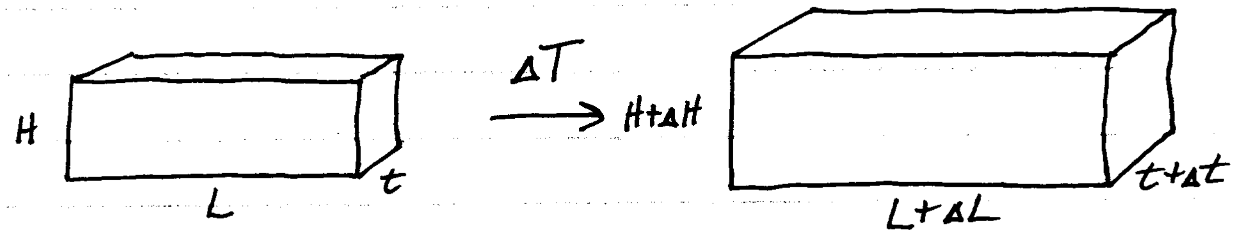
In addition to their response to stress materials also strain when exposed to a change in temperature.



For an isotropic material the axial strain in each direction due to the temperature change is the same.

$$\varepsilon_L = \frac{\Delta L}{L} = \varepsilon_H = \frac{\Delta H}{H} = \varepsilon_t = \frac{\Delta t}{t} = \alpha \Delta T$$

Coefficient of Thermal Expansion

α is a material property

Under a combination of load and temperature change the constitutive behavior becomes:

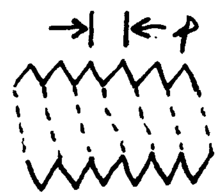
$$\varepsilon = \frac{\sigma}{E} + \alpha \Delta T \quad \text{for uniaxial loading}$$

elastic strain

thermal strain

Nuts and Bolts

For the analysis of nuts and bolts we need to know the position of the nut and how this position changes as it is turned with respect to the bolt.



threads

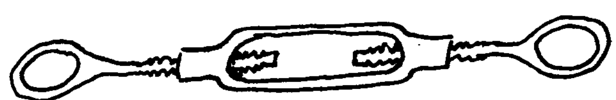
p = pitch of the threads

A mating nut will move a distance of np along the bolt as it is turned.

$$\delta = np \quad \text{for a nut/bolt connection}$$

n ← number of revolutions/turns

Double-acting turnbuckles are used to join cables together and act like a nut/bolt connection on each end.

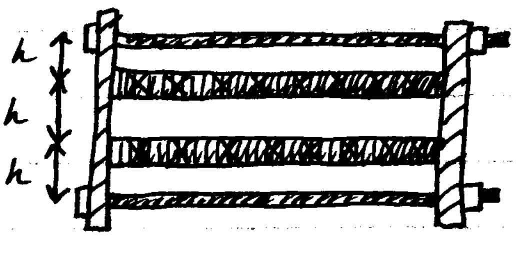


double-acting turnbuckle

In this case each side contributes np such that

$$\delta = 2np \quad \text{for a double-acting turnbuckle}$$

Example



Ⓐ E_b, A_b, p, α_b
Ⓑ E, A, L, α
Ⓒ $E, A, L+\Delta, \alpha$
Ⓓ E_b, A_b, p, α_b

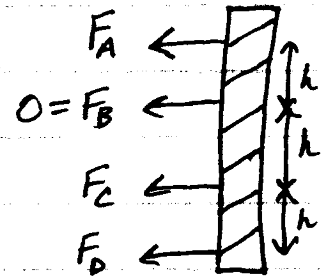
Bar Ⓒ has a misfit of Δ with respect to bar Ⓑ.

a) Determine the number of turns required in the bolts at Ⓐ and Ⓓ to keep the rigid plates perpendicular to the bars and just bring them into contact with bar Ⓑ.

b) Determine the number of turns in the bolts to keep the plates perpendicular and bring bar Ⓑ to a pre-stress of [redacted] $-\beta E$.

c) Determine the state of deformation of the system if the left plate remains perpendicular and a temperature change ΔT is imposed.

a) Right plate :



$$\Sigma F_x = -F_A - F_C - F_D = 0$$

$$\Sigma M_z^A = -F_D(3h) - F_C(2h) = 0$$

$$F_C = -\tfrac{3}{2} F_D$$

Some overall kinematics of the system tells us that $\delta_C = -\Delta$.

$$\varepsilon_c = \frac{\delta_c}{L+\Delta} \approx \frac{\delta_c}{L} \quad \text{for } L >> \Delta$$

This is a valid approximation for small misfits.

$$\varepsilon_c = \frac{-\Delta}{L} = \frac{\sigma_c}{E} \rightarrow \sigma_c = -\frac{E\Delta}{L}$$

$$F_c = \sigma_c A = -\frac{EA}{L}\Delta$$

$$\rightarrow F_D = \frac{2}{3}\frac{EA}{L}\Delta$$

$$\rightarrow F_A = -F_c - F_D = \frac{1}{3}\frac{EA}{L}\Delta$$

Bolt A : $F_A = \frac{1}{3}\frac{EA}{L}\Delta \rightarrow \sigma_A = \frac{F_A}{A_B} = \frac{1}{3}\frac{EA}{LA_b}\Delta$

We need $\delta_A^{\sigma} + \delta_A^{turns} = -\Delta$

$$\delta_A^{\sigma} = \varepsilon_A L = \frac{\sigma_A}{E_b}L = \frac{1}{3}\frac{EA}{E_b A_b}\Delta$$

again using $L >> \Delta$

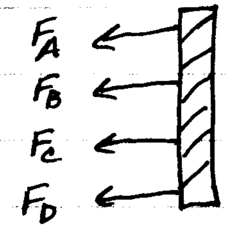
$$\rightarrow \delta_A^{turns} = -\left(1 + \frac{1}{3}\frac{EA}{E_b A_b}\right)\Delta = n_A p$$

$$\rightarrow \boxed{n_A = -\left(1 + \frac{1}{3}\frac{EA}{E_b A_b}\right)\frac{\Delta}{p}}$$

Using exactly the same steps for Bolt D we get

$$\boxed{n_D = -\left(1 + \frac{2}{3}\frac{EA}{E_b A_b}\right)\frac{\Delta}{p}}$$

b)



Before we analyze equilibrium, let's use what we know about F_B and F_C.

$$\sigma_B = -\beta E = \frac{F_B}{A} \rightarrow F_B = -\beta E A$$

$$\varepsilon_B = \sigma_B / E = -\beta \rightarrow \delta_B = -\beta L$$

In order to keep the plate perpendicular this means that $\delta_c = -\Delta - \beta L$

$$\rightarrow \varepsilon_c = -\frac{\Delta}{L} - \beta \quad \text{(again using } \Delta \ll L\text{)}$$

$$\rightarrow \sigma_c = -E\left(\beta + \frac{\Delta}{L}\right) \rightarrow F_c = -EA\left(\beta + \frac{\Delta}{L}\right)$$

Now let's look at equilibrium.

$$\Sigma F_x = -F_A - F_B - F_C - F_D = 0$$

$$-F_A + \beta E A + \beta E A + \frac{\Delta}{L} E A - F_D = 0$$

$$\Sigma M_z^A = -F_D(3h) + EA\left(\beta + \frac{\Delta}{L}\right)(2h) + EA\beta(h) = 0$$

$$\rightarrow F_D = EA\beta + \frac{2}{3} EA \frac{\Delta}{L}$$

$$\rightarrow F_A = EA\beta + \frac{1}{3} EA \frac{\Delta}{L}$$

Bolt (A): $\underbrace{\delta_A^{\sigma}}_{\frac{F_A L}{E_b A_b}} + \underbrace{\delta_A^{turns}}_{n_A p} = -\Delta - \beta L$

$$\frac{EA}{E_b A_b}\beta L + \frac{1}{3}\frac{EA}{E_b A_b}\Delta + n_A p = -\Delta - \beta L$$

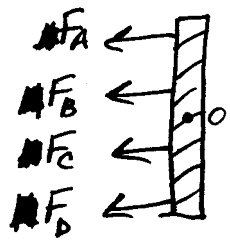
$$n_A p = -\left(1 + \frac{1}{3}\frac{EA}{E_b A_b}\right)\Delta - \left(1 + \frac{EA}{E_b A_b}\right)\beta L$$

$$\boxed{n_A = -\left(1 + \frac{1}{3}\frac{EA}{E_b A_b}\right)\frac{\Delta}{p} - \left(1 + \frac{EA}{E_b A_b}\right)\frac{\beta L}{p}}$$

Same steps $\longrightarrow$ $\boxed{n_D = -\left(1 + \frac{2}{3}\frac{EA}{E_b A_b}\right)\frac{\Delta}{p} - \left(1 + \frac{EA}{E_b A_b}\right)\frac{\beta L}{p}}$

c) Everything is prestressed as in part (b). After the temperature change the right plate does not necessarily remain perpendicular. To analyze this part of the problem we can look at the changes from the state in part (b) such that the total forces and deflections are those in (b) plus the changes.

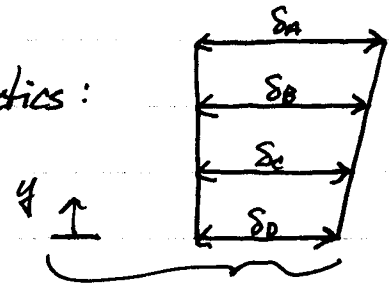
Equilibrium of the ΔF's



$$\Sigma F_x = -F_A - F_B - F_C - F_D = 0 \quad (1)$$

$$\Sigma M_z^o = (F_A - F_D)\frac{3h}{2} + (F_B - F_C)\frac{h}{2} = 0 \quad (2)$$

System kinematics:



$$\delta_B = \frac{2}{3}\delta_A + \frac{1}{3}\delta_D \quad (3)$$

$$\delta_C = \frac{1}{3}\delta_A + \frac{2}{3}\delta_D \quad (4)$$

$$\delta(y) = \delta_D + \frac{\delta_A - \delta_D}{3h} y$$

$$\delta_A = \varepsilon_A L = L\left(\frac{F_A}{E_b A_b} + \alpha_b \Delta T\right) \quad (5)$$

$$\delta_B = \varepsilon_B L = L\left(\frac{F_B}{EA} + \alpha \Delta T\right) \quad (6)$$

$$\delta_C = \varepsilon_C L = L\left(\frac{F_C}{EA} + \alpha \Delta T\right) \quad (7)$$

$$\delta_D = \varepsilon_D L = L\left(\frac{F_D}{E_b A_b} + \alpha_b \Delta T\right) \quad (8)$$

8 Equations for $F_A, F_B, F_C, F_D, \delta_A, \delta_B, \delta_C, \delta_D$

(6), (5), (8) in (3) $\rightarrow$ $$\frac{F_B}{EA} + \alpha \Delta T = \frac{2}{3}\frac{F_A}{E_b A_b} + \frac{2}{3}\alpha_b \Delta T + \frac{1}{3}\frac{F_D}{E_b A_b} + \frac{1}{3}\alpha_b \Delta T$$

(7), (5), (8) in (4) $\rightarrow$ $$\frac{F_C}{EA} + \alpha \Delta T = \frac{1}{3}\frac{F_A}{E_b A_b} + \frac{2}{3}\frac{F_D}{E_b A_b} + \alpha_b \Delta T$$

$$F_B = \frac{2}{3}\frac{EA}{E_b A_b} F_A + \frac{1}{3}\frac{EA}{E_b A_b} F_D + EA(\alpha_b - \alpha)\Delta T$$

$$F_C = \frac{1}{3}\frac{EA}{E_b A_b} F_A + \frac{2}{3}\frac{EA}{E_b A_b} F_D + EA(\alpha_b - \alpha)\Delta T$$

$$① \rightarrow \left(1+\frac{EA}{E_bA_b}\right)F_A + \left(1+\frac{EA}{E_bA_b}\right)F_D + 2EA(\alpha_b-\alpha)\Delta T = 0$$

$$② \rightarrow 3F_A - 3F_D + \frac{1}{3}\frac{EA}{E_bA_b}F_A - \frac{1}{3}\frac{EA}{E_bA_b}F_D = 0$$

$$② \rightarrow F_A = F_D$$

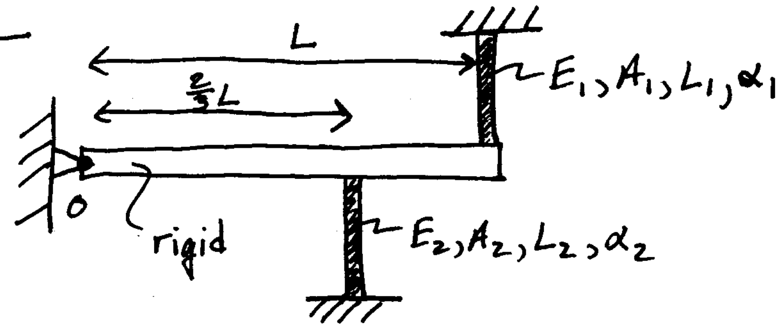
$$① \rightarrow F_A = F_D = \frac{EA(\alpha-\alpha_b)\Delta T}{1+\frac{EA}{E_bA_b}}$$

$$\rightarrow F_C = F_B = -\frac{EA(\alpha-\alpha_b)\Delta T}{1+EA/E_bA_b}$$

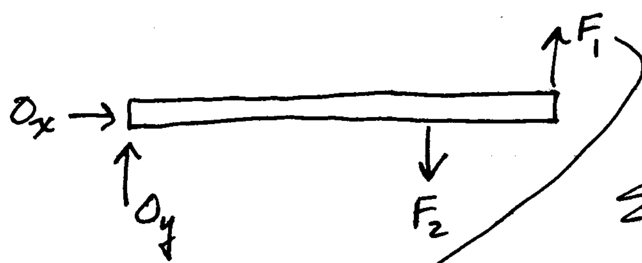
$$\rightarrow \boxed{\delta_A = \delta_B = \delta_C = \delta_D = \frac{(\alpha EA - \alpha_b E_bA_b)\Delta T L}{EA + E_bA_b}}$$

So, for this problem the plate does remain perpendicular after the temperature change. This would <u>not</u> have happened if each bar had different properties or if there was asymmetry in the geometric configuration.

Example



Determine the deformations due to a temperature change ΔT.



$$\Sigma F_x = \boxed{O_x = 0}$$

$$\Sigma M_z^O = F_1 L - F_2 \tfrac{2}{3} L = 0$$

$$\boxed{F_2 = \tfrac{3}{2} F_1}$$

$$\Sigma F_y = O_y - F_2 + F_1 = 0$$

$$\rightarrow \boxed{O_y = \tfrac{1}{2} F_1}$$

sign choice is important!

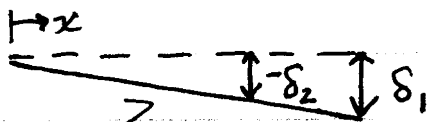
$$F_1 = \sigma_1 A_1, \quad \delta_1 = \varepsilon_1 L_1, \quad \varepsilon_1 = \underbrace{\frac{\sigma_1}{E_1}}_{\text{elastic strain}} + \underbrace{\alpha_1 \Delta T}_{\text{thermal strain}}$$

$$\rightarrow \boxed{\delta_1 = \underbrace{\frac{F_1 L_1}{E_1 A_1}}_{\text{elastic deflection}} + \underbrace{\alpha_1 L_1 \Delta T}_{\text{thermal deflection}}}$$

Similarly: $\delta_2 = \dfrac{F_2 L_2}{E_2 A_2} + \alpha_2 L_2 \Delta T$

$$\boxed{\delta_2 = \frac{3}{2}\frac{F_1 L_2}{E_2 A_2} + \alpha_2 L_2 \Delta T}$$

System kinematics:



$$\boxed{\delta(x) = \delta_1 \frac{x}{L}}$$

$$\delta_2 = -\delta\left(\tfrac{2}{3}L\right) = -\frac{2}{3}\delta_1$$

$$\delta_2 = -\frac{2}{3}\delta_1 \rightarrow \frac{3}{2}\frac{F_1 L_2}{E_2 A_2} + \alpha_2 L_2 \Delta T = -\frac{2}{3}\frac{F_1 L_1}{E_1 A_1} - \frac{2}{3}\alpha_1 L_1 \Delta T$$

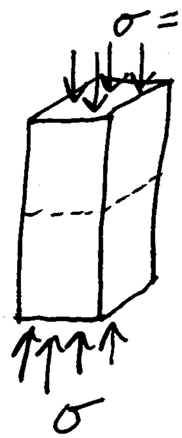
$$\frac{9E_1A_1L_2 + 4E_2A_2L_1}{6E_1A_1E_2A_2} F_1 = -\left(\alpha_2 L_2 + \frac{2}{3}\alpha_1 L_1\right)\Delta T$$

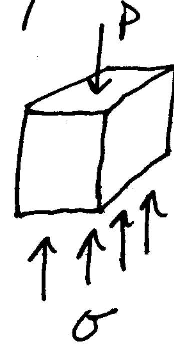
$$\boxed{F_1 = \frac{-6E_1A_1E_2A_2}{9E_1A_1L_2 + 4E_2A_2L_1}\left(\alpha_2 L_2 + \frac{2}{3}\alpha_1 L_1\right)\Delta T}$$

Everything else can be determined from the boxed equations.

Stresses on inclined sections

Now that we know a little bit about stresses, let's consider the following problem.
Take a brick and place a compressive force on it. Let's say you know that the brick material has a compressive/crushing strength σ_c and a shear strength τ_c.
Will the brick fail by compression or shear?



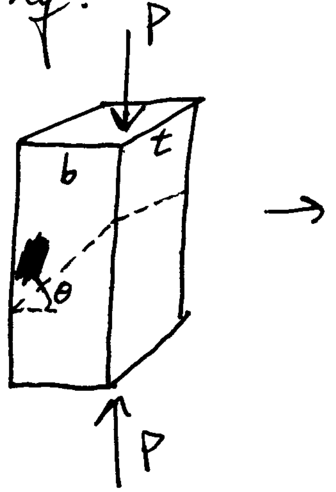


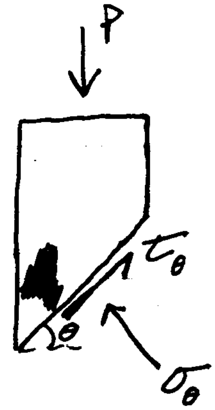
$$\Sigma F_y = -P + \sigma A = 0$$
$$\sigma = P/A$$

↑ we have used a convention when + is compression

So there is no shear, right?

Wrong!

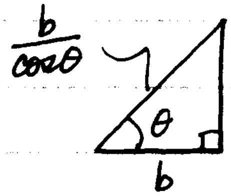




$$\Sigma F_y = -P + (\sigma_\theta \cos\theta) A_\sigma + (\tau_\theta \sin\theta) A_\tau = 0$$

$$\Sigma F_x = (-\sigma_\theta \sin\theta) A_\sigma + (\tau_\theta \cos\theta) A_\tau = 0$$

What are A_σ and A_τ? Well, first you should recognize that σ and τ act on the same area, so $A_\sigma = A_\tau$. Then, a little geometry gives the answer.



$$\rightarrow A_\sigma = A_\tau = \frac{bt}{\cos\theta} = \frac{A}{\cos\theta}$$

$$\Sigma F_x \rightarrow \tau_\theta = \sigma_\theta \frac{\sin\theta}{\cos\theta}$$

$$\Sigma F_y \rightarrow \sigma_\theta \cos\theta \frac{A}{\cos\theta} + \sigma_\theta \frac{\sin\theta}{\cos\theta} \sin\theta \frac{A}{\cos\theta} = P$$

$$\sigma_\theta \cos^2\theta + \sigma_\theta \sin^2\theta = \frac{P}{A}\cos^2\theta$$

$$\boxed{\begin{aligned} \sigma_\theta &= \frac{P}{A}\cos^2\theta \\ \tau_\theta &= \frac{P}{A}\sin\theta\cos\theta \end{aligned}}$$

Note: $\sigma_\theta^{max} = \sigma_\theta(\theta=0) = \frac{P}{A}$

$$\tau_\theta^{max} = \tau_\theta(\theta=45°) = \frac{1}{2}\frac{P}{A}$$

So if $\tau_c < \frac{\sigma_c}{2}$ then the brick will actually fail due to shear!

We will revisit this issue in more detail later.