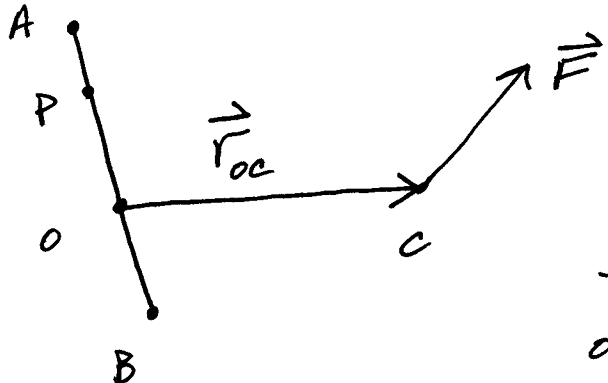
3-D Force Systems

- Moment of a force about an axis



The moment of the force $\vec{F}$ about the axis AB is a vector quantity that gives the amount of the moment due to $\vec{F}$ about any point on the axis, in the direction of the axis.

$\vec{M}_O = \vec{r}_{OC} \times \vec{F}$ moment of $\vec{F}$ about O (a point on the axis)

$\vec{e}_{AB}$ = a unit vector along the axis

$\vec{M}_{AB}$ = moment due to $\vec{F}$ about axis AB

$$\vec{M}_{AB} = \underbrace{(\vec{M}_O \cdot \vec{e}_{AB})}_{\pm \text{ magnitude}} \underbrace{\vec{e}_{AB}}_{\text{direction}}$$

$\pm$ comes from the fact that you could choose $\vec{e}_{AB}$ or $\vec{e}_{BA}$ and this formula will always produce the correct vector.

What if we had picked point P instead?

$$\vec{M}_{AB} = (\vec{M}_P \cdot \vec{e}_{AB})\vec{e}_{AB}$$

$$= (\vec{r}_{PC} \times \vec{F} \cdot \vec{e}_{AB})\,\vec{e}_{AB}$$

$$= \left[(\vec{r}_{PO} + \vec{r}_{OC}) \times \vec{F} \cdot \vec{e}_{AB}\right]\vec{e}_{AB}$$

$$= (\vec{r}_{PO} \times \vec{F} \cdot \vec{e}_{AB})\vec{e}_{AB} + (\vec{r}_{OC} \times \vec{F} \cdot \vec{e}_{AB})\vec{e}_{AB}$$

$$= (r_{PO}\,\underbrace{\vec{e}_{AB} \times \vec{F}}_{\substack{\text{This vector}\\ \text{is perpendicular}\\ \text{to } \vec{e}_{AB}}} \cdot \vec{e}_{AB})\vec{e}_{AB} + (\vec{M}_O \cdot \vec{e}_{AB})\vec{e}_{AB}$$

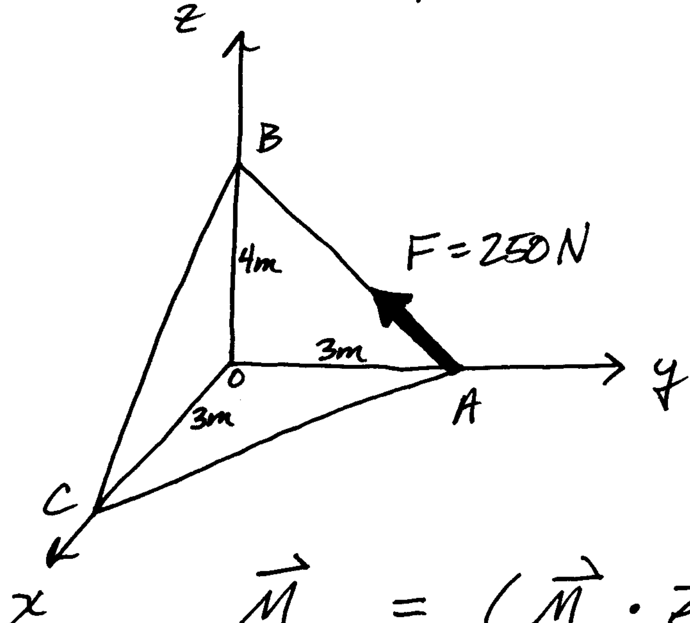
$\therefore$ This vector dotted with $\vec{e}_{AB} = 0$

$$= \boxed{(\vec{M}_O \cdot \vec{e}_{AB})\vec{e}_{AB} = (\vec{M}_P \cdot \vec{e}_{AB})\vec{e}_{AB}}$$

O & P are both on the axis AB

→ The moment due to a force about an axis does not depend on what point you decide to take moments about on the axis.

<u>Example</u> : Determine the moment of the force $\vec{F}$ [redacted] about the axis that is perpendicular to the plane ABC and passes through O.



$$\vec{M}_{O\perp} = (\underbrace{\vec{M}_O}_{\vec{r}_{OA} \times \vec{F}} \cdot \vec{e}_\perp)\,\vec{e}_\perp$$

$$\vec{r}_{OA} = 3\vec{j} \quad , \quad \vec{F} = 250\,\vec{e}_{AB} = 250\,\frac{-3\vec{j} + 4\vec{k}}{\sqrt{3^2+4^2}}$$

$$\vec{F} = -150\vec{j} + 200\vec{k}$$

$$\vec{M}_O = \vec{r}_{OA} \times \vec{F} = 3\vec{j} \times (-150\vec{j} + 200\vec{k})$$

$$= 3(200)\,\vec{j} \times \vec{k}$$

$$\vec{M}_O = 600\,\vec{i}$$

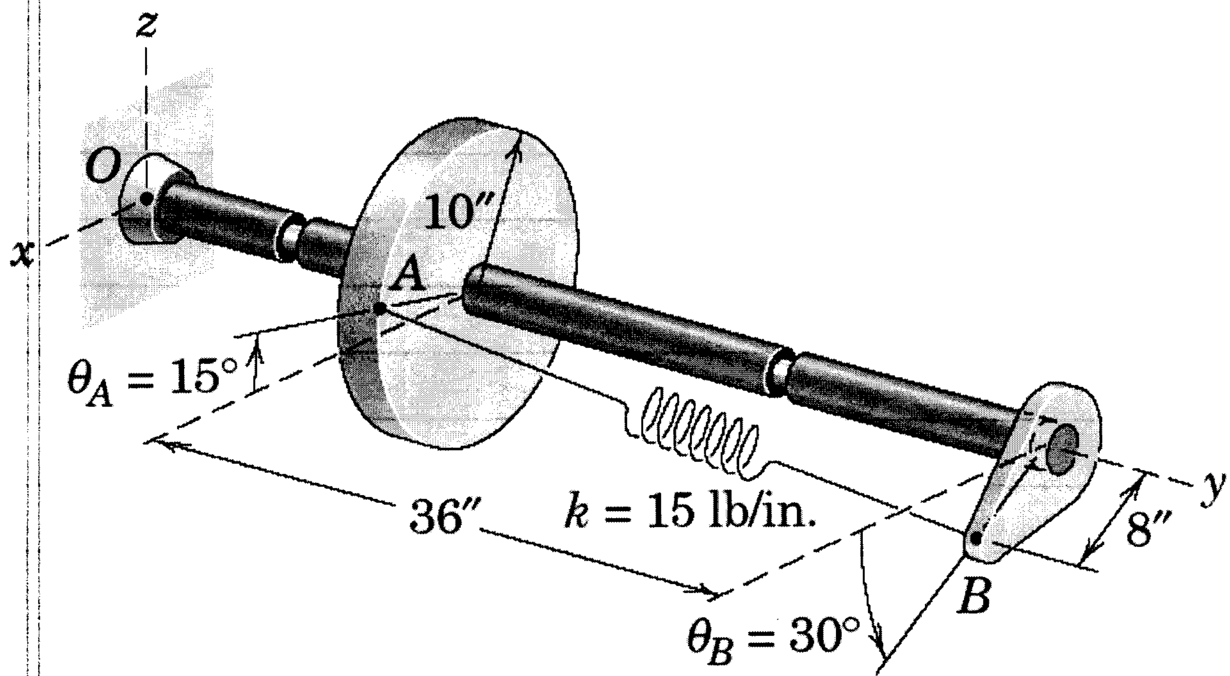
$$\vec{e}_{\perp} = \frac{\vec{CA} \times \vec{CB}}{|\vec{CA} \times \vec{CB}|} = \frac{(-3\vec{i} + 3\vec{j}) \times (-3\vec{i} + 4\vec{k})}{|\vec{CA} \times \vec{CB}|}$$

$$= \frac{-3(4)\overbrace{\vec{i} \times \vec{k}}^{-\vec{j}} + 3(-3)\overbrace{\vec{j} \times \vec{i}}^{-\vec{k}} + 3(4)\overbrace{\vec{j} \times \vec{k}}^{\vec{i}}}{|\vec{CA} \times \vec{CB}|}$$

$$= \frac{12\vec{i} + 12\vec{j} + 9\vec{k}}{\sqrt{144 + 144 + 81}} = 0.625\vec{i} + 0.625\vec{j} + 0.467\vec{k}$$

$$\rightarrow \vec{M}_{o\perp} = 600(0.625)(0.625\vec{i} + 0.625\vec{j} + 0.467\vec{k})$$

$$\boxed{\vec{M}_{o\perp} = (234\vec{i} + 234\vec{j} + 175\vec{k})\ N\cdot m}$$



The spring is unstretched when $\theta_A = \theta_B = 0°$. In the position shown, determine the vector representation of the force $\vec{F}$ that the spring exerts on point A. Also, determine the moment of this force about the axis that passes through the center of the shaft. The magnitude of the spring force is $F = k\delta$ where $\delta =$ deflection of the spring, i.e. $\delta = L - L_o$.

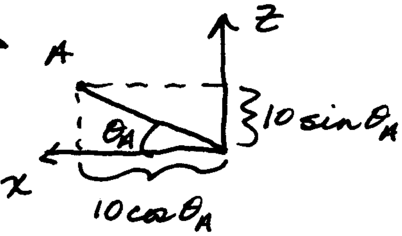
Let's determine δ. The spring is unstretched when $\theta_A = \theta_B = 0$, so let's determine the length of spring for this state.

$$\vec{A}_o = 10\vec{i} + y_A\vec{j} \quad , \quad \vec{B}_o = 8\vec{i} + (y_A + 36)\vec{j}$$

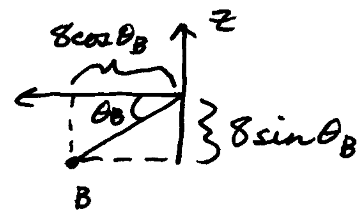
$$\vec{L}_o = \overrightarrow{A_oB_o} = -2\vec{i} + 36\vec{j} \rightarrow L_o = \sqrt{2^2 + 36^2} = 36.06 \text{ in.}$$

Now let's determine the stretched length.

$$\vec{A} = 10\cos 15^\circ \vec{i} + y_A \vec{j} + 10 \sin 15^\circ \vec{k}$$
$$= 9.66\vec{i} + y_A\vec{j} + 2.59\vec{k}$$



$$\vec{B} = 8\cos 30^\circ \vec{i} + (y_A + 36)\vec{j} - 8\sin 30^\circ \vec{k}$$
$$= 6.93\vec{i} + (y_A+36)\vec{j} - 4\vec{k}$$



$$\vec{AB} = -2.73\vec{i} + 36\vec{j} - 6.59\vec{k}$$

$$\rightarrow L = \sqrt{2.73^2 + 36^2 + 6.59^2} = 36.70 \text{ in.}$$

$$\rightarrow \delta = 0.64 \text{ in}$$

$$F = k\delta = 15 \tfrac{\text{lb}}{\text{in}} \cdot 0.64 \text{ in} = 9.6 \text{ lb.}$$

Note that the direction of $\vec{F}$ is from A towards B

$$\vec{e}_{AB} = \frac{\vec{AB}}{|\vec{AB}|} = \frac{\vec{AB}}{L} = -0.074\vec{i} + 0.981\vec{j} - 0.180\vec{k}$$

$$\rightarrow \vec{F} = F\vec{e}_{AB} = (-0.71\vec{i} + 9.42\vec{j} - 1.73\vec{k}) \text{ lb.}$$

Moment about the shaft axis:

$$\vec{M}_{Axis} = (\vec{r} \times \vec{F} \cdot \vec{e}_{Axis})\,\vec{e}_{Axis}$$

$$\vec{e}_{Axis} = \vec{j}$$

$$\vec{r} = 9.66\,\vec{i} + y\,\vec{j} + 2.59\,\vec{k}$$

This can be anything since you can pick any point on the y-axis to take moments about

$$\left(\vec{r}\times\vec{F}\right)_y = 9.66(-1.73)\underbrace{\vec{i}\times\vec{k}}_{-\vec{j}} + 2.59(-0.71)\underbrace{\vec{k}\times\vec{i}}_{\vec{j}}$$

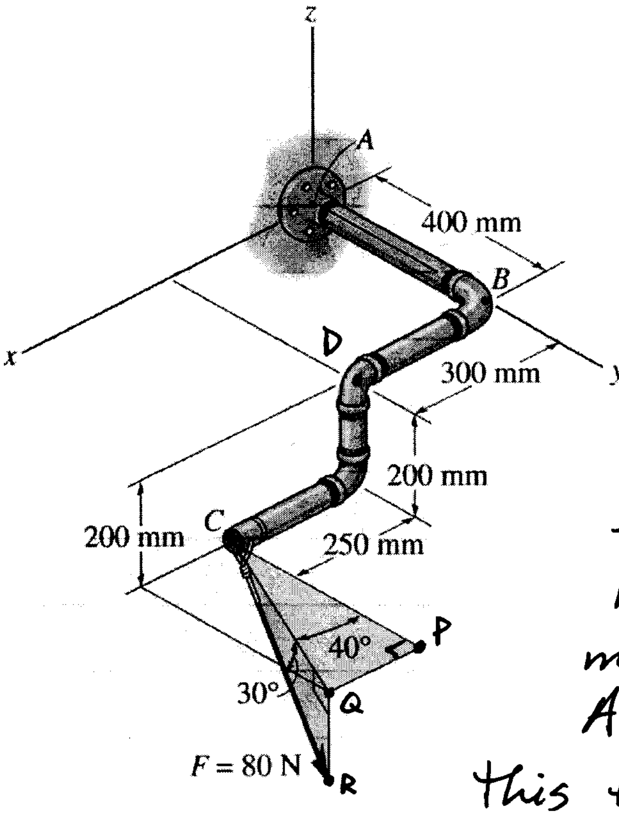
$$= (16.71 - 1.84)\,\vec{j} = 14.87\,\vec{j}$$

We only need the y-component because we will be dotting the result with $\vec{j}$

$$\rightarrow \vec{r}\times\vec{F} = M_x\vec{i} + 14.87\,\vec{j} + M_z\vec{k}$$

$$\vec{r}\times\vec{F}\cdot\vec{e}_{Axis} = 14.87$$

$$\vec{M}_{Axis} = 14.87\,\vec{j}$$

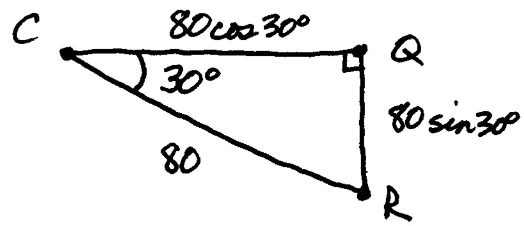


The pipe assembly is subjected to the 80-N force. The angle between the projection of the force on to the x-y plane and the y-axis is 40°. Then the angle between the force and this direction is 30°. Determine the moment of this force about A, and the moment of this force about an axis passing through A and D.

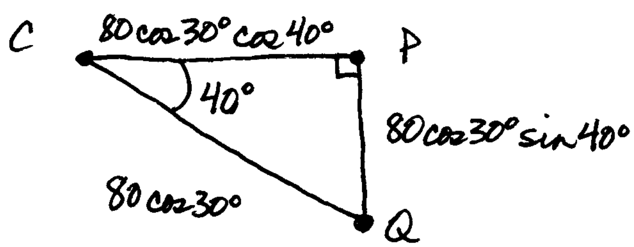
First we need a vector representation of $\vec{F}$.

$$\vec{F} = F\,\vec{e}_F$$

Consider triangle CRQ



Now consider triangle CQP



PQ is in the x-direction $\rightarrow F_x = 80\cos 30^\circ \sin 40^\circ$
CP is in the y-direction $\rightarrow F_y = 80\cos 30^\circ \cos 40^\circ$
QR is in the −z-direction $\rightarrow F_z = -80\sin 30^\circ$

$$\vec{F} = (44.5\vec{i} + 53.1\vec{j} - 40\vec{k})\,N$$

check $F = \sqrt{44.5^2 + 53.1^2 + 40^2} = 80$ ✓

$$\vec{M}_A = \vec{r}_{AC} \times \vec{F}$$

$$\vec{r}_{AC} = (550\vec{i} + 400\vec{j} - 200\vec{k})\,mm$$

$$\vec{M}_A = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 550 & 400 & -200 \\ 44.5 & 53.1 & -40 \end{vmatrix} \begin{aligned} &(-16000 + 200\cdot 53.1)\vec{i} \\ &= +(-200\cdot 44.5 + 40\cdot 550)\vec{j} \\ &+(550\cdot 53.1 - 400\cdot 44.5)\vec{k} \end{aligned}$$

$$\boxed{\vec{M}_A = (-5380\vec{i} + 13100\vec{j} + 11405\vec{k})\,N\cdot mm}$$ ← Moment about point A

$$\vec{M}_{AD} = (\vec{M}_A \cdot \vec{e}_{AD})\vec{e}_{AD}$$

$$\vec{e}_{AD} = \frac{300\vec{i} + 400\vec{j}}{\sqrt{300^2 + 400^2}} = 0.6\vec{i} + 0.8\vec{j}$$

$$\vec{M}_{AD} = \underbrace{(-5380\cdot 0.6 + 13100\cdot 0.8)}_{7252}(0.6\vec{i} + 0.8\vec{j})$$

$$\boxed{\vec{M}_{AD} = (4351\vec{i} + 5802\vec{j})\,N\cdot mm}$$ ← Moment about axis AD

Equilibrium of a Rigid Body

There are two key components to every equilibrium analysis.

1) Equilibrium equations

Vector form: $\Sigma \vec{F} = 0$

$\Sigma \vec{M}_A = 0$ (A is any arbitrary point)

Scalar form:

$\Sigma F_x = 0$ * $\Sigma M_x^A = 0$

$\Sigma F_y = 0$ * $\Sigma M_y^A = 0$

$\Sigma F_z = 0$ $\Sigma M_z^A = 0$ *

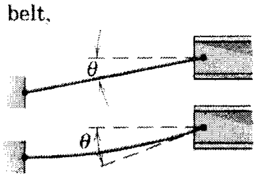
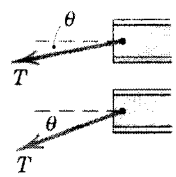
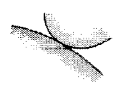
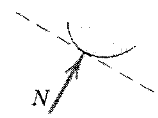
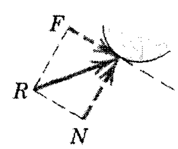
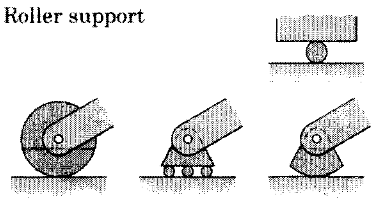
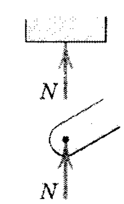
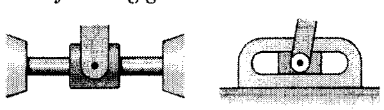
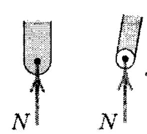
The * equations are necessary for 2-D (planar) analyses.

2) Correct and well-labeled Free Body Diagrams.

A correct free body diagram has all possible reaction forces drawn at locations where external supports or connections to other components or parts of the structure have been removed.

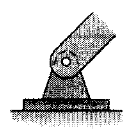
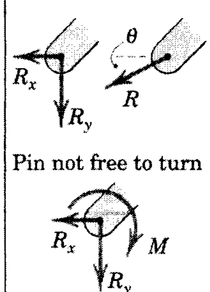
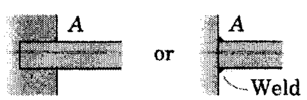
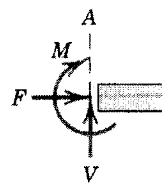
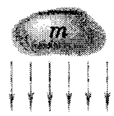
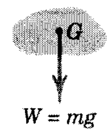
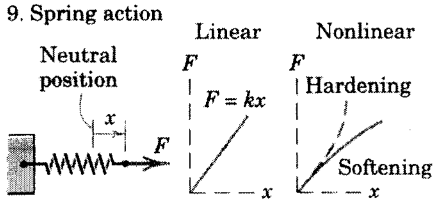
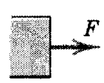
If a previous analysis has indicated that a connection or support force is zero, then and only then is it acceptable to omit that force from a subsequently drawn free body diagram.

When drawing a FBD we are isolating an object from its surroundings. In doing so, we must replace the surroundings with the forces that it applys to the object.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS		
Type of Contact and Force Origin	Action on Body to Be Isolated	
1. Flexible cable, belt, chain, or rope Weight of cable negligible Weight of cable not negligible 	 Force exerted by a flexible cable is always a tension away from the body in the direction of the cable.	Cables forces should be drawn assuming the cable is in tension. Unknown force magnitude direction known.
2. Smooth surfaces 	 Contact force is compressive and is normal to the surface.	Unknown magnitude, direction known.
3. Rough surfaces 	 Rough surfaces are capable of supporting a tangential component F (frictional force) as well as a normal component N of the resultant contact force R.	Unknown normal & tangential magnitudes.
4. Roller support 	 Roller, rocker, or ball support transmits a compressive force normal to the supporting surface.	Unknown magnitude, known direction.
5. Freely sliding guide 	 Collar or slider free to move along smooth guides; can support force normal to guide only.	Unknown magnitude, known direction.

Illustrated above and on the next two pages are several different types of structural supports and how the forces due to those supports would be drawn on a free body diagram.

If a support can resist motion in a given direction then it can provide a force in that direction (to oppose the motion). If a support resists rotation then it can apply a moment/couple to the object.

MODELING THE ACTION OF FORCES IN TWO-DIMENSIONAL ANALYSIS (*cont.*)	
Type of Contact and Force Origin	Action on Body to Be Isolated
6. Pin connection 	Pin free to turn; Pin not free to turn  A freely hinged pin connection is capable of supporting a force in any direction in the plane normal to the pin axis. We may either show two components R_x and R_y or a magnitude R and direction θ. A pin not free to turn also supports a couple M.
7. Built-in or fixed support 	 A built-in or fixed support is capable of supporting an axial force F, a transverse force V (shear force), and a couple M (bending moment) to prevent rotation.
8. Gravitational attraction 	 The resultant of gravitational attraction on all elements of a body of mass m is the weight $W = mg$ and acts toward the center of the earth through the center mass G.
9. Spring action 	 Spring force is tensile if spring is stretched and compressive if compressed. For a linearly elastic spring the stiffness k is the force required to deform the spring a unit distance.

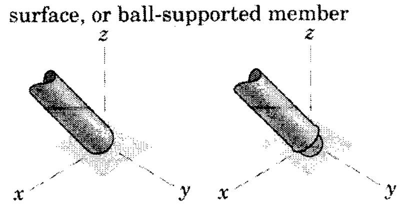
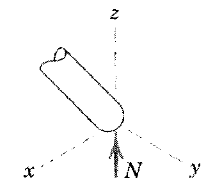
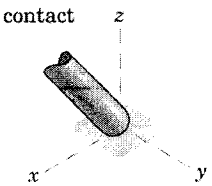
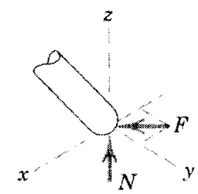
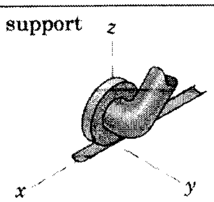
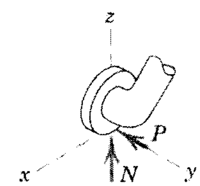
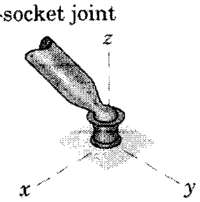
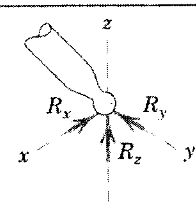
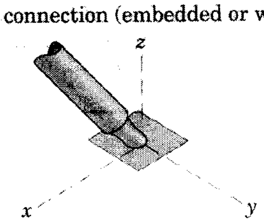
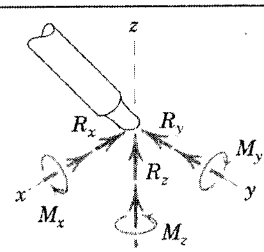
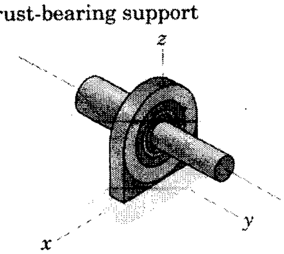
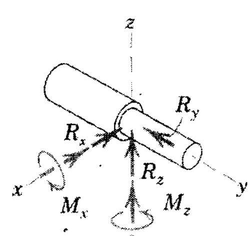
Usually pins are assumed to rotate freely.
Unknown x & y components of force.

Unknown x (axial), y (shear) forces, and moment.

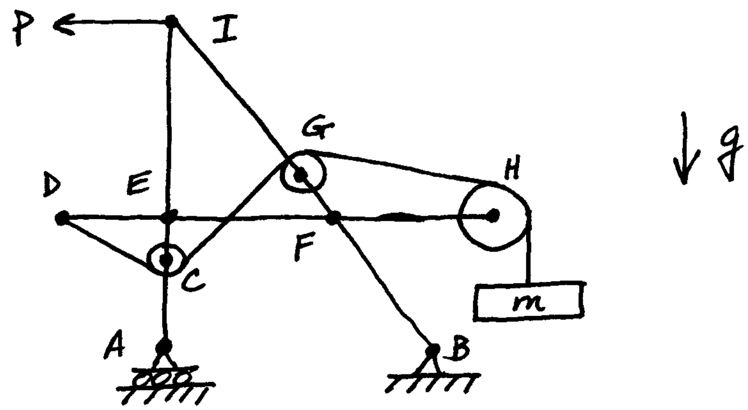
Remove the earth and replace it with the force it exerts. Replace the distributed weight with a point force at the center of mass.

The next page contains 3-D examples. If a support does not resist ~~[illegible]~~ translation in a given direction then it can not supply a force in that direction. If a support does not resist rotation in a given direction then it can not supply a moment/couple in that direction.

MODELING THE ACTION OF FORCES IN THREE-DIMENSIONAL ANALYSIS

Type of Contact and Force Origin	Action on Body to Be Isolated	
1. Member in contact with smooth surface, or ball-supported member 	 Force must be normal to the surface and directed toward the member.	1 Unknown
2. Member in contact with rough surface 	 The possibility exists for a force F tangent to the surface (friction force) to act on the member, as well as a normal force N.	3 Unknowns (magnitude & direction unknown in the plane)
3. Roller or wheel support with lateral constraint 	 A lateral force P exerted by the guide on the wheel can exist, in addition to the normal force N.	2 Unknowns
4. Ball-and-socket joint 	 A ball-and-socket joint free to pivot about the center of the ball can support a force **R** with all three components.	3 Unknowns
5. Fixed connection (embedded or welded) 	 In addition to three components of force, a fixed connection can support a couple **M** represented by its three components.	6 Unknowns
6. Thrust-bearing support 	 Thrust bearing is capable of supporting axial force R_y as well as radial forces R_x and R_z. Couples M_x and M_z must, in some cases, be assumed zero in order to provide statical determinacy.	5 Unknowns with the collar, 4 Unknowns without the collar

Example:

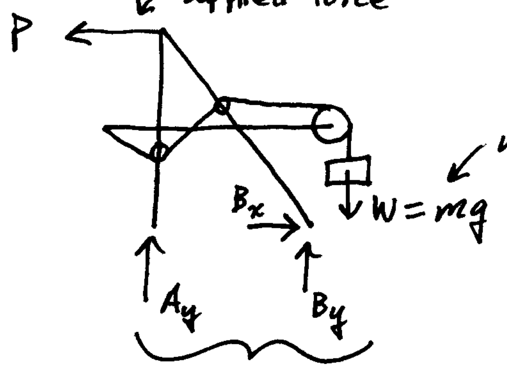


Bars IA, DH and IB are solid and joined together at E, I and F by frictionless pins. Weightless pulleys are attached at C, G and H by frictionless pins. Draw FBDs of the entire structure and each component of the structure.

Entire structure

applied force

weight due to "removal" of earth



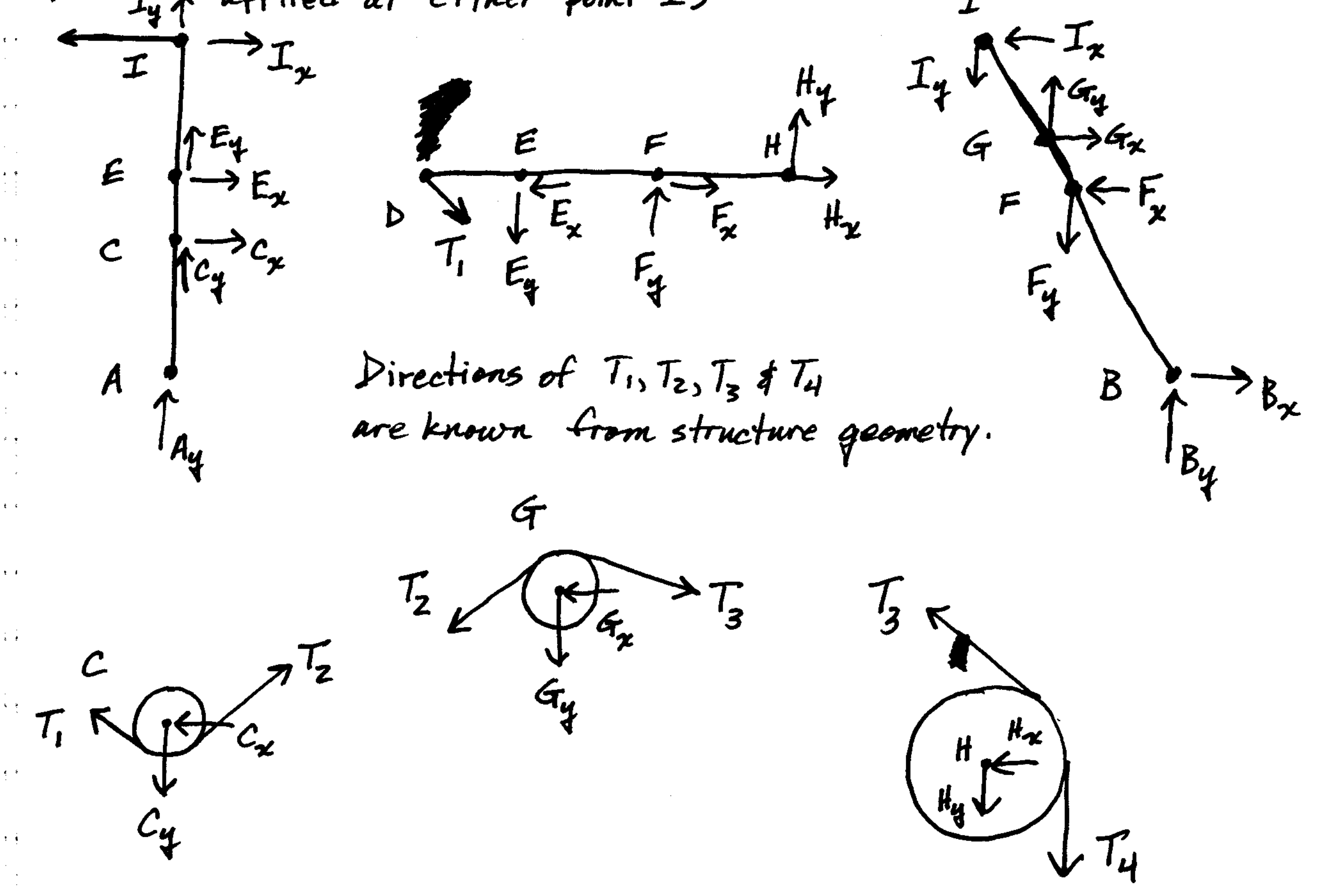
↳ support reactions due to "removal" of ground

A is free to traslate in the x-direction and free to rotate so there is no A_x or M_z^A (couple at A).

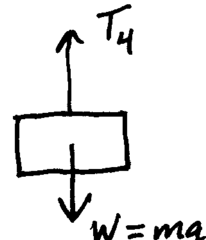
B is free to rotate so there is no M_z^B (couple at B).

Components

(Note that P, or a combination of loads that sums to P, can be applied at either point I)



Directions of T_1, T_2, T_3 & T_4 are known from structure geometry.

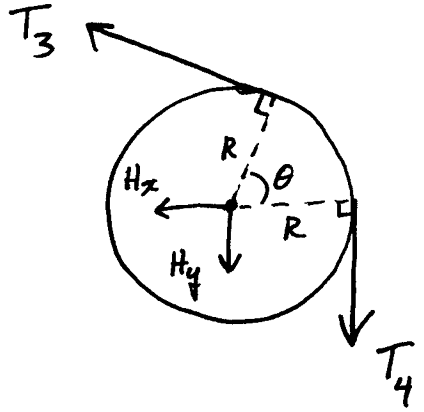


All internal forces come in equal but opposite pairs on the corresponding component FBDs. When you "add" all of the component FBDs together the result must be the FBD of the entire structure.

Also, notice that there are 19 unknown forces and 19 equilibrium equations from the 7 FBDs ($\Sigma F_x = 0$ and $\Sigma M_z = 0$ are trivially satisfied for the hanging mass).

Analysis of a weightless pulley on a frictionless pin with only cable loads and pin forces. No applied couples or forces applied away from the pin are allowed if the conclusions from this analysis are to be valid.

Consider the pulley at H



$$\Sigma M_z^H = 0$$

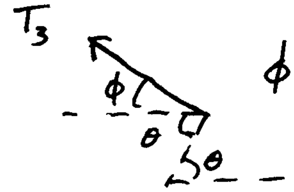
$$-T_4 R + T_3 R = 0$$

$$\rightarrow \quad T_3 = T_4$$

Note that if there were a couple C_H applied at point H (or to any point on the pulley) then this moment equation would be:

$$-T_4 R + T_3 R + C_H = 0 \rightarrow T_3 \neq T_4$$

We can then also solve for H_x and H_y.



$$\phi = 90^\circ - \theta$$

$$\Sigma F_x = -H_x - T_3 \cos\phi = 0$$

$$\Sigma F_y = -H_y - T_4 + T_3 \sin\phi = 0$$

$$\rightarrow H_x = -T_3 \cos\phi \quad , \quad H_y = T_3 \sin\phi - T_4$$