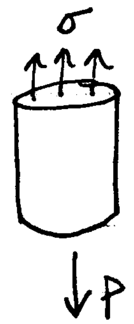
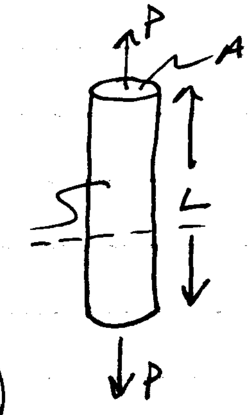
Nonuniform Axially Loades Members

Recall for the uniform case:





$$\Sigma F_y = \sigma A - P = 0$$

$$\rightarrow \sigma = \frac{P}{A}$$

Material $\rightarrow$ $\sigma = E\varepsilon$

Kinematics $\rightarrow$ $\varepsilon = \frac{\delta}{L}$

$$\frac{P}{A} = E\frac{\delta}{L}$$

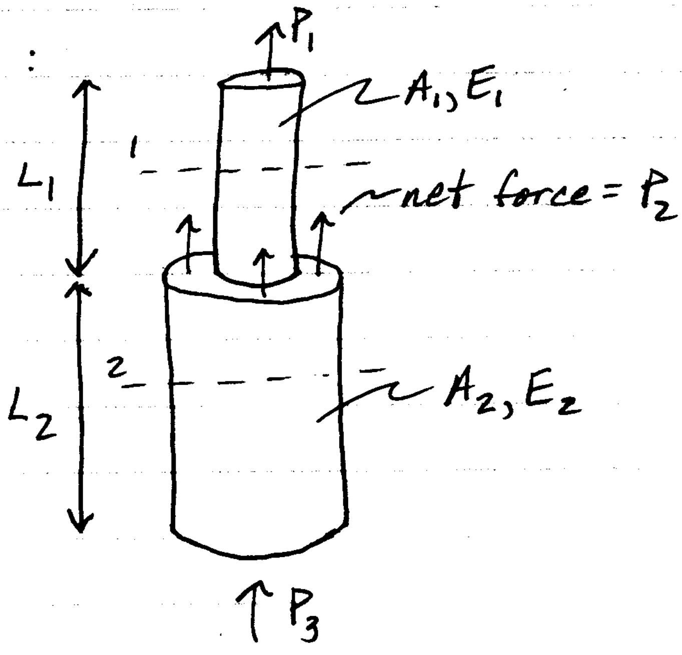
$$P = \frac{EA}{L}\delta$$

or $\delta = \frac{PL}{EA}$

For the nonuniform case we can introduce variations in the area A, Young's modulus E, and the distribution of load along the bar.

Discrete changes:

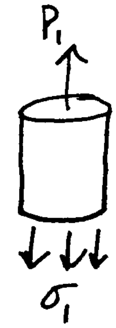
Total deflection?



Overall equilibrium: $\Sigma F_y = P_1 + P_2 + P_3 = 0$

$$\rightarrow P_3 = -P_1 - P_2$$

Section 1:



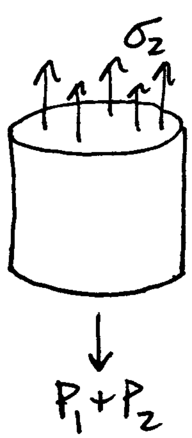
$$\Sigma F_y = P_1 - \sigma_1 A_1 = 0$$
$$\sigma_1 = P_1/A_1$$

Note that this formula is valid for σ_1 at every location in L_1. This means that σ_1 is uniform within L_1. Since E_1 is also homogeneous in L_1 we can determine the strain and deflection for this part of the bar.

$$\varepsilon_1 = \frac{\sigma_1}{E_1} = \frac{P_1}{E_1 A_1}$$

$$\delta_1 = \varepsilon_1 L_1 = \frac{P_1 L_1}{E_1 A_1}$$

We can analyze section 2 similarly.



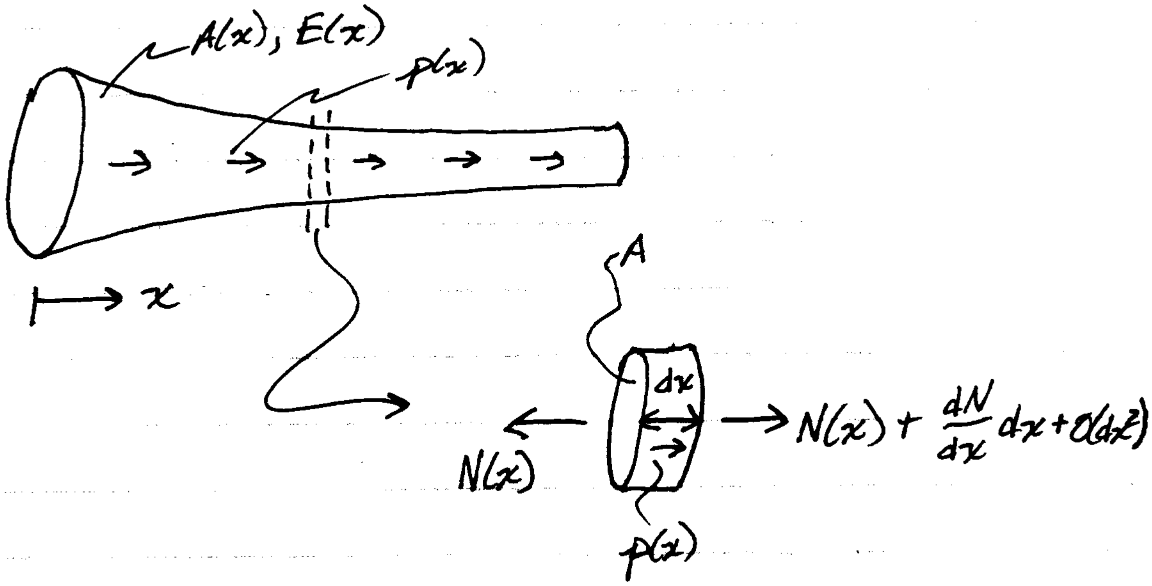
$$\Sigma F_y = \sigma_2 A_2 - P_1 - P_2 = 0$$
$$\sigma_2 = \frac{P_1 + P_2}{A_2}$$

$$\varepsilon_2 = \frac{\sigma_2}{E_2} = \frac{P_1 + P_2}{E_2 A_2}$$

$$\delta_2 = \varepsilon_2 L_2 = \frac{(P_1 + P_2) L_2}{E_2 A_2}$$

Finally $\boxed{\delta_{total} = \delta_1 + \delta_2 = \frac{P_1 L_1}{E_1 A_1} + \frac{(P_1 + P_2) L_2}{E_2 A_2}}$

Continuous Variations $p(x)$, $A(x)$, $E(x)$

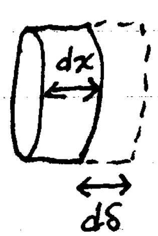


N = axial force/normal force. It's positive sense is directed outward from the area. The axial stress is then simply $\sigma = N/A$.* (* Actually an approximation for gradual tapering.)

Equilibrium of our FBD: $\Sigma F_x = \cancel{N(x)} + \frac{dN}{dx}\cancel{dx} + O(dx^2) - \cancel{N(x)} + p(x)\cancel{dx} = 0$

$$\lim_{dx \to 0} \longrightarrow \frac{dN}{dx} + p = 0 \quad \text{or} \quad \frac{dN}{dx} = -p(x)$$

Next consider the deflection of our differential element.



Kinematics $\rightarrow$ $\varepsilon = \frac{d\delta}{dx}$

$\longrightarrow$ * $\delta_{total} = \int_0^L d\delta$

Constitutive law $\rightarrow$ $\sigma = E\,\varepsilon$

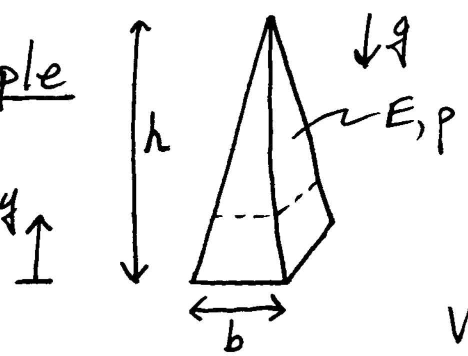
E $\leftarrow$ can be function of x

$$\sigma = \frac{N(x)}{A(x)} = E(x)\frac{d\delta}{dx}$$

$$\rightarrow \quad d\delta = \frac{N(x)}{A(x)E(x)}\,dx$$

$$\boxed{\delta_{total} = \int_0^L d\delta = \int_0^L \frac{N(x)}{A(x)E(x)}\,dx}$$

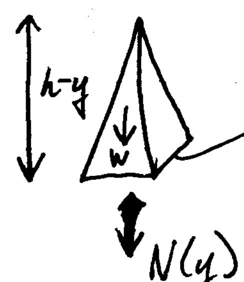
<u>Example</u>



Determine the shortening of the slender square pyramid due to its own weight.

$$V_{pyramid} = \frac{1}{3}A_{base}\,h$$

FBD of top :



$$A(y) = \left[b\left(1-\frac{y}{h}\right)\right]^2 = \left(\frac{b}{h}\right)^2(h-y)^2$$

$$V(y) = \frac{1}{3}\left[b\left(1-\frac{y}{h}\right)\right]^2(h-y) = \frac{1}{3}\left(\frac{b}{h}\right)^2(h-y)^3$$

$$\Sigma F_y = -N - \underbrace{\rho g\,\frac{1}{3}\left(\frac{b}{h}\right)^2(h-y)^3}_{W} = 0$$

$$N = -\frac{\rho g}{3}\left(\frac{b}{h}\right)^2(h-y)^3$$

$$\delta_{total} = \int_0^h -\frac{\rho g}{3E}\cancel{\left(\frac{b}{h}\right)^2}\frac{(h-y)^{\cancel{3}}}{\cancel{\left(\frac{b}{h}\right)^2}\cancel{(h-y)^2}}\,dy = \frac{\rho g}{3E}\,\frac{1}{2}(y-h)^2\Big]_0^h$$

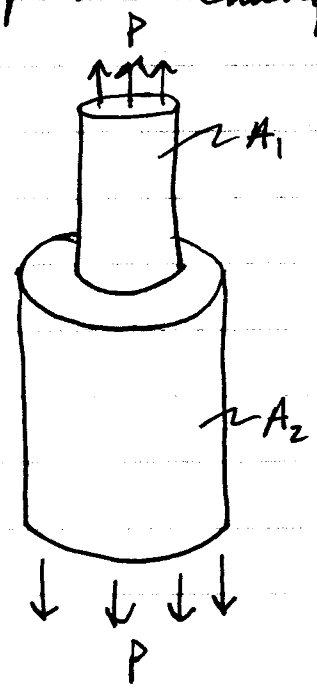
$$\boxed{\delta_{total} = -\frac{\rho g h^2}{6E}}$$

Why do we say $\sigma = \frac{N}{A}$ is an approximation when the bar is not prismatic?

First, note that $\sigma_{Average} = \frac{N}{A}$ is not an approximation. So the issue is that the stress is not uniformly distributed over the area. Some points in A will have $\sigma > \sigma_{Ave}$ and some points will have $\sigma < \sigma_{Ave}$. Overall we must have:

$$\underbrace{\sigma_{Ave} A}_{\text{Net force associated with } \sigma_{Average}} = \underbrace{\int_A \sigma \, dA}_{\text{Net force due to } \sigma}$$

Let's look at the extreme case where there is a step-like change in the area.

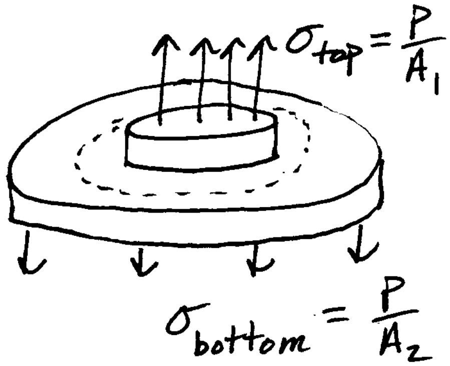


$$\left.\begin{aligned} \sigma_{Ave}^{top} &= \frac{P}{A_1} \\ \sigma_{Ave}^{bottom} &= \frac{P}{A_2} \end{aligned}\right\} \text{These are perfectly valid.}$$

Let's assume that the stresses are uniform in the top with $\sigma_{top} = \sigma_{Ave}^{top}$ and in the bottom with $\sigma_{bottom} = \sigma_{Ave}^{bottom}$.

Next, let's show that this cannot be correct.

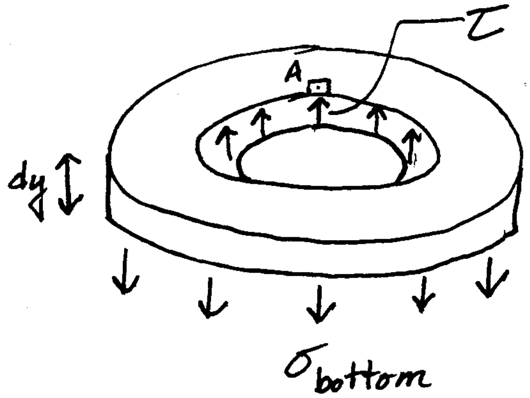
Let's look at some cuts around the joint.



$$\Sigma F_y = \sigma_{top} A_1 - \sigma_{bottom} A_2 = 0$$

$$P - P = 0 \checkmark$$

This is OK.

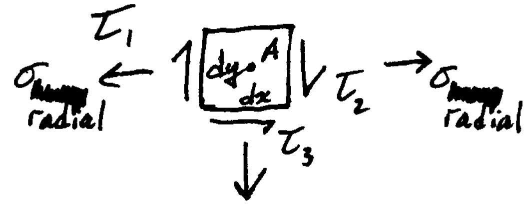


$$\Sigma F_y = \tau (2\pi R_i \, dy) - \sigma_{bottom} (\pi R_o^2 - \pi R_i^2) = 0$$

$$\rightarrow \tau = \sigma_{bottom} \frac{\pi (R_o^2 - R_i^2)}{2\pi R_i \, dy}$$

Our FBD is valid in the limit as $dy \rightarrow 0$, so unless $\tau \rightarrow \infty$ at the surface equilibrium cannot be satisfied with these uniform stress states. In fact, we can show that $\tau \rightarrow 0$ at the surface!

Look at point A from the side.



σ (actual axial stress at A)

$$\Sigma F_x = \tau_3 \, dx \, dz = 0$$

$$\rightarrow \tau_3 = 0$$

* σ_{radial} balances itself

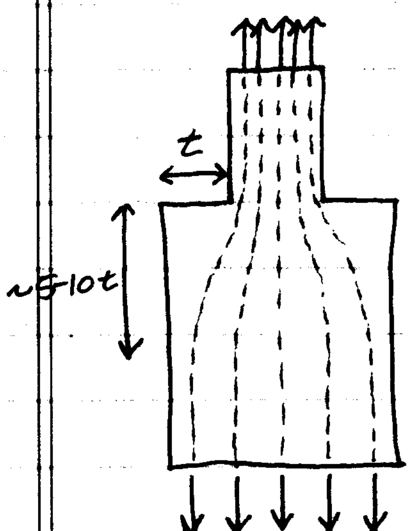
$$\Sigma M_z^A \rightarrow \tau_2 = -\tau_1, \quad \Sigma F_y = 2\tau_1 \, dy \, dz - \sigma \, dx \, dz = 0$$

$dx \rightarrow 0$ implies $\tau_1 = 0$, $dy \rightarrow 0$ implies $\sigma = 0$

This little analysis shows that if there is no direct load on a surface then both the $\underbrace{\text{axial stress}}$ and shear stress are zero.

*The one normal to the surface.

So what happens?



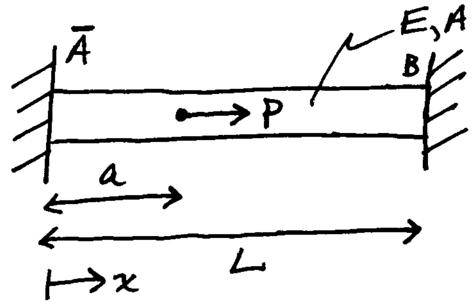
One intuitive and inexact way to think about it is with "lines of force". Lines of force want to separate from one another as much as possible, but must do so gradually. (There are mathematical equations that describes this, but these are beyond the scope of this course.)

If the bottom part of the bar is long enough then the spacing between the lines of force become uniform again and this means the stress is uniform again. If the length L of the bar is much longer than t then we can neglect the effects of the non-uniform contributions to the stresses, strains, and deflections.

Similarly if $\frac{dA}{dx} \ll L$ (for smooth $A(x)$) then we can safely approximate the stress as being uniformly distributed over the area at each location along L.

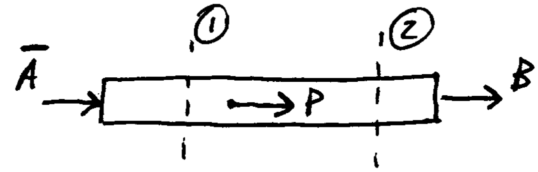
Statically Indeterminate Problems

Simple:



Determine the stresses and strains in the bar.

Equilibrium:



$$\Sigma F_x = \bar{A} + B + P = 0 \quad (a)$$

cut 1 $\quad 0 < x < a$: $\quad \xrightarrow{\bar{A}} \square \rightarrow N_1$

$$\Sigma F_x = \bar{A} + N_1 = 0 \rightarrow N_1 = -\bar{A}$$

cut 2 $\quad a < x < L$: $\quad N_2 \leftarrow \square \rightarrow B$

$$\Sigma F_x = B - N_2 = 0 \rightarrow N_2 = B$$

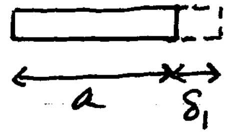
N_1 = internal axial force in the domain $0 < x < a$
N_2 = internal axial force in the domain $a < x < L$

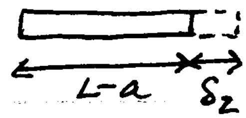
Constitutive Response: $\quad 0 < x < a$: $\quad \sigma_1 = E\varepsilon_1$

$$\varepsilon_1 = \frac{\sigma_1}{E} = \frac{N_1}{EA} = \frac{-\bar{A}}{EA} \quad (b)$$

Similarly in $a < x < L$: $\quad \varepsilon_2 = \frac{\sigma_2}{E} = \frac{B}{EA} \quad (c)$

Kinematics:





$$\varepsilon_1 = \frac{\delta_1}{a} \qquad\qquad \varepsilon_2 = \frac{\delta_2}{L-a}$$

But there is also a constraint imposed by the two walls. First note that the total deflection is $\delta_{total} = \delta_1 + \delta_2$. The walls are rigid $\rightarrow \delta_{total} = 0$.

$$\rightarrow \delta_1 + \delta_2 = 0 \rightarrow \delta_2 = -\delta_1$$

$$\therefore \varepsilon_1 = \frac{\delta_1}{a} \ \text{(d)} \quad \text{and} \quad \varepsilon_2 = \frac{-\delta_1}{L-a} \ \text{(e)}$$

(b) & (d) $\rightarrow \varepsilon_1 = \dfrac{\delta_1}{a} = \dfrac{-\bar{A}}{EA} \rightarrow \bar{A} = \dfrac{-\delta_1 EA}{a}$

(c) & (e) $\rightarrow \varepsilon_2 = \dfrac{-\delta_1}{L-a} = \dfrac{B}{EA} \rightarrow B = \dfrac{-\delta_1 EA}{L-a}$

(a) $\rightarrow -\dfrac{\delta_1 EA}{a} - \dfrac{\delta_1 EA}{L-a} + P = 0$

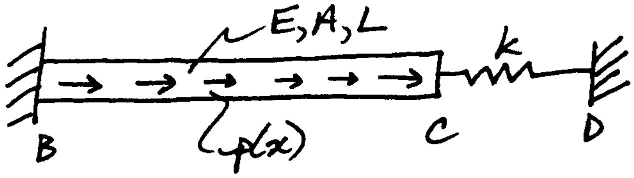
$$\boxed{\delta_1 = \frac{P}{EA}\frac{a(L-a)}{L}}$$

Now we can find: $\bar{A} = -P\dfrac{L-a}{L} \rightarrow \boxed{\sigma_1 = \dfrac{P}{A}\dfrac{L-a}{L}}$

$B = -P\dfrac{a}{L} \rightarrow \boxed{\sigma_2 = \dfrac{-P}{A}\dfrac{a}{L}}$

Then: $\boxed{\varepsilon_1 = \dfrac{\sigma_1}{E}}$, $\boxed{\varepsilon_2 = \dfrac{\sigma_2}{E}}$

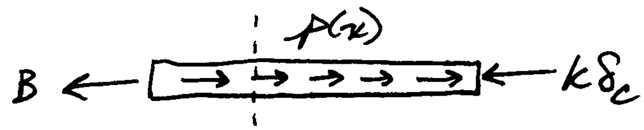
Difficult:



$$p(x) = p_0 \frac{x}{L}, \quad k = \frac{EA}{4L}$$

Determine $\sigma(x)$ in the bar.

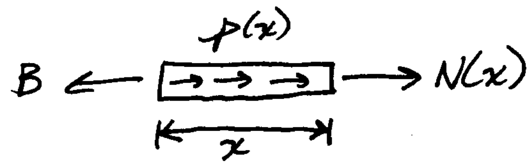
Equilibrium:



$$\Sigma F_x = -B + \int_0^L p_0 \frac{x}{L}\, dx - k\delta_c = 0$$

$$-B + \frac{1}{2} p_0 L - k\delta_c = 0 \quad (a)$$

cut at x:



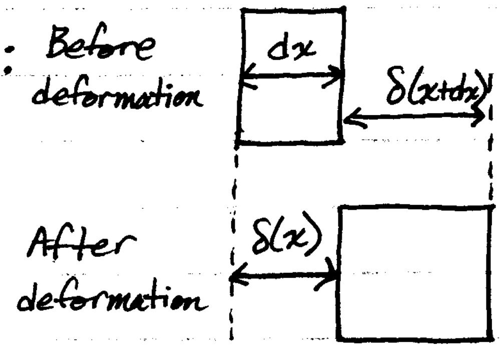
$$\Sigma F_x = -B + \underbrace{\int_0^x p_0 \frac{x'}{L}\, dx'}_{\frac{1}{2} p_0 \frac{x^2}{L}} + N(x) = 0$$

$$N(x) = B - \frac{1}{2} p_0 \frac{x^2}{L} \quad \leftarrow \text{axial force}$$

$$\sigma(x) = \frac{N(x)}{A} \quad \leftarrow \text{axial/normal stress}$$

Constitutive response: $\varepsilon(x) = \dfrac{\sigma(x)}{E} = \dfrac{B}{EA} - \dfrac{p_0}{2EA}\dfrac{x^2}{L}$ (b)

Kinematics :



$$\varepsilon(x) = \frac{\Delta L}{L} = \frac{\delta(x+dx) - \delta(x)}{dx} \overset{\lim_{dx\to 0}}{=} \frac{d\delta}{dx}$$

$$\varepsilon(x) = \frac{d\delta}{dx} \quad (c)$$

$$(b)\ \&\ (c) \rightarrow \quad \varepsilon(x) = \frac{d\delta}{dx} = \frac{B}{EA} - \frac{p_0}{2EA}\frac{x^2}{L}$$

$$\int_0^L \frac{d\delta}{dx}\,dx = \int_0^L \frac{B}{EA} - \frac{p_0}{2EA}\frac{x^2}{L}\,dx$$

$$\delta\Big]_0^L = \frac{B}{EA}x - \frac{p_0}{6EA}\frac{x^3}{L}\Big]_0^L$$

$$\underbrace{\delta(x=L)}_{=\delta_c} - \underbrace{\delta(x=0)}_{=0} = \frac{BL}{EA} - \frac{p_0 L^2}{6EA}$$

$$\boxed{\delta_c = \frac{6BL - p_0 L^2}{6EA}}$$

We still have B as an unknown in this equation. We need to use (a).

$$(a) \rightarrow \ -B + \frac{1}{2}p_0 L - \underbrace{\frac{EA}{4L}}_{k}\ \frac{6BL - p_0 L^2}{6EA} = 0$$

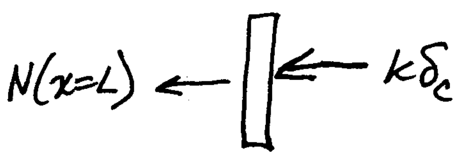
$$-B + \frac{1}{2} p_o L - \frac{B}{4} + \frac{1}{24} p_o L = 0$$

$$\frac{5B}{4} = \frac{13}{24} p_o L \rightarrow \boxed{B = \frac{13}{30} p_o L}$$

$$\rightarrow \boxed{\delta_c = \frac{8}{30} \frac{p_o L^2}{EA}}$$

$$\boxed{\begin{aligned} \sigma(x) &= \left[\frac{13}{30} - \frac{1}{2} \frac{x^2}{L^2} \right] \frac{p_o L}{A} \\ \varepsilon(x) &= \left[\frac{13}{30} - \frac{1}{2} \frac{x^2}{L^2} \right] \frac{p_o L}{EA} \end{aligned}}$$

What can we check? At C, the spring puts a force on the bar. Let's draw a FBD of a sliver at the end of the bar.



Note that the distributed load p does not contribute a net force because the length of the sliver is effectively zero.

$$N(x=L) = \sigma(x=L) A = -\frac{1}{15} p_o L$$

$$k\delta_c = \frac{EA}{4L} \frac{8}{30} \frac{p_o L^2}{EA} = \frac{1}{15} p_o L$$

$$\Sigma F_x = -N(x=L) - k\delta_c = 0$$

$$\frac{1}{15} p_o L - \frac{1}{15} p_o L = 0 \checkmark$$