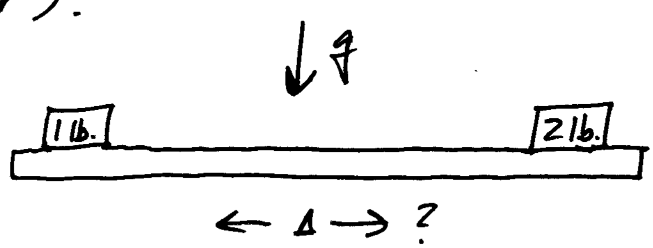
Moments

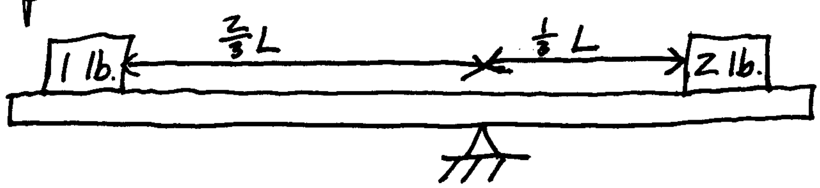
In your physics class the physical quantity of <u>moment</u> of a force was most likely called a <u>torque</u>. They are the same thing.

When we talk about a moment we need to specify a force, the point of application of that force, <u>and</u> the point which we are considering the moment of the force about.

Consider the following lever system (a teeter-totter).



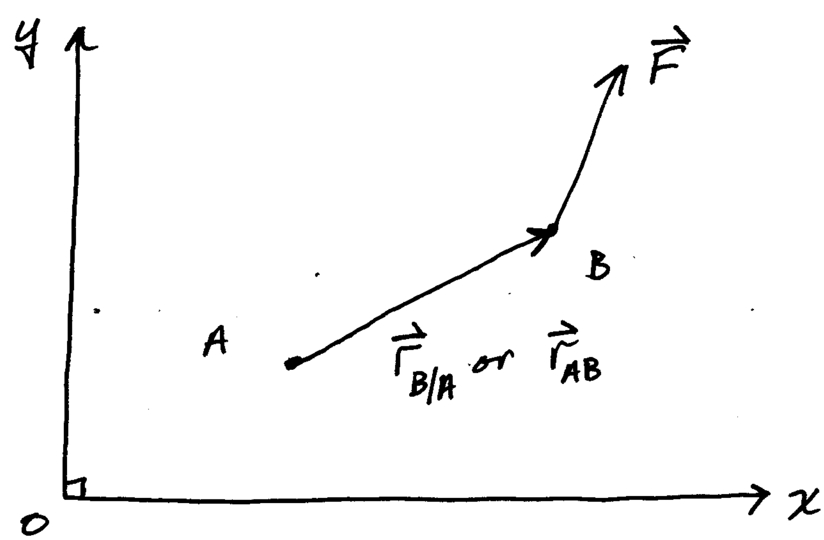
Where do we place the fulcrum for equilibrium? For those who have seen the balance of moments/torques before, we know that the fulcrum has to be placed closer to the 2 lb. weight such that the ratio of the distances to the fulcrum is equal to the ratio of the weights.



The moment of a force about a point gives the propensity of the force to cause a rotation about the point. The dimensions of the moment of a force are force multiplied by distance.

Looking at the lever, the 2 lb. weight tends to rotate the bar clockwise, and its moment about the fulcrum is 2 lb. $\times \frac{1}{3}L$. The 1 lb. weight tends to rotate the bar counterclockwise, and its moment about the fulcrum is 1 lb. $\times \frac{2}{3}L$. So, ultimately, the moments of these two forces about the fulcrum are equal in magnitude but tend to cause rotation in the opposite directions.

Let's consider a more general 2-D scenario.

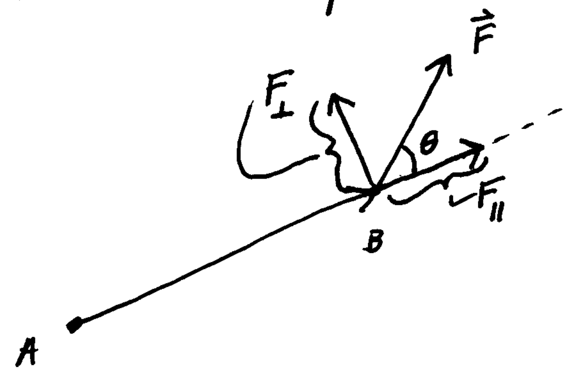


$\vec{r}_{B/A}$ = "position of B with respect to A"
$\vec{r}_{AB}$ = "position vector from A to B"

The question is, what is the moment of the force $\vec{F}$, which is applied at point B, about point A.

In general there is no need to have any structure connecting points A and B, but sometimes considering a bar (or a right-angled frame) connecting the two points is useful.

Let's break up the force $\vec{F}$ into components perpendicular and parallel to $\vec{r}_{AB}$.



Does the $F_{\parallel}$ component tend to rotate a bar connecting A and B? No, this component of the force tends to stretch $\vec{r}_{AB}$.

Then, the perpendicular component does tend to rotate such a bar and the magnitude of the moment is

$$M_A = r_{AB} \cdot F_\perp$$

Next, notice that $F_\perp = F \sin\theta$, so

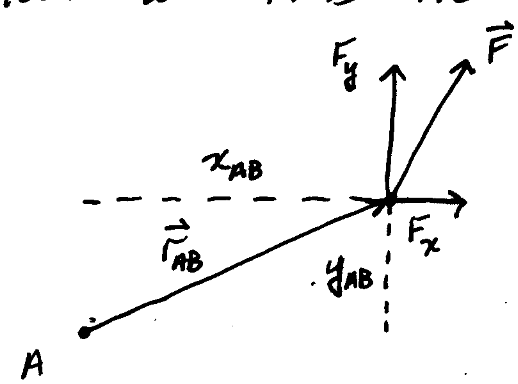
$$M_A = r_{AB} F \sin\theta$$

Finally, the direction of M is counterclockwise and if we use the right-hand rule this direction is perpendicular to both $\vec{r}_{AB}$ and $\vec{F}$ and is out of the page. So in vector form we have

$$\vec{M}_A = r_{AB} F \sin\theta \, \vec{e}_\perp$$

This looks like the cross product $\vec{M} = \vec{r}_{AB} \times \vec{F}$, but we need to analyze this in 3-D before we formally make this definition.

Let's look at this in components.



$$\vec{F} = F_x \vec{i} + F_y \vec{j}$$

$$\vec{r}_{AB} = x_{AB} \vec{i} + y_{AB} \vec{j}$$

For the sign of the moment, using the right hand rule, we will call the direction out of the page the positive z-direction.

First consider the moment about point A due to the x-component of $\vec{F}$.

F_x tends to "stretch" the x_{AB} component of the vector $\vec{r}_{AB}$ but "rotates" the y_{AB} component in the clockwise $(-z)$ direction.

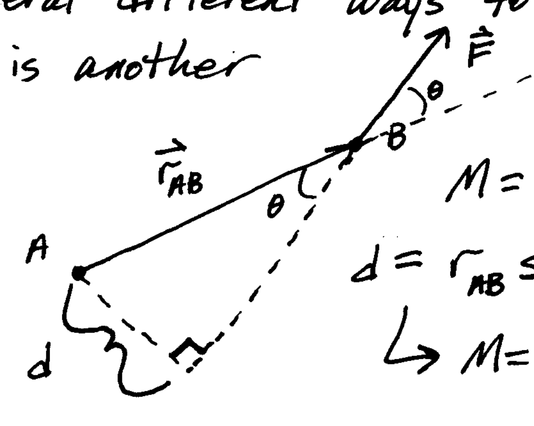
Similarly, F_y "stretches" y_{AB} and rotates x_{AB} counterclockwise $(+z)$.

The total moment due to $\vec{F}$ is the sum of these two contributions.

$$M = -F_x\, y_{AB} + F_y\, x_{AB}$$

Terminology: y_{AB} is called the "moment arm" of the F_x component of the force about point A.
x_{AB} is then the moment arm of the F_y component.

There can be several different ways to visualize moments in 2-D. Here is another

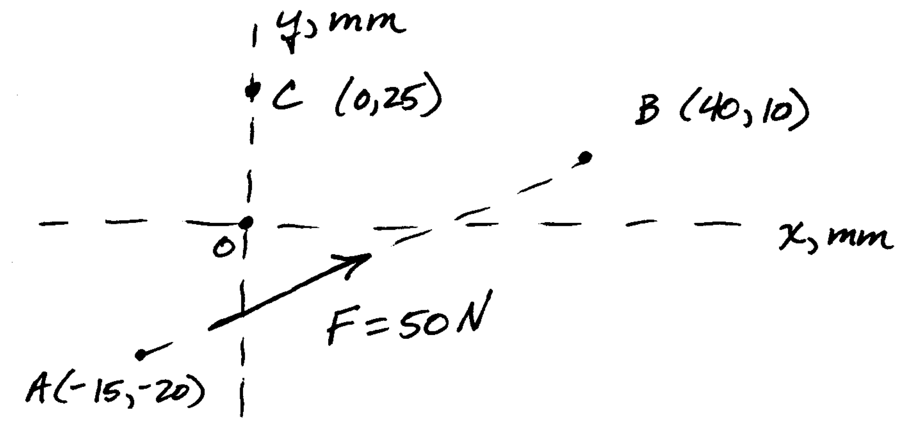


$$M = Fd$$

$$d = r_{AB} \sin\theta$$

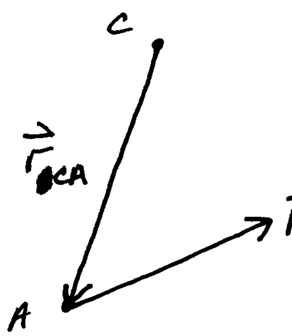
$$\hookrightarrow M = F r_{AB} \sin\theta$$

Example:



Determine the moment of the 50 N force (a) about the point C as if it acts at A and (b) as if it acts at B.

a)

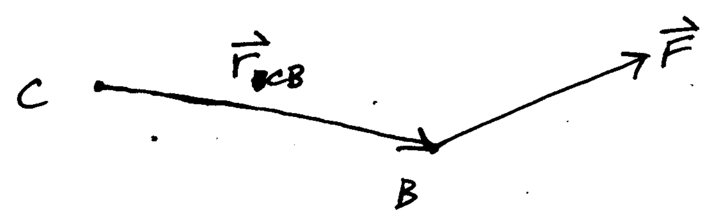


$$\vec{F} = 50 \frac{55\vec{i} + 30\vec{j}}{\sqrt{55^2+30^2}} = 43.9\vec{i} + 23.9\vec{j} \text{ N}$$

$$\vec{r}_{CA} = -15\vec{i} - 45\vec{j} \text{ mm}$$

$$M_z^C = 43.9(45) - 23.9(15) = 1617 \text{ N}\cdot\text{mm}$$

b)



$$\vec{r}_{CB} = 40\vec{i} - 15\vec{j}$$

$$M_z^C = 23.9(40) + 43.9(15) = 1615 \text{ N}\cdot\text{mm}$$

Difference is due to truncation error.

We will ultimately show that in general vector form, the moment due to a force applied at point B about point A is

$$\vec{M}^{(A)} = \vec{r}_{AB} \times \vec{F}^{(B)} = \vec{r}_{B/A} \times \vec{F}^{(B)}$$

We can used this to prove the equality of our previous results.

(a) $\vec{M}^{(C)} = \vec{r}_{CA} \times \vec{F}^{(A)}$

(b) $\vec{M}^{(C)} = \vec{r}_{CB} \times \vec{F}^{(B)}$

Now recognize that the vector form of both $\vec{F}^{(A)}$ and $\vec{F}^{(B)}$ are the same, and let's just call this $\vec{F}$, i.e.

$$\vec{F}^{(A)} = \vec{F}^{(B)} = \vec{F}$$

Next, let's write $\vec{r}_{CB} = \vec{r}_{CA} + \vec{r}_{AB}$

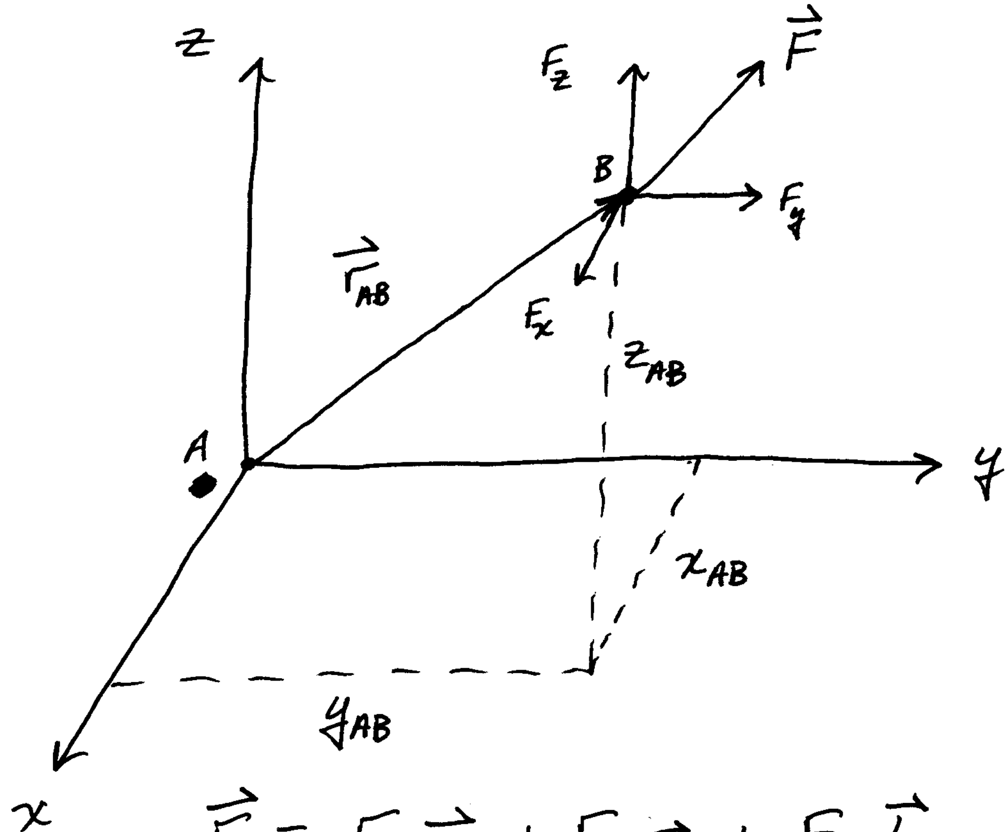
$$(b) \rightarrow \vec{M}^{(C)} = (\vec{r}_{CA} + \vec{r}_{AB}) \times \vec{F}$$

$$= \vec{r}_{CA} \times \vec{F} + \vec{r}_{AB} \times \vec{F}$$

but $\vec{F} = F \dfrac{\vec{r}_{AB}}{r_{AB}}$ and $\vec{r}_{AB} \times \vec{F} = \dfrac{F}{r_{AB}} \underbrace{(\vec{r}_{AB} \times \vec{r}_{AB})}_{=0}$

$\rightarrow \vec{M}^{(C)} = \vec{r}_{CA} \times \vec{F}$ = result from (a)

Moments in 3-D (translate origin to A)



$$\vec{F} = F_x \vec{i} + F_y \vec{j} + F_z \vec{k}$$

$$\vec{r}_{AB} = x_{AB} \vec{i} + y_{AB} \vec{j} + z_{AB} \vec{k}$$

The F_x component of the force rotates the z_{AB} component of the moment arm in the positive y-direction, and the y_{AB} component of the moment arm in the negative z-direction. This component of the force only tends to "stretch" x_{AB}.

$$\rightarrow F_x : \; F_x z_{AB} \vec{j} - F_x y_{AB} \vec{k}$$

Similarly, $F_y: \quad F_y\, x_{AB}\, \vec{k} - F_y\, z_{AB}\, \vec{i}$

$$F_z: \quad F_z\, y_{AB}\, \vec{i} - F_z\, x_{AB}\, \vec{j}$$

The sum of all of these contributions is the total moment of the force $\vec{F}$ about point A.

$$\vec{M}^{(A)} = (F_z\, y_{AB} - F_y\, z_{AB})\,\vec{i} + (F_x\, z_{AB} - F_z\, x_{AB})\,\vec{j} + (F_y\, x_{AB} - F_x\, y_{AB})\,\vec{k}$$

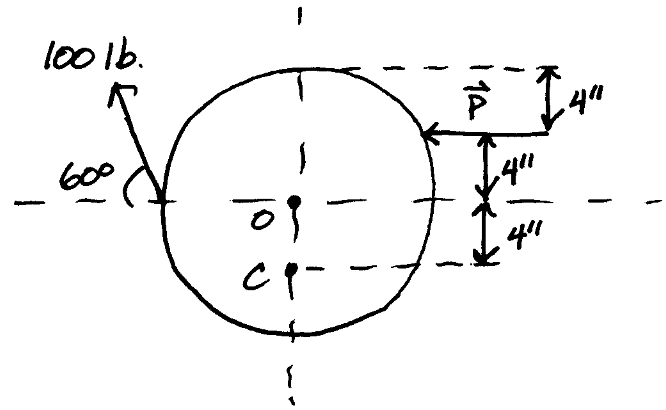
Now consider $\vec{r}_{AB} \times \vec{F}$

$$\vec{r}_{AB} \times \vec{F} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ x_{AB} & y_{AB} & z_{AB} \\ F_x & F_y & F_z \end{vmatrix} = \begin{array}{l} F_z\, y_{AB}\,\vec{i} + F_x\, z_{AB}\,\vec{j} + F_y\, x_{AB}\,\vec{k} \\ - F_y\, z_{AB}\,\vec{i} - F_z\, x_{AB}\,\vec{j} - F_x\, y_{AB}\,\vec{k} \end{array}$$

$$= \vec{M}^{(A)}$$

$$\therefore \boxed{\vec{M}^{(A)} = \vec{r}_{AB} \times \vec{F}^{(B)}}$$

Back to a 2-D Example:



If the combined moment of the two forces about point C is zero, determine

a) the magnitude of $\vec{P}$

b) the magnitude of $\vec{R}$ the resultant of the two forces

c) the x & y coordinates of a point A on the rim of the wheel about which the combined moment of the two forces is a maximum

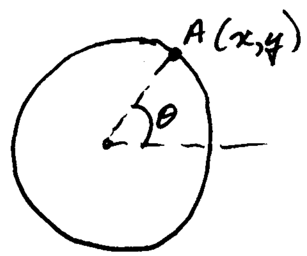
d) the combined moment $\vec{M}_A$ of the two forces about A

a) $M_z^C = 0 = P \cdot 8 + 50 \cdot 4 - 86.6 \cdot 8 \rightarrow \boxed{P = 61.6 \text{ lb.}}$

b) $\vec{R} = -50\vec{i} + 86.6\vec{j} - 61.6\vec{i} = -111.6\vec{i} + 86.6\vec{j}$

$\rightarrow R = \sqrt{111.6^2 + 86.6^2} = \boxed{141.3 \text{ lb.} = R}$

c)



$x = 8\cos\theta$

$y = 8\sin\theta$

$M_z^A = P(4 - y) - 50y - 86.6(8 + x)$, $P = 61.6$

$$\rightarrow M_z^A = -446.4 - 111.6y - 86.6x$$

$$= -446.4 - 111.6(8\sin\theta) - 86.6(8\cos\theta)$$

$$M_z^A = -446.4 - 892.8\sin\theta - 692.8\cos\theta$$

Maximize M_z^A with respect to θ

$$\frac{dM_z^A}{d\theta} = -892.8\cos\theta + 692.8\sin\theta = 0$$

$$\frac{\sin\theta}{\cos\theta} = \frac{892.8}{692.8}$$

$$\rightarrow \theta = 52.2^\circ,\ 232.2^\circ$$

$M_z^A(\theta = 52.2^\circ) = -1577$ lb·in (min value but max magnitude)

$M_z^A(\theta = 232.2^\circ) = 684$ lb·in (max value but <u>not</u> max magnitude)

Note: There will be a point on the rim where the moment is between these two extreme <u>values</u> with magnitude equal to zero.

Max magnitude at $x = 8\cos 52.2^\circ = 4.9$ in

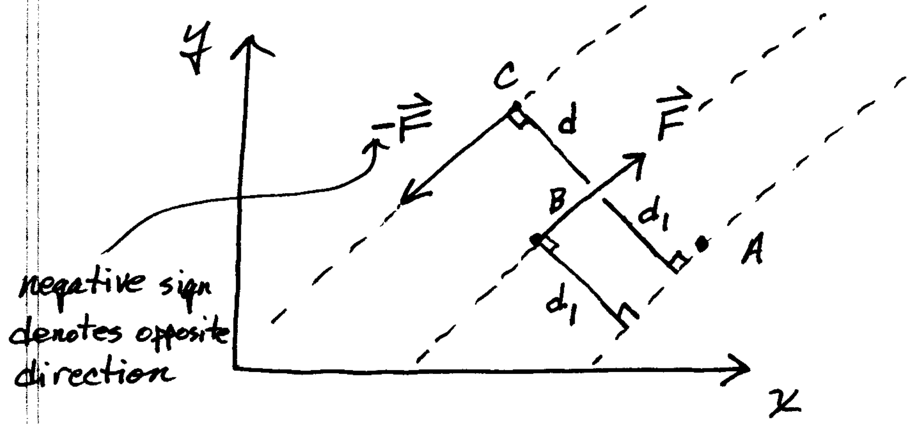
$y = 8\sin 52.2^\circ = 6.3$ in

d) Max magnitude is 1577 lb·in in the negative z-direction (clockwise).

Couples

A couple is a set of two $\underbrace{\text{equal}}_{\text{in magnitude}}$ and $\underbrace{\text{opposite}}_{\text{in direction}}$ non-colinear forces.

2-D : Consider the total moment due to a couple about an arbitrary point.



$$|\vec{F}| = |-\vec{F}| = F$$

$$\Sigma M_z^A = -F d_1 + F(d_1 + d)$$

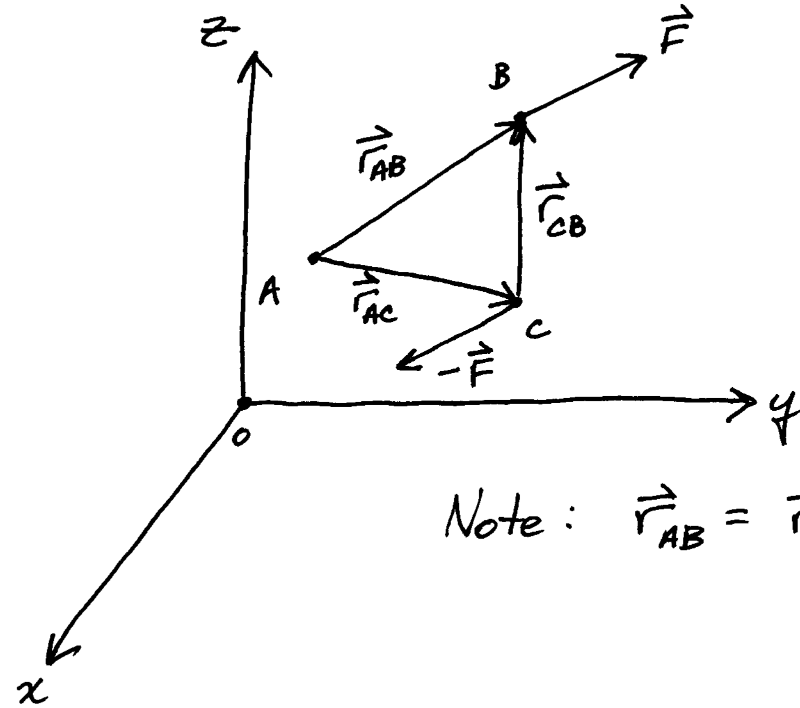
$$= F d$$

* Note that this result has absolutely nothing to do with the location of point A.

** Therefore, the moment due to a couple is the same about any point in space.

Equivalent representations of a couple:

3-D



Note: $\vec{r}_{AB} = \vec{r}_{AC} + \vec{r}_{CB}$

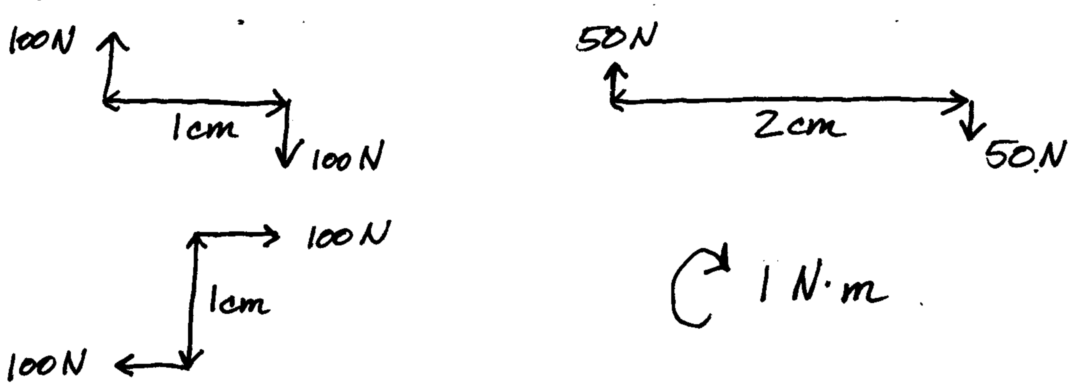
$$\Sigma \vec{M}^{(A)} = \vec{r}_{AB} \times \vec{F} + \vec{r}_{AC} \times (-\vec{F})$$

$$= (\vec{r}_{AB} - \vec{r}_{AC}) \times \vec{F}$$

$$\vec{C} = \vec{r}_{CB} \times \vec{F}$$

$\vec{r}_{CB}$ has nothing to do with the location of point A.

Equivalent Couples



Equivalent Systems

Two sets of force systems are equivalent (as far as their effects on a rigid body are concerned) if the sum of all of the forces and the sum of all of the moments due to the forces and couples about <u>any</u> point in space are identical.

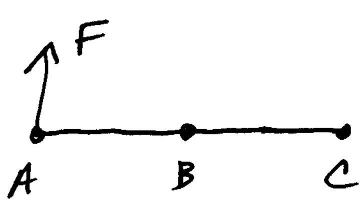
$$\Sigma \vec{F} = \sum_{i=1}^{N} \vec{F}_i$$

$$\Sigma \vec{M}^{(A)} = \sum_{i=1}^{M} \vec{C}_i + \sum_{i=1}^{N} \vec{r}_{AB_i} \times \vec{F}_i$$

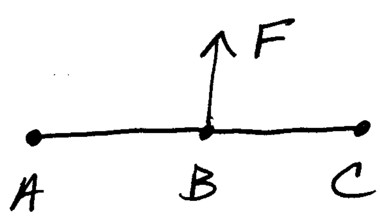
where there are N forces in the system acting at the points B_i and there are M arbitrarily positioned couples.

Recall that the net force of a couple is zero and the moment due to a couple about any point in space is the same. Therefore, a couple can be moved to any point in space without changing its effects on a rigid body.

However, changing the position of a force (not along its line of action) does change its moment about every point in space.

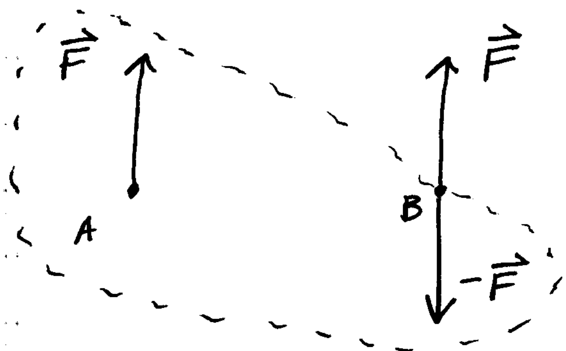


$\neq$



So, if we insist on moving a force, perhaps because it simplifies an analysis, we must compensate for the change in moment. Here is how this can be done:

We want to move $\vec{F}$ from A to B.

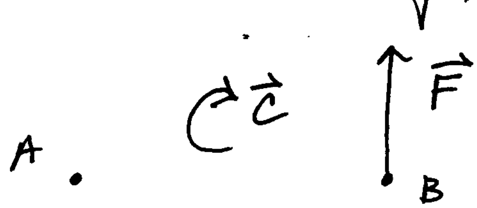


At B place both $\vec{F}$ and $-\vec{F}$. Essentially we are adding zero to point B.

Now, consider the two forces contained within the dashed curve. This is a couple that mathematically is represented as

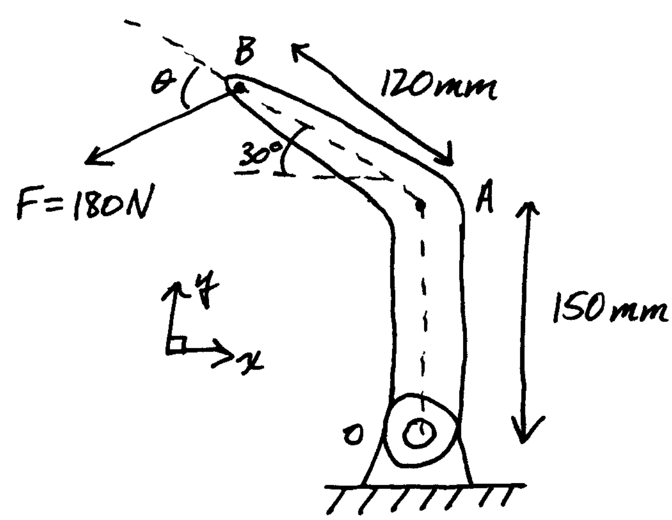
$$\vec{C} = \vec{r}_{BA} \times \vec{F}$$

So the following is also an equivalent system



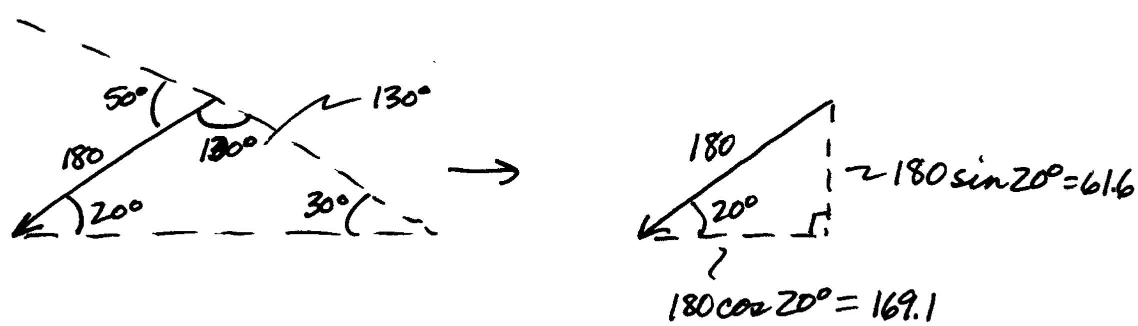
$\vec{F}$ acting at point B and a compensating couple $\vec{C} = \vec{r}_{BA} \times \vec{F}$ acting at any point in space.

Example:



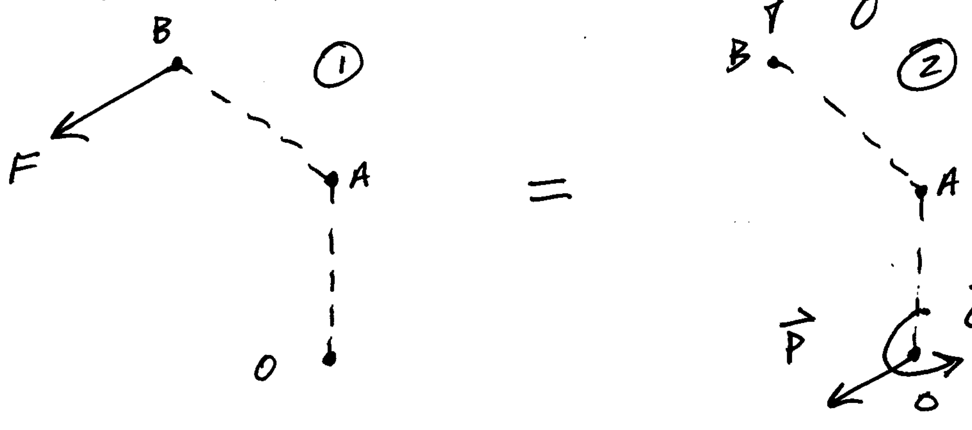
The 180-N force is applied to the end of the body OAB. If $\theta = 50°$, determine the equivalent force-couple system at the shaft axis O.

First, let's find the components of $\vec{F}$.



$$\rightarrow \quad \vec{F} = -169.1\,\vec{i} - 61.6\,\vec{j}$$

We want to find the following equivalent system



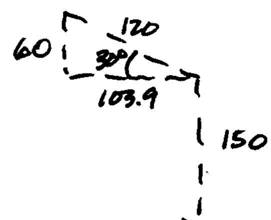
(Note: you can put $\vec{C}$ anywhere)

First, the sum of the forces in each system has to be the same,

$$\rightarrow \quad \vec{F} = \vec{P} \rightarrow \boxed{\vec{P} = -169.1\vec{i} - 61.6\vec{j}}\ N$$

Next, the moments about any point in space must be equivalent. Since the moment due to $\vec{P}$ about O is zero it is easiest to choose this point.

$$\vec{r}_{OB} \times \vec{F} = \vec{C}$$

Vector method: 

$\vec{r}_{OB} = (-103.9\vec{i} + 210\vec{j})\,mm$

$$\vec{C} = (-103.9\vec{i} + 210\vec{j}) \times (-169.1\vec{i} - 61.6\vec{j})$$
$$= -103.9(-61.6)\underbrace{\vec{i} \times \vec{j}}_{\vec{k}} + 210(-169.1)\underbrace{\vec{j} \times \vec{i}}_{-k}$$

$$= 41{,}911\vec{k}\,N \cdot mm \quad \text{or} \quad 41.9\vec{k}\,N \cdot m$$

or $C = 41.9\ N \cdot m$ counterclockwise (CCW)

Scalar metod: F_x contribution $= 169.1(210)$ CCW
F_y contribution $= 61.6(103.9)$ CCW

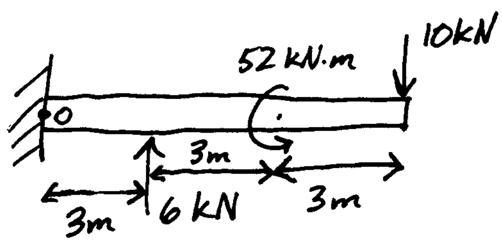
$$\rightarrow \quad C = 169.1(210) + 61.6(103.9) = 41.9\ N \cdot m \text{ CCW}$$

Resultants

In my opinion finding the resultant of a set of forces is not a particularly useful procedure. However it is informative to know what the concept means. Definition: The <u>resultant</u> of a system of forces and couples is the simplest force-couple combination which can replace the original system without altering the effects on a rigid body. (Except for a distributed load.)

For 2-D systems the resultant is either a single force or a single couple.

Example: Determine and locate the resultant $\vec{R}$ of the two forces and one couple acting on the beam.

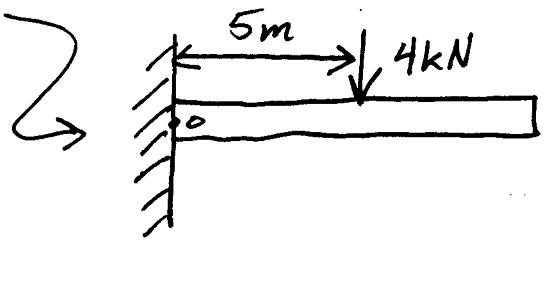


$$\Sigma F_x = 0$$

$$\Sigma F_y = -4\ kN \text{ or } 4kN\downarrow$$

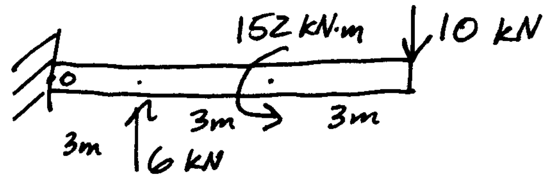
$$\Sigma M_z^o = 52 - 10\cdot 9 + 6\cdot 3$$
$$= -20\ kN\cdot m$$
$$\text{or } 20\ kN\cdot m \curvearrowright \text{ (CW)}$$

Resultant



$$\Sigma F_x = 0 \checkmark$$
$$\Sigma F_y = -4 \checkmark$$
$$\Sigma M_z^o = -20 \checkmark$$

What if we had:

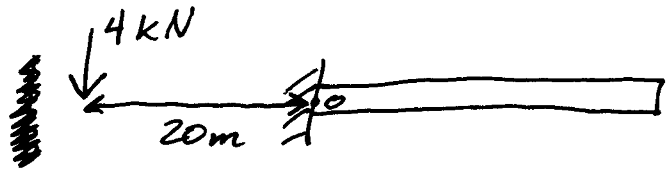


$\Sigma F_x = 0$

Then $\Sigma F_y = -4$

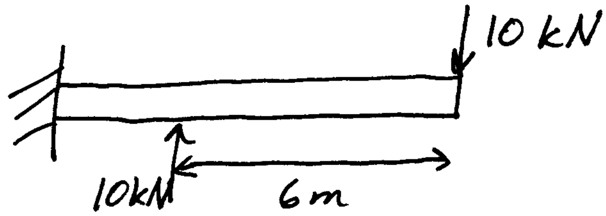
$\Sigma M_z^o = 152 - 10\cdot 9 + 6\cdot 3 = 80$

Resultant $\rightarrow$



The resultant does not necessarily act at a point on the structure.

Last case:



$\Sigma F_x = 0$

$\Sigma F_y = 0$

$\Sigma M_z^o = -60$ kN·m $\rightarrow$ Resultant is a couple acting anywhere in space