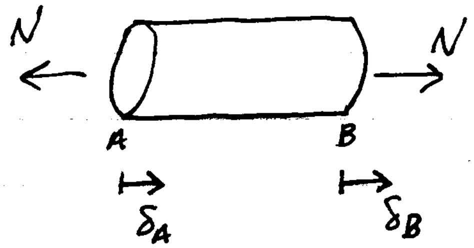
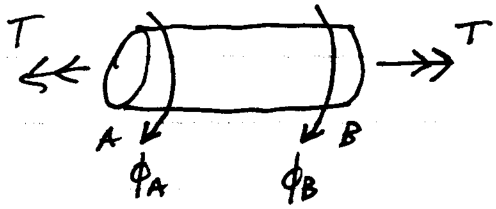
Analogies between torsion of circular shafts and tension/compression of bars.

If the following conventions are used for positive internal axial force and torque.



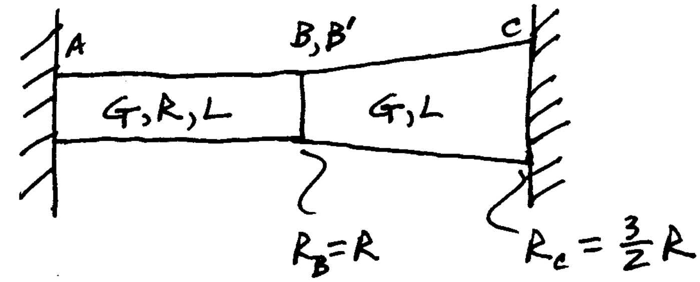


$$\underbrace{\delta = \delta_B - \delta_A = \frac{NL}{EA}}_{\text{Uniform } N, E, A} \qquad \underbrace{\phi = \phi_B - \phi_A = \frac{TL}{GI_p}}_{\text{Uniform } T, G, I_p}$$

$$\underbrace{\delta(x) - \delta_A = \int_{x_A}^{x} \frac{N(x')}{E(x')A(x')}\,dx'}_{\text{Non-uniform } N, E, A} \qquad \underbrace{\phi(x) - \phi_A = \int_{x_A}^{x} \frac{T(x')}{G(x')I_p(x')}\,dx'}_{\text{Non-uniform } T, G, I_p}$$

Generally it is good practice to use this convention for tension/compression of bars because we have other sources of axial deflection like $\delta_T = \alpha\Delta T L$ and $\delta_p = n\,p$ that are defined with the convention in mind. For torsion problems intuition can be an ally, but it must be used carefully.

Example



Shaft AB is given an initial twist ϕ_0 ↠, it is then bonded to the tapered shaft BC and then the original torque to produce ϕ_0 is removed.

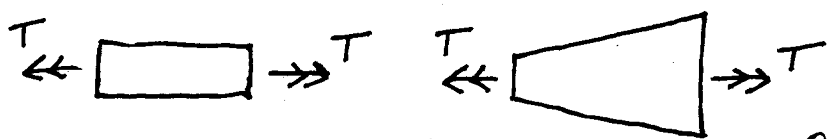
a) Determine the torque required to produce ϕ_0.
b) Determine the final $\phi(x)$ in the structure.
c) Determine τ_{max} in the structure.

a)

$$\phi_0 = \frac{T_0 L}{G I_p} \rightarrow T_0 = G I_p \frac{\phi_0}{L}$$

$$I_p = \frac{\pi}{2} R^4$$

b)



Initially $\underbrace{\phi_B - \cancel{\phi_A}^{\,0} = \phi_0 \quad \text{and} \quad \cancel{\phi_C}^{\,0} - \phi_{B'} = 0}$

$$\boxed{\phi_B - \phi_{B'} = \phi_0}$$

* This condition holds at all times after the two sides are bonded together.

Shaft AB: $\phi_B - \cancel{\phi_A}^{\,0} = \dfrac{TL}{G I_p}$

$$\phi_B = \frac{2TL}{\pi G R^4} \rightarrow \phi(x) = \frac{2TL}{\pi G R^4} \frac{(x - x_A)}{L}$$

Shaft BC : This shaft has a non-uniform I_p.

$$R(x) = R + \frac{1}{2}R\,\frac{x - x_B}{L}$$

$$\hookrightarrow I_p(x) = \frac{\pi}{2}R^4(x)$$

[redacted] $$\cancelto{0}{\phi_C} - \phi(x) = \int_x^{x_C} \frac{2T}{G\pi\left[R + \frac{1}{2}R\frac{x' - x_B}{L}\right]^4}\,dx'$$

$$u = R + \frac{1}{2}R\,\frac{x' - x_B}{L}$$

$$du = \frac{1}{2}\frac{R}{L}\,dx'$$

$$-\phi(x) = \int_{R + \frac{1}{2}R\frac{x - x_B}{L}}^{\frac{3}{2}R} \frac{2T}{G\pi}\,\frac{2L}{R}\,\frac{1}{u^4}\,du$$

$$= \frac{4TL}{\pi G R}\left[-\frac{1}{3}u^{-3}\right]_{R + \frac{1}{2}R\frac{x - x_B}{L}}^{\frac{3}{2}R}$$

$$= -\frac{4}{3\pi}\left(\frac{3}{2}\right)^{-3}\frac{TL}{GR^4} + \frac{4}{3\pi}\,\frac{TL}{GR\left(R + \frac{1}{2}R\frac{x - x_B}{L}\right)^3}$$

$$\boxed{\phi(x) = \frac{-4}{3\pi}\,\frac{TL}{GR}\left[\frac{1}{\left(R + \frac{1}{2}R\frac{x - x_B}{L}\right)^3} - \frac{8}{27}\,\frac{1}{R^3}\right]}$$

But we still need T.

$$\phi_{B'} = \phi(x_B) = \frac{-4}{3\pi}\,\frac{TL}{GR}\underbrace{\left[\frac{1}{R^3} - \frac{8}{27}\,\frac{1}{R^3}\right]}_{\frac{19}{27}\frac{1}{R^3}}$$

$$\phi_B - \phi_{B'} = \phi_0$$

$$\frac{2TL}{\pi G R^4} - \left(-\frac{4}{3\pi}\,\frac{19}{27}\right)\frac{TL}{GR^4} = \phi_0$$

$$\rightarrow \boxed{T = \frac{\pi G R^4}{2L}\overbrace{\left(1 + \frac{38}{81}\right)^{-1}}^{\frac{81}{119}}\phi_0}$$

Checks: We should expect,

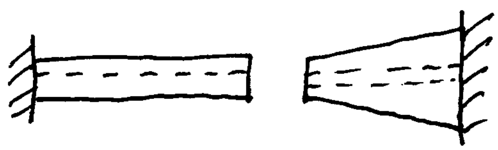
$$\phi_{B'} < 0 \;, \quad T > 0 \rightarrow \phi_{B'} < 0 \;\checkmark$$

$$0 < \phi_B < \phi_0 \;, \quad 0 < T < \frac{\pi G R^4}{2L}\phi_0 \rightarrow 0 < \phi_B < \phi_0 \;\checkmark$$

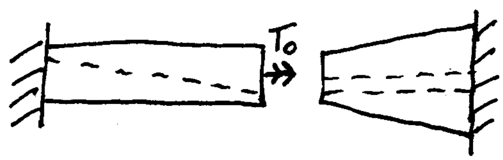
c) $$\tau_{max} = \frac{T\,R(x)}{I_p(x)} = \frac{2T}{\pi R^3(x)}$$

→ max occurs where $R(x)$ is smallest, which is in the straight bar.

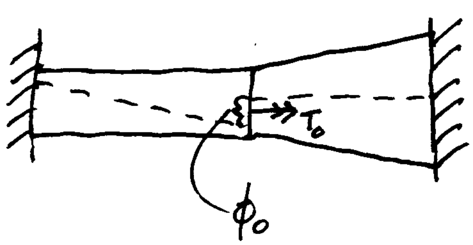
$$\therefore \boxed{\tau_{max} = \frac{2T}{\pi R^3} = \frac{81}{119} G \frac{R}{L} \phi_0}$$



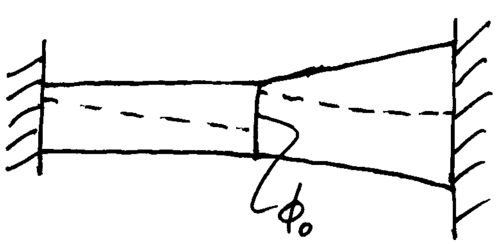
Initially unbonded and un-torqued



Unbounded & torqued

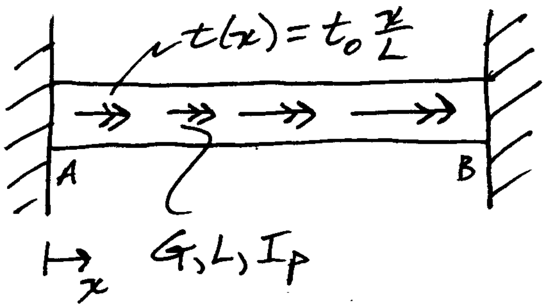


Bonded and torqued



Bonded and torque removed

Review for Test 1



Reactions at A & B?
ϕ_{max}?

$$T_A \twoheadrightarrow \underset{t(x)}{\boxed{\twoheadrightarrow \twoheadrightarrow \twoheadrightarrow}} \twoheadrightarrow T_B \qquad \Sigma M_x = T_A + T_B + \underbrace{\int_0^L t_0 \frac{x}{L}\,dx}_{\frac{1}{2}t_0 L} = 0$$

$$\boxed{T_A + T_B + \frac{1}{2}t_0 L = 0}$$

$$\cancelto{0}{\phi_B} - \cancelto{0}{\phi_A} = \int_0^L \frac{T(x)}{GI_p}\,dx$$

$$T_A \twoheadrightarrow \underset{t(x)}{\boxed{\twoheadrightarrow \twoheadrightarrow}} \twoheadrightarrow T(x) \qquad \Sigma M_x = T_A + T(x) + \int_0^x t_0 \frac{x'}{L}\,dx' = 0$$

$$T(x) = -T_A - \frac{1}{2}t_0 \frac{x^2}{L}$$

$$0 = \int_0^L \frac{-T_A - \frac{1}{2}t_0 \frac{x^2}{L}}{GI_p}\,dx$$

$$0 = \frac{1}{GI_p}\left[-T_A L - \frac{1}{6}t_0 L^2\right]$$

$$\rightarrow \boxed{T_A = -\frac{1}{6}t_0 L}$$

$$\rightarrow \boxed{T_B = -\frac{1}{3}t_0 L}$$

To find ϕ_{max} we need $\phi(x)$.

$$\phi(x) - \cancel{\phi_A}^{\,0} = \int_0^x \frac{T(x')}{G I_p}\, dx'$$

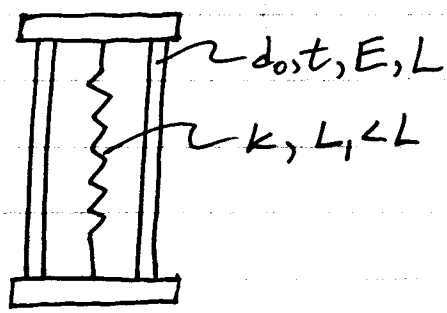
$$= \int_0^x \frac{\frac{1}{6} t_0 L - \frac{1}{2} t_0 \frac{x^2}{L}}{G I_p}\, dx$$

$$\phi(x) = \frac{1}{G I_p}\left[\frac{1}{6} t_0 L x - \frac{1}{6} t_0 \frac{x^3}{L}\right]$$

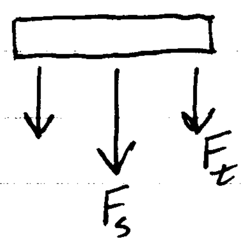
$$\frac{d\phi}{dx} = \frac{1}{G I_p}\left[\frac{1}{6} t_0 L - \frac{1}{2} t_0 \frac{x^2}{L}\right] = 0$$

$$\rightarrow x = \frac{L}{\sqrt{3}}$$

$$\boxed{\phi_{max} = \phi\left(\frac{L}{\sqrt{3}}\right) = \frac{1}{G I_p} \frac{1}{9\sqrt{3}} t_0 L^2}$$



The spring pulls on the rigid end caps to pre-stress the system. Determine the forces.



$$\Sigma F_y = -F_s - F_t = 0$$

$$F_s = -F_t$$

$$F_s = k\delta_s = k(L_2 - L_1)$$

$$F_t = \sigma_t A_t = \underbrace{E\varepsilon_t}_{\sigma_t}\underbrace{\pi d_o t}_{A_t} = E\underbrace{(\delta_t/L)}_{\varepsilon_t}\pi d_o t$$

$$F_t = (\pi d_o t/L)E\underbrace{\delta_t}_{L_2 - L}$$

Kinematic Constraint: Either recognize that (a) L_2 is the same for both the spring and the tube or (b) that the spring deflection and the tube deflection are related through the misfit, $\delta_s - \delta_t = L - L_1$.

(a) $\rightarrow$ $F_s = k(L_2 - L_1) = -F_t = -(\pi d_o t/L)E(L_2 - L)$

$$(k + (\pi d_o t/L)E)L_2 = kL_1 + \pi d_o t E$$

$$\rightarrow \boxed{L_2 = \frac{kL_1 + \pi d_o t E}{kL + \pi d_o t E}L} \quad \begin{cases} \rightarrow F_s = k(L_2 - L_1) \\ \rightarrow F_t = \frac{\pi d_o t E}{L}(L_2 - L) \end{cases}$$

(b) $\rightarrow$ $F_s = k\delta_s = -F_t = -\frac{\pi d_o t}{L}E\underbrace{(\delta_s - L + L_1)}_{\delta_t}$

$$\left(k + \frac{\pi d_o t E}{L}\right)\delta_s = \frac{\pi d_o t E}{L}(L - L_1)$$

$$\boxed{\delta_s = \frac{\pi d_o t E}{kL + \pi d_o t E}(L - L_1)}$$

$\rightarrow$ $F_s = k\delta_s$

$\rightarrow$ $F_t = \frac{\pi d_o t}{L}E(\delta_s - L + L_1)$

Algebra will show that these are equivalent.