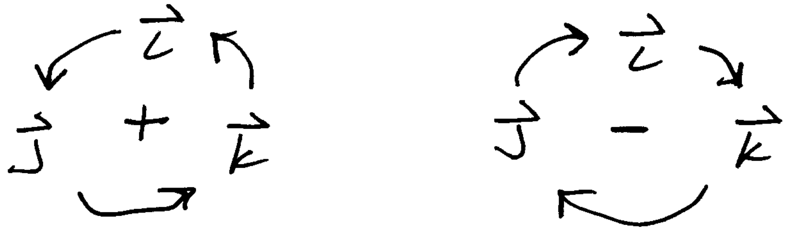
Determining the components of $\vec{C}$

$$\vec{C} = \vec{A} \times \vec{B} = (A_x \vec{i} + A_y \vec{j} + A_z \vec{k}) \times (B_x \vec{i} + B_y \vec{j} + B_z \vec{k})$$

$$\begin{aligned} &= A_x B_x \, \vec{i} \times \vec{i} + A_x B_y \, \vec{i} \times \vec{j} + A_x B_z \, \vec{i} \times \vec{k} \\ &+ A_y B_x \, \vec{j} \times \vec{i} + A_y B_y \, \vec{j} \times \vec{j} + A_y B_z \, \vec{j} \times \vec{k} \\ &+ A_z B_x \, \vec{k} \times \vec{i} + A_z B_y \, \vec{k} \times \vec{j} + A_z B_z \, \vec{k} \times \vec{k} \end{aligned}$$

but note that $\vec{i} \times \vec{i} = \vec{j} \times \vec{j} = \vec{k} \times \vec{k} = 0$
because $\theta = 0$ and $\sin \theta = 0$

$$\vec{i} \times \vec{j} = \vec{k} \qquad \vec{j} \times \vec{k} = \vec{i} \qquad \vec{k} \times \vec{i} = \vec{j}$$
$$\vec{j} \times \vec{i} = -\vec{k} \qquad \vec{k} \times \vec{j} = -\vec{i} \qquad \vec{i} \times \vec{k} = -\vec{j}$$



then arranging terms we get

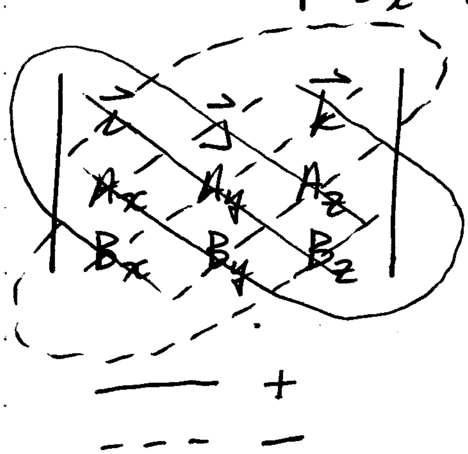
$$\boxed{\vec{C} = (A_y B_z - A_z B_y)\vec{i} + (A_z B_x - A_x B_z)\vec{j} + (A_x B_y - A_y B_x)\vec{k}}$$

Note that if $\vec{D} = \vec{B} \times \vec{A}$ then $\vec{D} = -\vec{C}$, i.e. the cross product is <u>not</u> commutative.

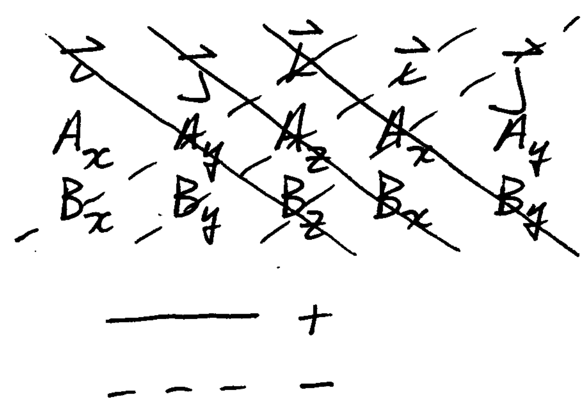
<u>Matrix Method for cross product</u>

Determinant

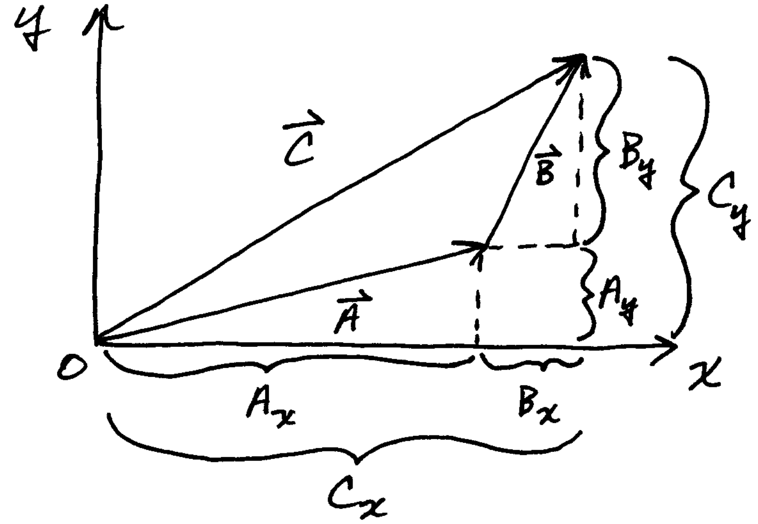
$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_y B_z - A_z B_y)\vec{i} + (A_z B_x - A_x B_z)\vec{j} + (A_x B_y - A_y B_x)\vec{k}$$



or



Vector Addition



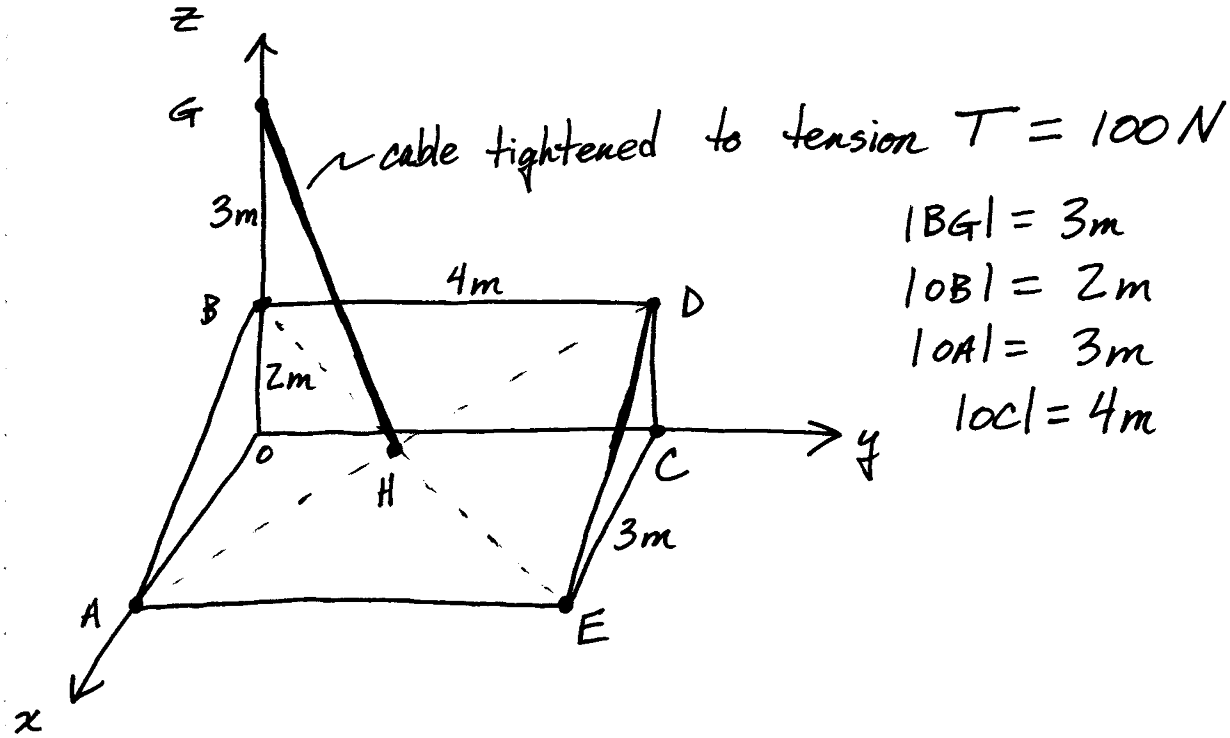
$$\vec{C} = \vec{A} + \vec{B} = \underbrace{(A_x + B_x)}_{C_x}\vec{i} + \underbrace{(A_y + B_y)}_{C_y}\vec{j} + \underbrace{(A_z + B_z)}_{C_z}\vec{k}$$

We have already used this principle when we grouped $\vec{i}$, $\vec{j}$ and $\vec{k}$ components in the cross product.

Note that 1 vector equation like the one above constitutes $\underline{\underline{3}}$ scalar equations, one in $\vec{i}$, one in $\vec{j}$, and one in $\vec{k}$.

$$C_x = A_x + B_x$$
$$C_y = A_y + B_y$$
$$C_z = A_z + B_z$$

Application of cross and dot products to problems in mechanics:



Determine the part of the tension in the cable that is perpendicular to the inclined plane and the part parallel to the plane.

First we recognize that the force due to the cable on the plane can be represented as a vector with magnitude $T = 100\,N$ and in the direction from H towards G.

$$\vec{T} = T\,\vec{e}_{HG}$$

Our first task is to determine $\vec{e}_{HG}$

To do this we will construct the vector from H to G and then normalize it.

$$\vec{HG} = -1.5\vec{i} - 2\vec{j} + 4\vec{k}$$

(Note: $\vec{OG} = \vec{OH} + \vec{HG} \rightarrow \vec{HG} = \vec{OG} - \vec{OH}$ where $\vec{OG} = 5\vec{k}$ and $\vec{OH} = 1.5\vec{i} + 2\vec{j} + 1\vec{k}$)

Then : $$\vec{e}_{HG} = \frac{-1.5\vec{i} - 2\vec{j} + 4\vec{k}}{\sqrt{1.5^2 + 2^2 + 4^2}} = \underbrace{-0.318}_{-0.318}\vec{i} - 0.424\vec{j} + \underbrace{0.848}_{0.848}\vec{k}$$

So $$\vec{T} = 100\left(-0.318\vec{i} - 0.424\vec{j} + 0.848\vec{k}\right) N$$

Next, the question asks us to decompose the tension into two parts, i.e.

$$\vec{T} = \underbrace{\vec{T}_{\perp}}_{\text{perpendicular to the plane}} + \underbrace{\vec{T}_{\parallel}}_{\text{parallel to the plane}}$$

If we can determine either $\vec{T}_{\perp}$ or $\vec{T}_{\parallel}$ then the other can be found by simple vector subtraction.

Here we must recognize that there are infinitely many directions parallel to the plane and it will be practically impossible for us to choose the one that allows for the decomposition ~~[illegible]~~ above.

However, there are only two (one opposite to the other) directions perpendicular to the plane.

So we will write $\vec{T}_\perp = T_\perp \vec{e}_\perp$

First let's find $\vec{e}_\perp$ by finding any vector normal to the plane and then normalizing it.

$$\vec{e}_\perp = \frac{\vec{AB} \times \vec{AE}}{|\vec{AB} \times \vec{AE}|} = \frac{(-3\vec{i}+2\vec{k})\times(4\vec{j})}{|\vec{AB}\times\vec{AE}|} = \frac{-12\vec{i}\times\vec{j} + 8\vec{k}\times\vec{j}}{|\vec{AB}\times\vec{AE}|}$$

$$= \frac{-12\vec{k} - 8\vec{i}}{\sqrt{144+64}} = -0.5547\vec{i} - 0.8321\vec{k}$$

Now $T_\perp$ is the amount of $\vec{T}$ in the $\vec{e}_\perp$ direction $\Rightarrow$ $T_\perp = \vec{T}\cdot\vec{e}_\perp$

$$\rightarrow T_\perp = 100(-0.318)(-0.5547) + 100(0.848)(-0.8321)$$
$$= -52.9$$

Notice the $-$ sign. This is because we chose $\vec{e}_\perp$ in the wrong direction. This does not matter because the signs work themselves out.

$$\rightarrow \vec{T}_\perp = -52.9\left(-0.5547\vec{i} - 0.8321\vec{k}\right)$$
$$= 29.3\vec{i} + 44.0\vec{k}$$

$$\rightarrow \vec{T}_\parallel = \vec{T} - \vec{T}_\perp = -61.1\vec{i} - 42.4\vec{j} + 40.8\vec{k}$$

Force

Force is a vector quantity. A force is specified by its magnitude and the direction in which it acts. Mathematically we can represent it as

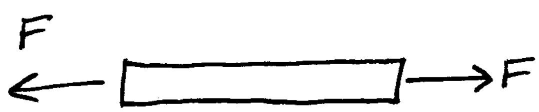
$$\vec{F} = F\vec{e}_F$$

or in component form

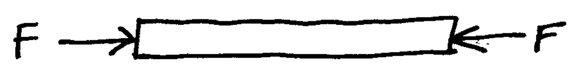
$$\vec{F} = F_x\vec{i} + F_y\vec{j} + F_z\vec{k}$$

Notice that $\vec{F} = F\underbrace{\left(\frac{F_x}{F}\vec{i} + \frac{F_y}{F}\vec{j} + \frac{F_z}{F}\vec{k}\right)}_{\text{So this must be } \vec{e}_F.}$

For deformable bodies the point of application of a force is important.



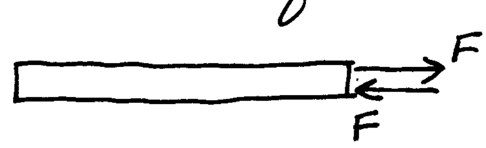
tension → stretching



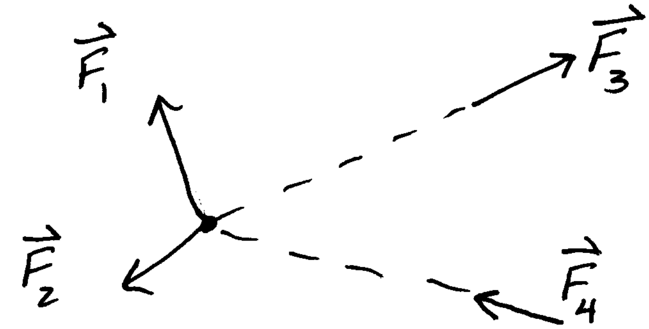
compression → contraction

However, if we are only concerned with the effects of a force on a rigid body, then we can move a force along its line of action without changing the net external effects on the rigid body.

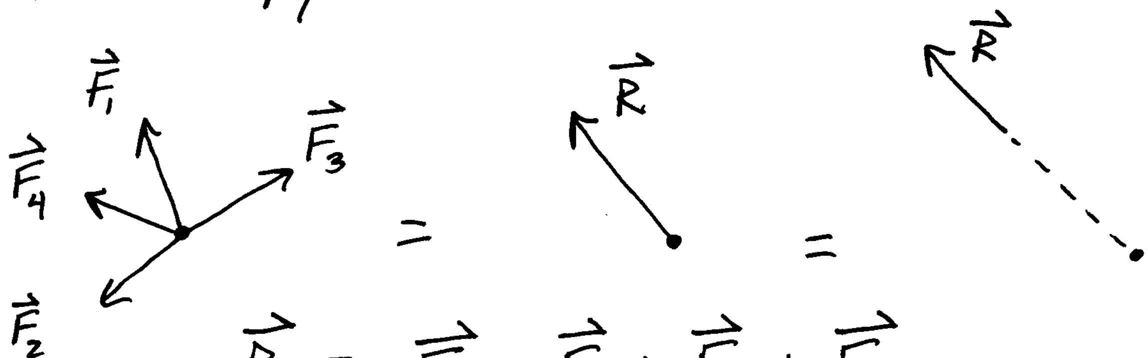
i.e. both of the bars above are in equilibrium. We could also do this



<u>Concurrent Forces</u> – Two or more forces are concurrent if their lines of action intersect at a common point.



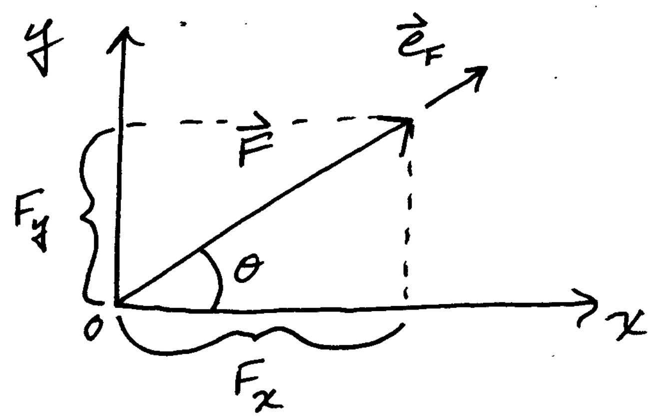
The resultant of a set of concurrent forces is then simply the sum of the set.



$$\vec{R} = \vec{F}_1 + \vec{F}_2 + \vec{F}_3 + \vec{F}_4$$

or generally $\quad \vec{R} = \sum_{i=1}^{N} \vec{F}_i$

<u>Two-Dimensional Force Systems</u>



$$\vec{F} = F\,\vec{e}_F$$

$$\vec{F} = F_x\,\vec{i} + F_y\,\vec{j}$$

$$F = \sqrt{F_x^2 + F_y^2}$$

$F_x = F\cos\theta$ (θ is the angle between $\vec{F}$ and the x-axis)

$F_y = F\sin\theta$ (Note: $\sin\theta = \cos(90° - \theta)$ and $90° - \theta$ is the angle between $\vec{F}$ and the y-axis)

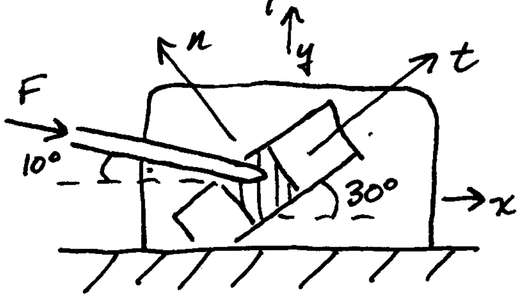
Finally $\dfrac{F\sin\theta}{F\cos\theta} = F_y/F_x$

$$\rightarrow \frac{\sin\theta}{\cos\theta} = \tan\theta = F_y/F_x$$

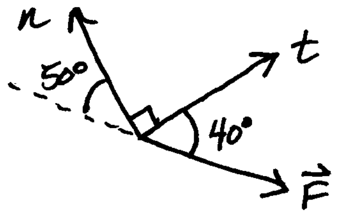
$$\theta = \arctan(F_y/F_x)$$

Note that the choice of the orientation of the coordinate system is arbitrary and the representation of a force in different coordinate systems will have different components. The best way to illustrate this is by example.

Ex)



The t-component of the force $\vec{F}$ is known to be 75 N. Determine the n-component, the magnitude, and the x-y components of $\vec{F}$.



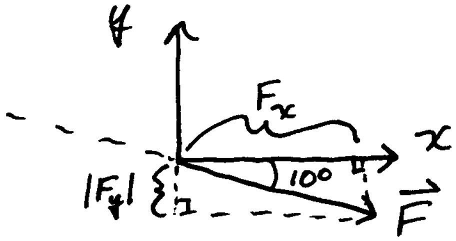
$$\vec{F} = \underbrace{F\cos 40°}_{F_t}\,\vec{e}_t \underbrace{- F\cos 50°}_{F_n}\,\vec{e}_n$$

but we are given that $F_t = 75\,N$

$$\rightarrow \quad F\cos 40° = 75\,N$$
$$F = 75/\cos 40° \; N$$
$$F = 97.9\,N$$

Then $F_n = -F\cos 50° = -97.9\cos 50° = -62.9\,N$

$$\rightarrow \quad \vec{F} = (75\,\vec{e}_t - 62.9\,\vec{e}_n)\,N$$



$$\vec{F} = F\cos 10°\,\vec{i} - F\sin 10°\,\vec{j}$$

$$\vec{F} = 96.4\,\vec{i} - 17.0\,\vec{j}$$

Note: $F = \sqrt{96.4^2 + 17^2} = 97.9$ ✓