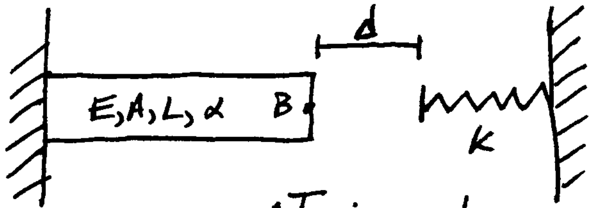
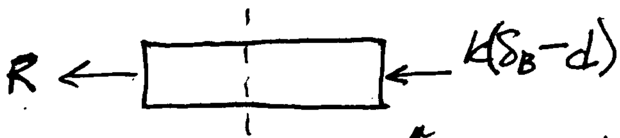
Misfit Example:



ΔT imposed

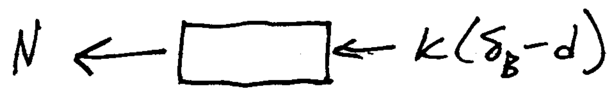
Determine δ_B [redacted] when the bar is in contact with the spring.



↑ Direction is important here because I am assuming δ_B is positive →

$$\Sigma F_x = -R - k(\delta_B - d) = 0$$

$$\rightarrow R = -k(\delta_B - d)$$



$$\Sigma F_x = -N - k(\delta_B - d) = 0 \Rightarrow N = -k(\delta_B - d)$$

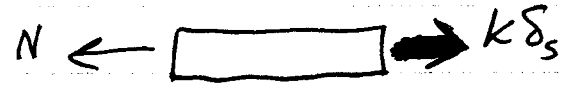
$$\delta_B = \frac{NL}{EA} + \alpha L \Delta T$$

$$\delta_B = \frac{-k(\delta_B - d)L}{EA} + \alpha L \Delta T$$

$$\delta_B \left(1 + \frac{kL}{EA}\right) = \frac{kdL}{EA} + \alpha L \Delta T$$

$$\boxed{\delta_B = \frac{\frac{kd}{EA} + \alpha \Delta T}{1 + kL/EA} L}$$

Another Approach



δ_s = deflection of spring $+ \leftarrow$

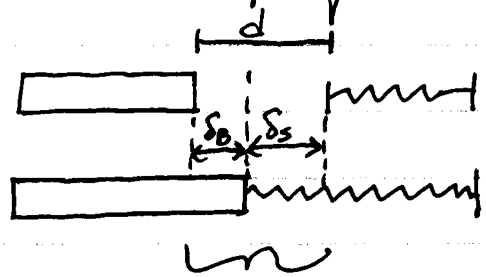
$$\Sigma F_x = -N + k\delta_s = 0$$

$$N = +k\delta_s$$

$$\delta_B = \frac{NL}{EA} + \alpha L \Delta T$$

$$\delta_B = \frac{+k\delta_s L}{EA} + \alpha L \Delta T$$

We need a relationship between δ_B and δ_s when the bar and spring are in contact.



$$\delta_B + \delta_s = d \rightarrow \delta_s = \delta_B - d$$

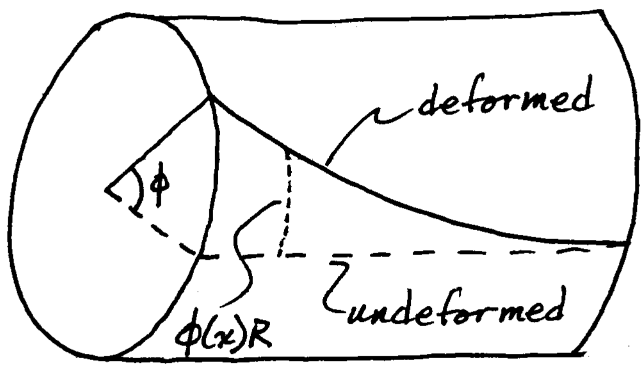
$$\rightarrow \delta_B = \frac{k(\delta_B - d)L}{EA} + \alpha L \Delta T$$

same equation as before.

$$\rightarrow \boxed{\delta_B = \frac{\frac{kd}{EA} + \alpha \Delta T}{1 + kL/EA} L}$$

Torsion of Circular Shafts

First, let's consider the deformation of a circular shaft of radius R due to a set of torques that tend to twist it. For now we will not specify the magnitude or distribution of the torques since we are only concerned with the kinematics of the deformation at this point.

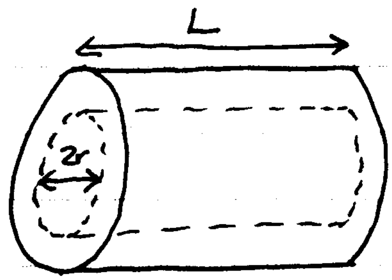


ϕ = angle of twist

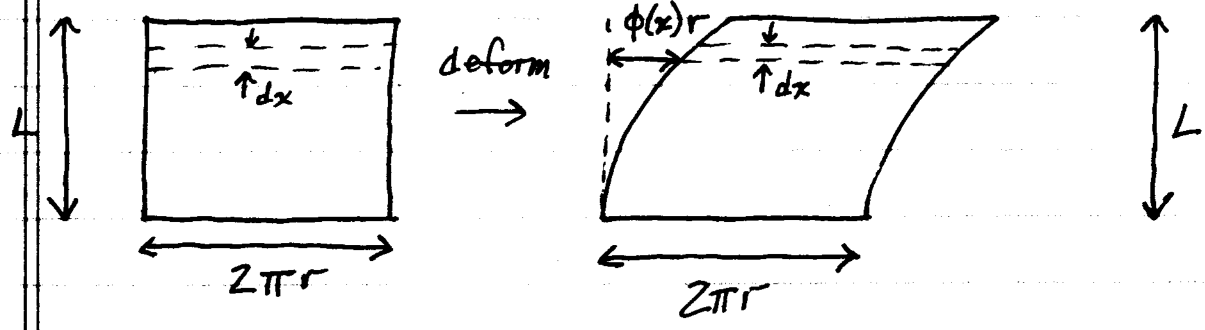
$\phi(x)$ = angle of twist at location x

$\rightarrow \phi(x)R$ = deflection due to the twist on the outer surface of the shaft

Next, let's consider a cylindrical "foil" of arbitrary radius r, and look at how it has been deformed.

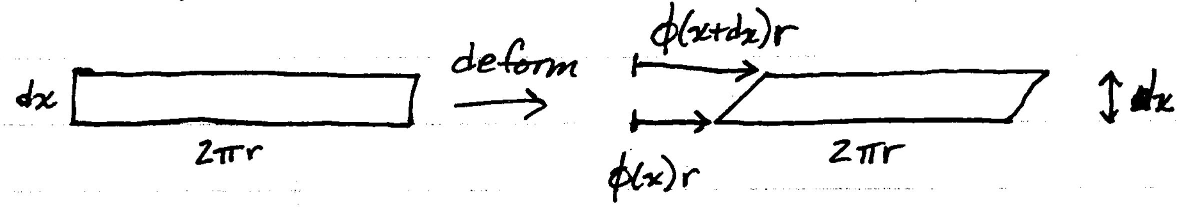


Unravel the foil



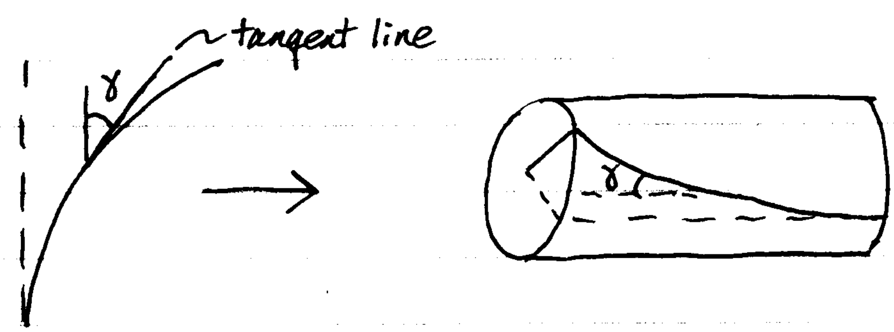
undeformed deformed

Look at the sliver:



$$\gamma = \frac{\delta_{horizontal}}{L_{vertical}} = \frac{\phi(x+dx)r - \phi(x)r}{dx}$$

$$\lim_{dx \to 0} \rightarrow \boxed{\gamma = \frac{d\phi}{dx} r}$$



At a given location x, $\gamma_{max} = \frac{d\phi}{dx} r_{max} = \frac{d\phi}{dx} R$

$$\rightarrow \gamma = \frac{d\phi}{dx} r = \frac{\gamma_{max}}{R} r = \gamma_{max} \frac{r}{R}$$

At the center of the shaft $r=0 \rightarrow \gamma = 0$.

So that is the kinematics for the deformation of a solid circular shaft with an angle of twist $\phi(x)$

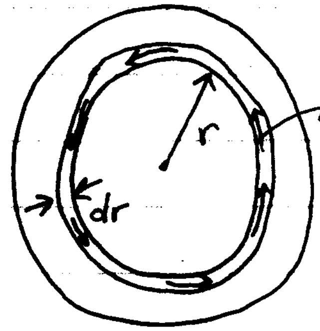
$$\boxed{\gamma(x,r) = \frac{d\phi}{dx} r}$$

What other principle involves strain?
Material constitutive response.

$$\boxed{\tau = G\gamma = G \frac{d\phi}{dx} r}$$

What's left? Equilibrium

Of course the equilibrium analysis will depend on how the shaft is loaded. First, let's consider a uniform torque.



$$\Sigma M_x = T - \int_0^R r(\tau \underbrace{2\pi r)\,dr}_{=dA \text{ for the ring}} = 0$$

$$\rightarrow \quad T = \int_0^R 2\pi \underbrace{\tau(r)}_{} r^2\, dr$$

τ is a function of r

Note that there is no x-dependence in τ for this loading.

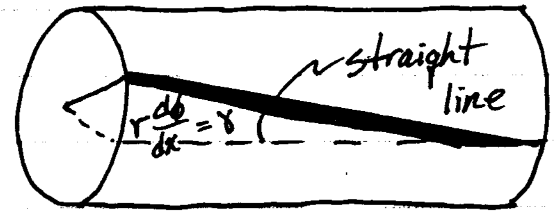
Recall: $\tau = G \frac{d\phi}{dx} r$

[redacted] if τ is independent of x then $\frac{d\phi}{dx}$ must be a constant.

$$\rightarrow \quad \frac{d\phi}{dx} = c_1 \rightarrow \phi = c_1 x + c_2$$

Then if $\phi(x=0) = \phi_0$ and $\phi(x=L) = \phi_L$

$$\rightarrow \quad \phi = \frac{\phi_L - \phi_0}{L} x + \phi_0$$



Let's take $\phi_0 = 0 \rightarrow \phi = \frac{\phi_L}{L} x$

Then: $T = \int_0^R 2\pi G \frac{\phi_L}{L} r^3 \, dr$

$$= G \frac{\phi_L}{L} \int_0^R 2\pi r^3 \, dr$$

$$= G \frac{\phi_L}{L} \int_0^{2\pi} \int_0^R r^3 \, dr \, d\theta$$

$$= G \frac{\phi_L}{L} \underbrace{\int_A r^2 \, dA}_{I_p}$$

This can be obtained from a general and more direct derivation

I_p polar moment of inertia of the area

$$\rightarrow T = \frac{G I_p}{L} \phi_L$$

or $\boxed{\phi_L = \frac{TL}{G I_p}}$

Compare this to $\delta = \frac{PL}{EA}$.

Also note that for a circular area,

$$I_p = \int_A r^2 \, dA = \int_0^{2\pi} \int_0^R r^3 \, dr \, d\theta = \frac{\pi}{2} R^4 = \frac{\pi}{32} D^4$$

What is the maximum shear stress in the shaft?

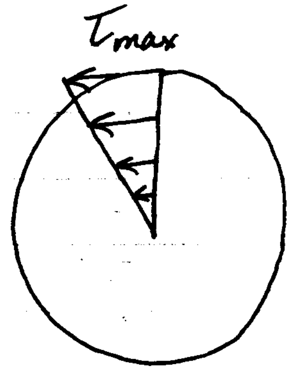
$$\tau = G \frac{d\phi}{dx} r$$

pure torsion $\rightarrow \frac{d\phi}{dx} = \frac{\phi}{L}$

$$\rightarrow \tau = G \frac{\phi}{L} r$$

$$= G \frac{T}{G I_p} r$$

$$\boxed{\tau = \frac{Tr}{I_p}}$$



$$\tau_{max} = \tau(r=R) = \frac{TR}{I_p}$$

$$\boxed{\tau_{max} = \frac{2T}{\pi R^3} = \frac{16T}{\pi D^3} \text{ for a circular shaft}}$$

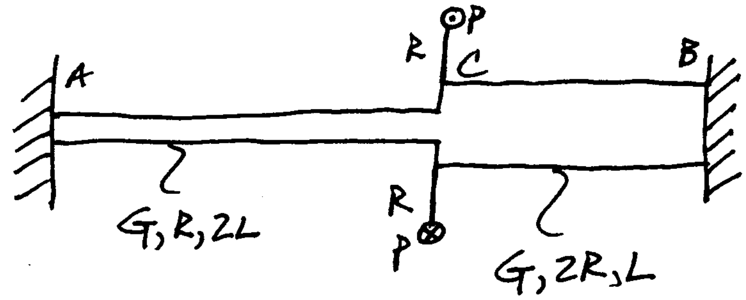
All derivations are also valid for tubes.

$$I_p = \int_A r^2 dA = \int_0^{2\pi} \int_{R_i}^{R_o} r^3 \, dr \, d\theta$$

$$I_p = \frac{\pi}{2}(R_o^4 - R_i^4) = \frac{\pi}{32}(D_o^4 - D_i^4)$$

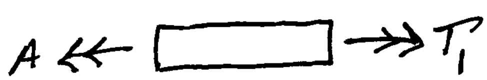
$R_o = R_i + t$ & $t \ll R_i$ $\rightarrow \underbrace{I_p \approx 2\pi R^3 t = \frac{\pi D^3 t}{4}}_{\text{Thin-walled tube approximations}}$

Example



$P\odot$ and $P\otimes$ are a set of forces applied out from and in to the page. These forces are applied to rigid rods of length R attached to the larger radius shaft. Determine the angle of twist of the force couple and the maximum shear stress in the system.

Thin shaft:



$\Sigma M_x \rightarrow T_1 = A$

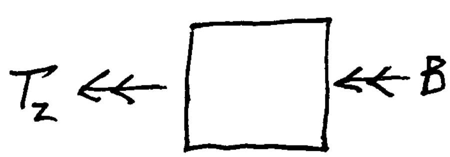
T_1 is independent of x in this part of the shaft.

$T_1, G, I_p = \frac{\pi}{2} R^4$ independent of x

$$\rightarrow \phi_c^{thin} = \frac{T_1 2L}{G I_p} = \frac{4L}{\pi G R^4} A$$

positive in the same direction as T_1 ($\twoheadrightarrow$)

Thick shaft:



$\Sigma M_x \rightarrow T_2 = -B$

Again $T_2, G, I_p = \frac{\pi}{2}(2R)^4$ independent of x $\rightarrow \phi_c^{thick} = \frac{T_2 L}{G I_p}$

positive $\twoheadleftarrow$

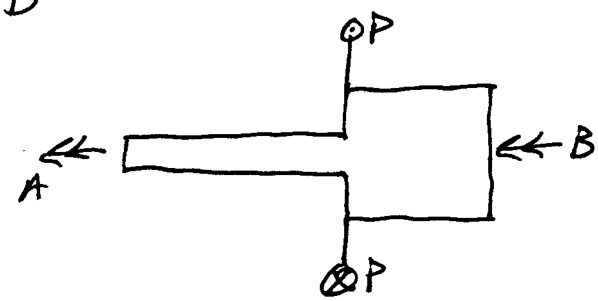
$$\phi_c^{thick} = \frac{L}{8\pi G R^4}(-B) \quad (+ \twoheadleftarrow)$$

Due to the sign difference $\phi_c = \phi_c^{thin} = -\phi_c^{thick}$

$$\rightarrow \quad \frac{4L}{\pi G R^4} A = \frac{L}{8\pi G R^4} B$$

$$32A = B$$

Overall equilibrium:



$$\Sigma M_x = -A - B + P(R+2R) + P(R+2R) = 0$$

$$A + B = 6PR$$

$$33A = 6PR$$

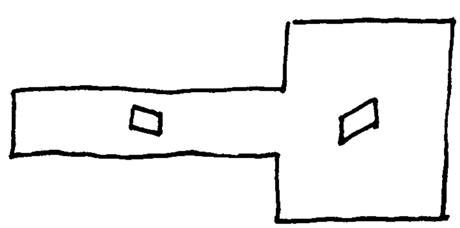
$$A = \frac{2}{11}PR \rightarrow B = \frac{64}{11}PR$$

$$\phi_c = \frac{4L}{\pi G R^4}\,\frac{2}{11}PR = \frac{8}{11\pi}\frac{PL}{GR^3}$$

Thin shaft: $\tau_{max} = \frac{T R_A}{I_p^A} = \frac{2}{11}PR\,\frac{2R}{\pi R^4} = \frac{4}{11\pi}\frac{P}{R^2}$

Thick shaft: $\tau_{max} = \frac{T R_B}{I_p^B} = -\frac{64}{11}PR\,\frac{2R}{8\pi R^4} = \frac{-16}{11\pi}\frac{P}{R^2}$

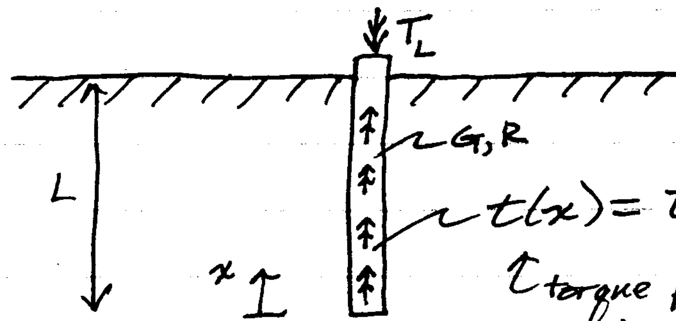
What does the minus sign mean?



Deformation (γ) & τ in opposite directions.

Thicker shaft will fail first when $\frac{16}{11\pi}\frac{P}{R^2} = \tau_{critical}$.

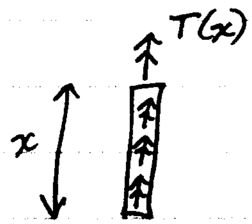
Example:



Determine $\phi(x)$ and τ_{max}.

$$\Sigma M_x = -T_L + \int_0^L t_0\left(1 - \frac{x}{L}\right)dx = 0$$

$$T_L = t_0\left(x - \frac{x^2}{2L}\right)\Big]_0^L = \frac{t_0 L}{2}$$



Direction choice implies positive angle of twist is also ↟.

$$\Sigma M_x = T(x) + \int_0^x t_0\left(1 - \frac{x'}{L}\right)dx' = 0$$

$$T(x) = -t_0\left(x - \frac{x^2}{2L}\right)$$

Note that $T(x)$ <u>is</u> dependent on x, so we <u>cannot</u> use $\phi = \frac{TL}{GI_p}$.

Recall: $\tau = G\frac{d\phi}{dx}r$ and $T = \int_0^R 2\pi\,\tau(r)\,r^2\,dr$

$$\rightarrow T = \int_0^R 2\pi r^3 G\frac{d\phi}{dx}\,dr$$, but $\frac{d\phi}{dx}$ does not depend on r

$$T = G\frac{d\phi}{dx}\underbrace{\int_0^R 2\pi r^2\,dr}_{I_p}$$

$$\rightarrow \frac{d\phi}{dx} = \frac{T(x)}{GI_p} \xrightarrow{\text{generalize}} \boxed{\frac{d\phi}{dx} = \frac{T(x)}{G(x)I_p(x)}}$$

For our problem: $\frac{d\phi}{dx} = \frac{-t_o}{GI_p}\left(x - \frac{x^2}{2L}\right)$

$$\int_0^x \frac{d\phi}{dx'}\,dx' = \int_0^x \frac{-t_o}{GI_p}\left(x' - \frac{x'^2}{2L}\right)dx'$$

$$\phi(x) - \phi(0) = \frac{-t_o}{GI_p}\left(\frac{x^2}{2} - \frac{x^3}{6L}\right)$$

We are interested in the relative angle of twist, so we can take $\phi(0)=0$, then

$$\phi(x) = \frac{-t_o}{GI_p}\left(\frac{x^2}{2} - \frac{x^3}{6L}\right)$$

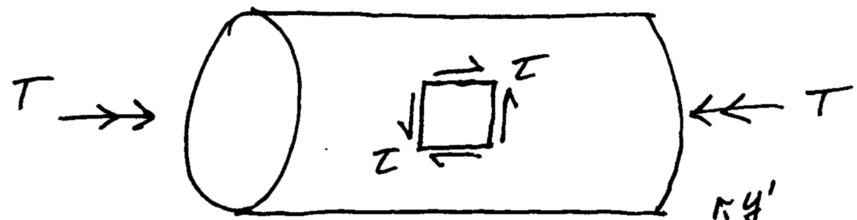
$$\phi_{max} = \phi(L) = \frac{-t_o}{GI_p}\frac{L^2}{3} = -\frac{2}{3\pi}\frac{t_o L^2}{GR^4}$$

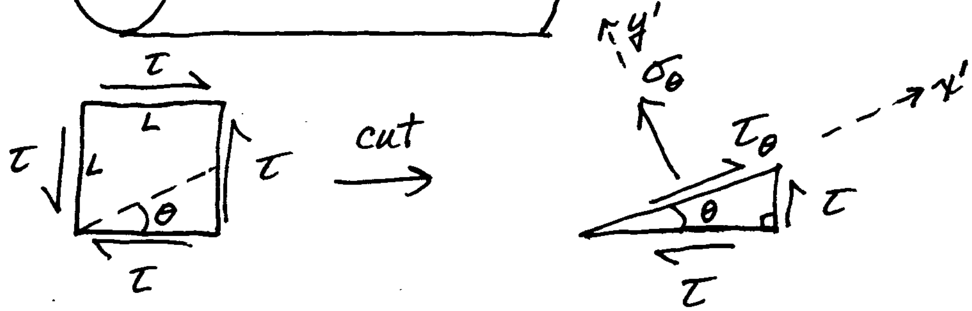
$$\tau_{max} = \frac{T_{max} R}{I_p}$$

$$T_{max} = T(L) = -\frac{t_o L}{2}$$

$$\tau_{max} = \frac{-t_o L}{2}\frac{2R}{\pi R^4} = \frac{-t_o L}{\pi R^3}$$

Stresses and strains in pure shear





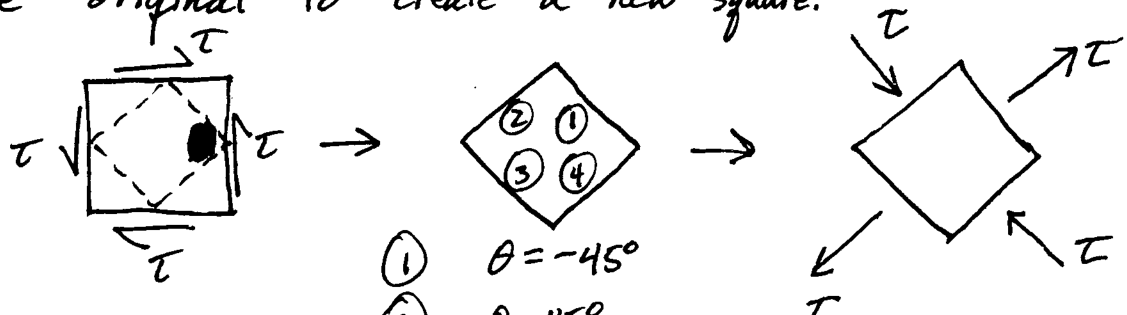
$$\sum F_{x'} = \tau_\theta \frac{Lt}{\cos\theta} - \tau\cos\theta\, Lt + \tau\sin\theta\, Lt\tan\theta \text{ [redacted] } = 0$$

$$\boxed{\tau_\theta = \tau\cos^2\theta - \tau\sin^2\theta = \tau\cos 2\theta}$$

$$\sum F_{y'} = \sigma_\theta \frac{Lt}{\cos\theta} + \tau\sin\theta\, Lt + \tau\cos\theta\, Lt\tan\theta \text{ [redacted] } = 0$$

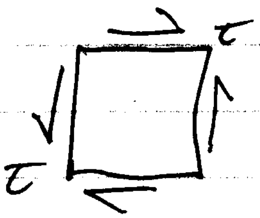
$$\boxed{\sigma_\theta = -\tau\sin\theta\cos\theta - \tau\cos\theta\sin\theta = -\tau\sin 2\theta}$$

Consider a rotation of our cuts of 45° from the original to "create" a new square.

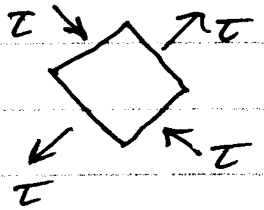


1. $\theta = -45°$
2. $\theta = 45°$
3. $\theta = 135°$
4. $\theta = -135°$

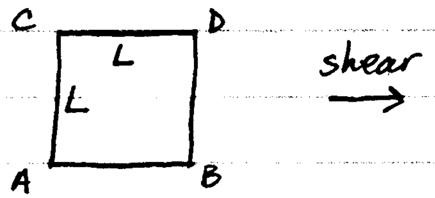
Shear in a coordinate system like $y \uparrow \rightarrow x$

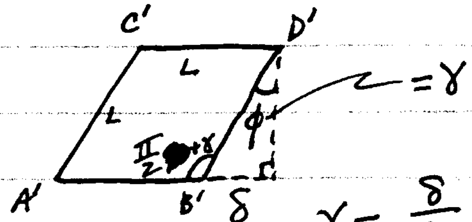


Looks like tension & compression in a system like $y' \nwarrow \nearrow x'$ 45°



What about strain?





$$\gamma = \frac{\delta}{L}$$

$$\tan\phi = \frac{\delta}{L}$$

for $\frac{\delta}{L} \ll 1$ $\quad \tan\phi = \phi = \frac{\delta}{L} = \gamma$

$$A'D' = \sqrt{L^2 + L^2 - 2L^2\cos\left(\frac{\pi}{2}+\gamma\right)} = \sqrt{2L^2(1+\sin\gamma)}$$

$$\gamma \ll 1 \rightarrow \sin\gamma \approx \gamma \rightarrow A'D' \approx \sqrt{2}\,L\sqrt{1+\gamma} \approx \sqrt{2}\,L\left(1+\frac{\gamma}{2}\right)$$

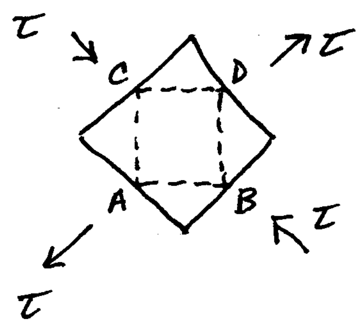
Taylor series for $\gamma \ll 1$

so $\quad A'D' \approx \sqrt{2}\,L\left(1+\frac{\gamma}{2}\right)$

$$\varepsilon_{AD} = \frac{A'D' - AD}{AD} = \frac{\sqrt{2}L\left(1+\frac{\gamma}{2}\right) - \sqrt{2}L}{\sqrt{2}L} = \frac{\gamma}{2}$$

Similarly for BC we can show

$$\varepsilon_{BC} = \frac{B'C' - BC}{BC} = \frac{\sqrt{2}L\left(1-\frac{\gamma}{2}\right) - \sqrt{2}L}{\sqrt{2}L} = -\frac{\gamma}{2}$$

Now consider 

Material Law: $\varepsilon_{AD} = \frac{\sigma_{Axial}}{E} - \frac{\nu \sigma_{transverse}}{E}$

$$= \frac{\tau}{E} - \frac{\nu}{E}(-\tau)$$

$$\varepsilon_{AD} = \tau\left(\frac{1+\nu}{E}\right) = \frac{\gamma}{2} = \frac{1}{2}\frac{\tau}{G}$$

$$\rightarrow \quad \boxed{G = \frac{E}{2(1+\nu)}}$$

You could also look at ε_{BC}:

$$\varepsilon_{BC} = \frac{\sigma_{Axial}}{E} - \frac{\nu \sigma_{transverse}}{E}$$

$$= \frac{1}{E}(-\tau) - \frac{\nu}{E}(\tau) = -\frac{\gamma}{2} = -\frac{1}{2}\frac{\tau}{G}$$

$$\rightarrow \quad G = \frac{E}{2(1+\nu)}$$