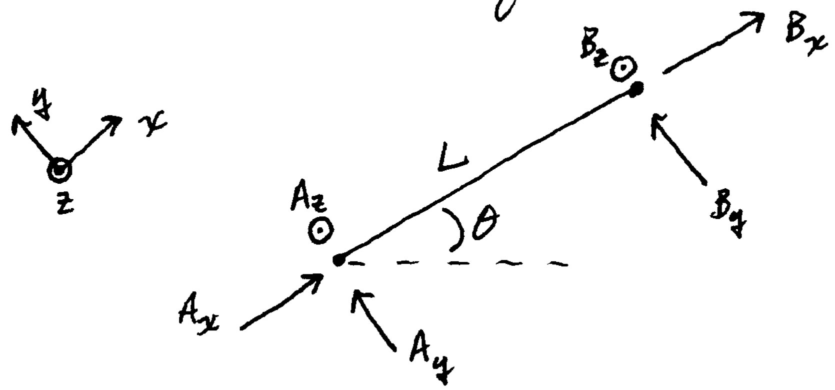
Trusses

A truss is a structure that is composed of members joined at their ends to form a rigid structure capable of supporting loads. In the analysis of trusses we are interested in the internal forces carried by each member so that a determination about the potential failure of the structure can be made. Generally, truss members are welded together or otherwise connected in a way such that the connection can clearly support a couple/moment. However, in the analysis of trusses it is usually satisfactory to assume that even these types of connections can be treated as pins.

If all of the truss members can be assumed to be (1) pin-ended, (2) weightless, and (3) loaded only at the pin ends, then the analysis of such structures can be simplified.

First, consider the equilibrium of one truss member.



Equilibrium analysis:

$$\sum F_x = A_x + B_x = 0$$
$$\sum F_y = A_y + B_y = 0$$
$$\sum F_z = A_z + B_z = 0$$

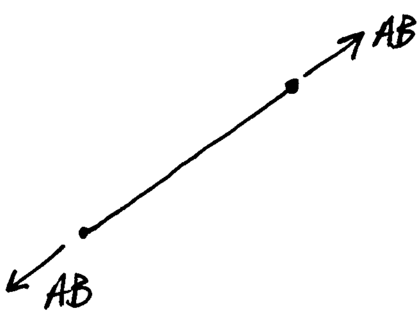
$$\sum \vec{M}_A = B_y L \vec{k} - B_z L \vec{j} = 0$$

$$\vec{j} \rightarrow B_z = 0 \quad \& \quad \sum F_z \rightarrow A_z = 0$$

$$\vec{k} \rightarrow B_y = 0 \quad \& \quad \sum F_y \rightarrow A_y = 0$$

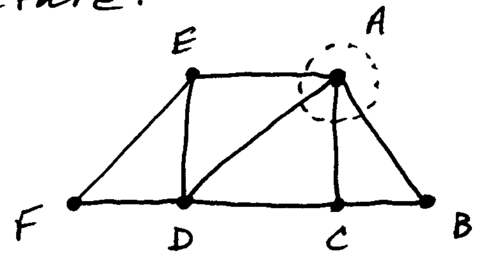
So, our solution is $B_y = A_y = B_z = A_z = 0$, and $B_x = -A_x$. In other words, any truss member that meets conditions 1 – 3 carries load along the line connecting the two pins.

A simplified FBD of truss member AB is then

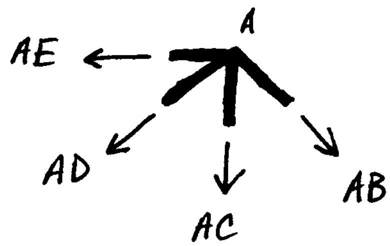


Here, AB is the unknown force magnitude, and we have chosen the direction such that we are assuming member AB is in tension. If our analysis then indicates that $AB < 0$, then this implies that member AB is in compression.

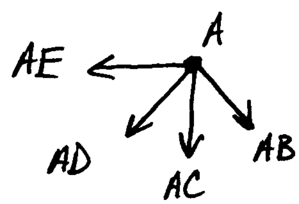
Next, let's consider one of the joints in a structure.



Imagine "cutting" this joint out from the rest of the structure. The FBD of the encircled region would look like,



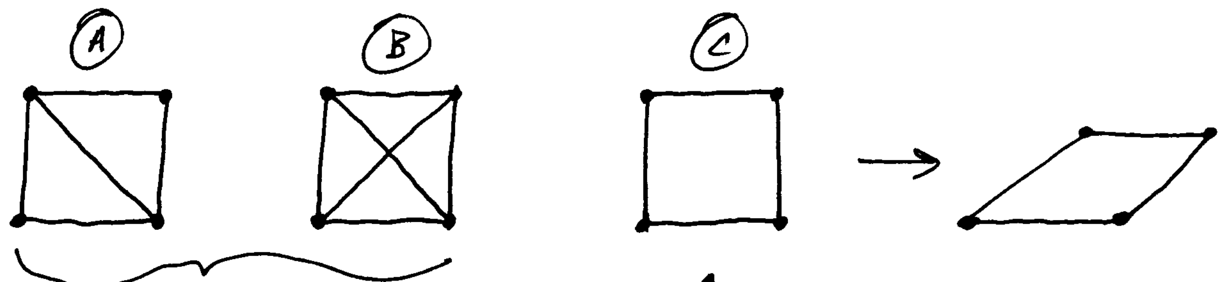
To avoid clutter we can imagine shrinking this circle such that the FBD of joint A looks like



Note that all of the truss member forces act through A. Hence moment equilibrium of joint A is satisfied trivially. Then, in 2-D there remains 2 (x and y) force balance equations, and in 3-D there are 3 (x, y and z) force balance equations.

Plane trusses (2-D) that are rigid require 3 external supports in order to prevent rigid body motions of the entire structure. Similarly, space trusses (3-D) that are rigid require 6 supports.

What is meant by rigid?



These are rigid.

This is not. It can collapse with no applied load. It is a <u>mechanism</u>.

How many unknowns are there in a general truss problem?

2-D: m members + 3 support reactions

3-D: m members + 6 support reactions

How many equations do we have to solve for these unknowns in a general truss problem?

Well, we can draw a FBD for each joint.

Then we have:

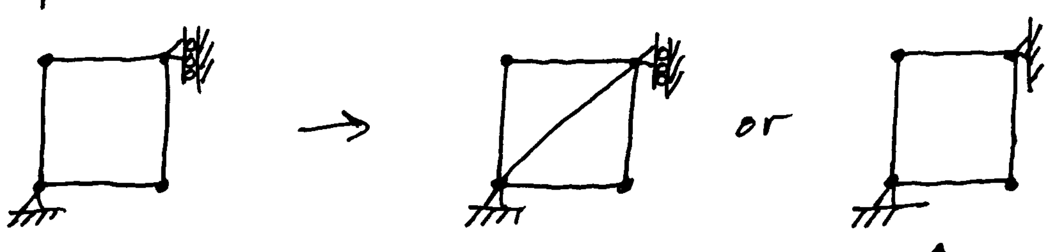
$$\left.\begin{array}{ll} \text{2-D:} & 2 \times j \text{ equations} \\ \text{3-D:} & 3 \times j \text{ equations} \end{array}\right\} j = \text{number of joints}$$

Let's look at these conditions for A, B and C

A: $m = 5$, $j = 4$ $\qquad m + 3 = 2j$ ✓
We could solve for all support reactions (assuming there are 3) and all member forces.

B: $m = 6$, $j = 4$ $\qquad m + 3 > 2j$
There are too many member forces to solve for. The structure is internally statically indeterminate.

C: $m = 4$, $j = 4$ $\qquad m + 3 < 2j$
More equations than unknowns. We either need to add truss members to rigidify (word?) the structure, or add external supports to prevent the mechanism motion.



4 unknown support reactions.

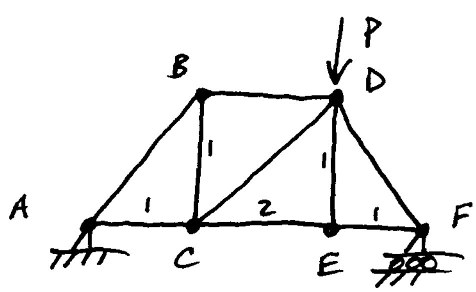
Method of Joints

Standard Procedure

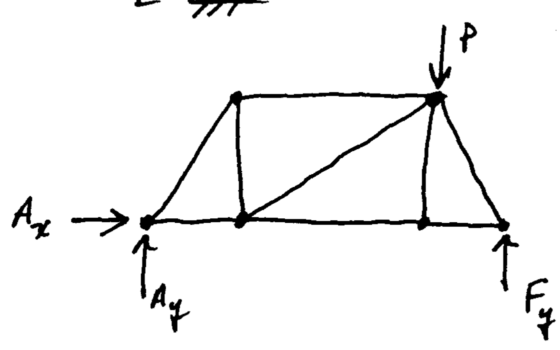
1) FBD of entire structure
2) Equilibrium analysis of entire structure
   This will solve for the external support reactions if there are 3 in 2D or 6 in 3D.
3) FBDs of each joint
4) Equilibrium analysis of each joint
   This will solve for all of the member forces leaving 3 (6) redundant equations in 2D (3D) to check for consistency in the solution.

Steps 1 and 2 can be skipped, but this usually makes the algebra more difficult and does not provide the redundant equations as checks.

Example:



Entire Structure:

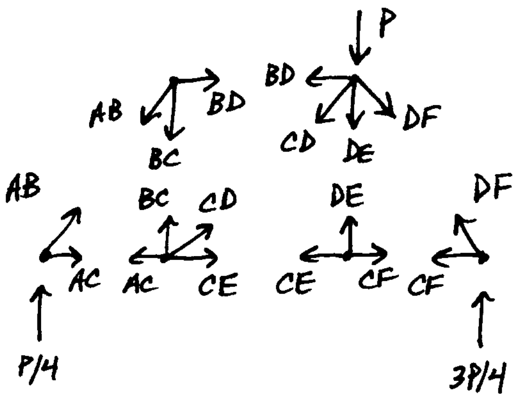


Equilibrium: $\Sigma F_x = A_x = 0$

$$\Sigma F_y = A_y + F_y - P = 0$$

$$\Sigma M_z^A = 4F_y - 3P = 0 \rightarrow F_y = \frac{3}{4}P, \quad A_y = \frac{1}{4}P$$

Joints:



Start at either end and move through the structure. Note, from inspection we can see that DE=0. Why?

(A) $\Sigma F_x = AB\frac{\sqrt{2}}{2} + AC = 0$

$\Sigma F_y = P/4 + AB\frac{\sqrt{2}}{2} = 0 \rightarrow AB = -\frac{\sqrt{2}}{4}P, \quad AC = \frac{1}{4}P$

(B) $\Sigma F_x = BD - AB\frac{\sqrt{2}}{2} = 0 \rightarrow BD = -\frac{1}{4}P$

$\Sigma F_y = -BC - AB\frac{\sqrt{2}}{2} = 0 \rightarrow BC = \frac{1}{4}P$

(C) $\Sigma F_x = -AC + CE + CD\frac{2}{\sqrt{5}} = 0$

$\Sigma F_y = BC + CD\frac{1}{\sqrt{5}} = 0 \rightarrow CD = -\frac{\sqrt{5}}{4}P, \quad CE = \frac{3}{4}P$

(E) $\Sigma F_x = -CE + CF = 0 \rightarrow CF = \frac{3}{4}P$

$\Sigma F_y = DE = 0 \rightarrow DE = 0$

(F) $\Sigma F_x = -CF - DF\frac{\sqrt{2}}{2} = 0 \rightarrow DF = -\frac{3\sqrt{2}}{4}P$

$\Sigma F_y = 3P/4 + DF\frac{\sqrt{2}}{2} = 3P/4 - 3P/4 = 0$ ✓

(D) $\Sigma F_x = -BD - CD\frac{2}{\sqrt{5}} + DF\frac{\sqrt{2}}{2} = 0$

$\frac{1}{4}P + \frac{\sqrt{5}}{4}P\frac{2}{\sqrt{5}} - \frac{3\sqrt{2}}{4}P\frac{\sqrt{2}}{2} = 0$ ✓

$\Sigma F_y = -P - CD\frac{1}{\sqrt{5}} - DE - DF\frac{\sqrt{2}}{2} = 0$

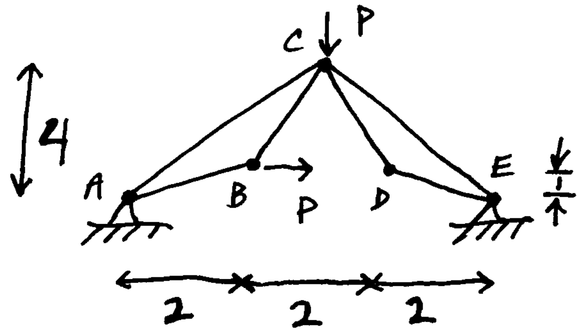
$-P + \frac{\sqrt{5}}{4}P\frac{1}{\sqrt{5}} - 0 + \frac{3\sqrt{2}}{4}P\frac{\sqrt{2}}{2} = 0$ ✓

Note: + → tension

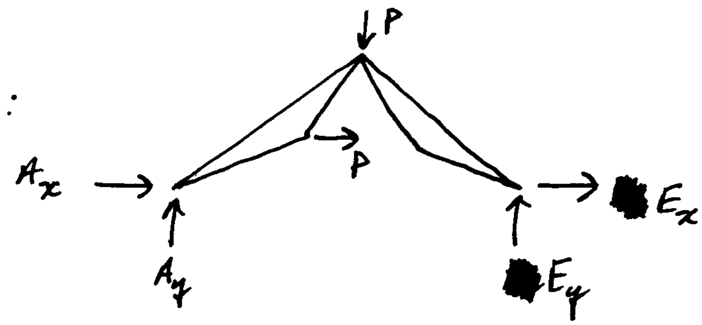
− → compression

For the forces as drawn above.

Example:

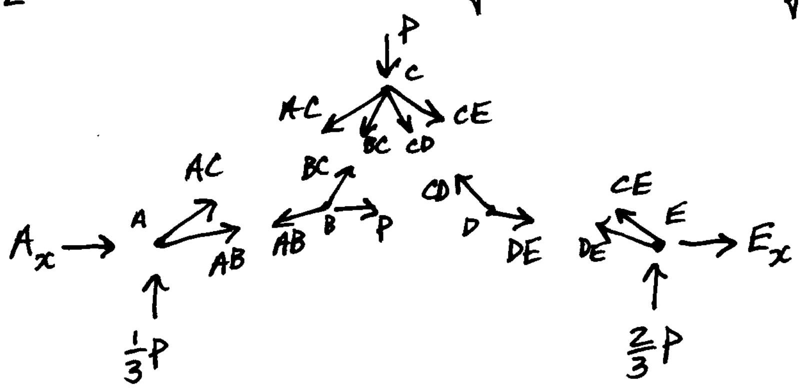


Entire Structure:



$$\Sigma F_x = A_x + E_x + P = 0$$
$$\Sigma F_y = A_y + E_y - P = 0$$
$$\Sigma M_z^A = -P\cdot 1 - P\cdot 3 + E_y\cdot 6 = 0 \rightarrow E_y = \frac{2}{3}P \rightarrow A_y = \frac{1}{3}P$$

Joints:



(A) $\Sigma F_x = A_x + AB\frac{2}{\sqrt{5}} + AC\frac{3}{\sqrt{25}} = 0$

$\Sigma F_y = \frac{1}{3}P + AB\frac{1}{\sqrt{5}} + AC\frac{4}{5} = 0$

(B) $\Sigma F_x = P - AB\frac{2}{\sqrt{5}} + BC\frac{1}{\sqrt{10}} = 0$

$\Sigma F_y = -AB\frac{1}{\sqrt{5}} + BC\frac{3}{\sqrt{10}} = 0$

$\rightarrow BC = \frac{\sqrt{2}}{3}AB \rightarrow P - AB\frac{2}{\sqrt{5}} + AB\frac{1}{3\sqrt{5}} = 0$

$\rightarrow AB = \frac{3}{\sqrt{5}}P$, $BC = \frac{\sqrt{2}}{\sqrt{5}}P$

Now we can go back to (A). We should have started at (B) and (D).

(A) $\rightarrow \frac{1}{3}P + \frac{3}{5}$ [redacted] $P + AC\frac{4}{5} = 0 \rightarrow AC =$ [redacted] $-\frac{7}{6}P$

$A_x + \frac{6}{5}P - \frac{7}{6}P\frac{3}{5} = 0 \rightarrow A_x = -\frac{1}{2}P$

Entire Structure $\rightarrow E_x = -\frac{1}{2}P$

(D) The two forces at D are not co-linear. This implies that both must be zero.

$$\left.\begin{aligned} \Sigma F_x &= DE\frac{2}{\sqrt{5}} - CD\frac{1}{\sqrt{10}} = 0 \\ \Sigma F_y &= -DE\frac{1}{\sqrt{5}} + CD\frac{3}{\sqrt{10}} = 0 \end{aligned}\right\} \rightarrow CD = DE = 0$$

(E) $\Sigma F_x = E_x - DE\frac{2}{\sqrt{5}} - CE\frac{3}{5} = 0$

$-\frac{1}{2}P - 0 - CE\,^3/_5 = 0 \rightarrow CE = -\frac{5}{6}P$

$\Sigma F_y = \frac{2}{3}P + DE\frac{1}{\sqrt{5}} + CE\frac{4}{5} = 0$

$\frac{2}{3}P + 0 + -\frac{5}{6}P\frac{4}{5} = 0$ ✓

(C) $\Sigma F_x = -AC\frac{3}{5} - BC\frac{1}{\sqrt{10}} + CD\frac{1}{\sqrt{10}} + CE\frac{3}{5} = 0$

$\frac{7}{6}\frac{3}{5}P - \frac{\sqrt{2}}{\sqrt{5}}\frac{1}{\sqrt{2}\sqrt{5}}P + 0 - \frac{5}{6}\frac{3}{5}P = 0$

$\frac{21}{30}P - \frac{6}{30}P - \frac{15}{30}P = 0$ ✓

$\Sigma F_y = -P - AC\frac{4}{5} - BC\frac{3}{\sqrt{10}} - CD\frac{3}{\sqrt{10}} - CE\frac{4}{5} = 0$

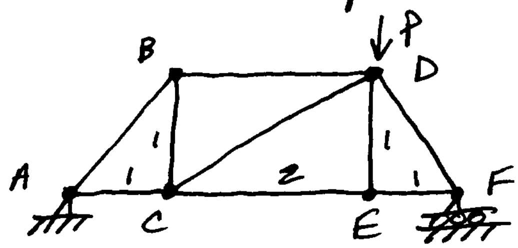
$-P + \frac{7}{6}\frac{4}{5}P - \frac{\sqrt{2}}{\sqrt{5}}\frac{3}{\sqrt{2}\sqrt{5}}P - 0 + \frac{5}{6}\frac{4}{5}P = 0$

$-\frac{30}{30}P + \frac{28}{30}P - \frac{18}{30}P + \frac{20}{30}P = 0$ ✓

Method of Sections

If you need the forces in each of the truss members, then the method of joints is as good as any other method. However, if you are only interested in a few of the member forces, then the method of sections might be the way to go.

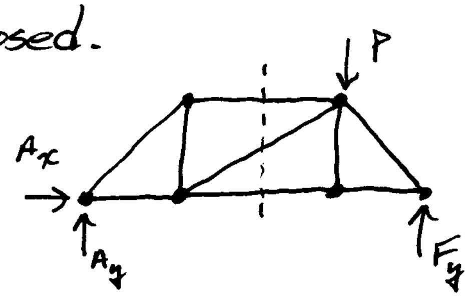
Consider our example on page 81.



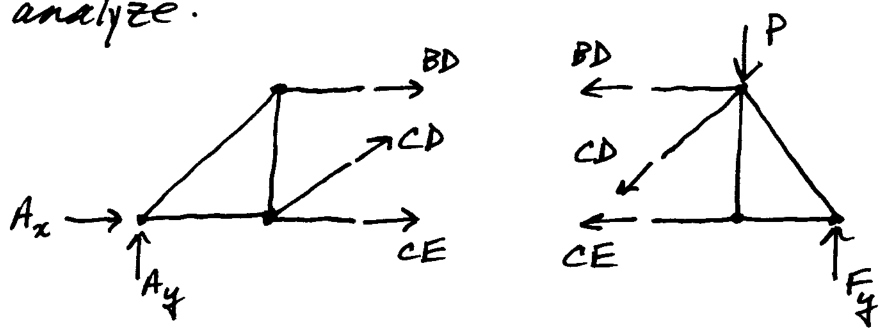
What if we are only interested in the forces in members BD, CD, and CE?

We still need to draw an FBD of the entire structure and perform an equilibrium analysis to determine A_x, A_y and F_y (see page 81). We found that $A_x = 0$, $A_y = \frac{1}{4}P$ and $F_y = \frac{3}{4}P$.

Next we "cut" the structure in such a way so that the internal forces BD, CD and CE are exposed.



Now draw a FBD of the side that you want to analyze.



Next, perform an equilibrium analysis on the FBD that you have drawn. If you draw both sides, then choose <u>one</u> side to analyze.

Left side:

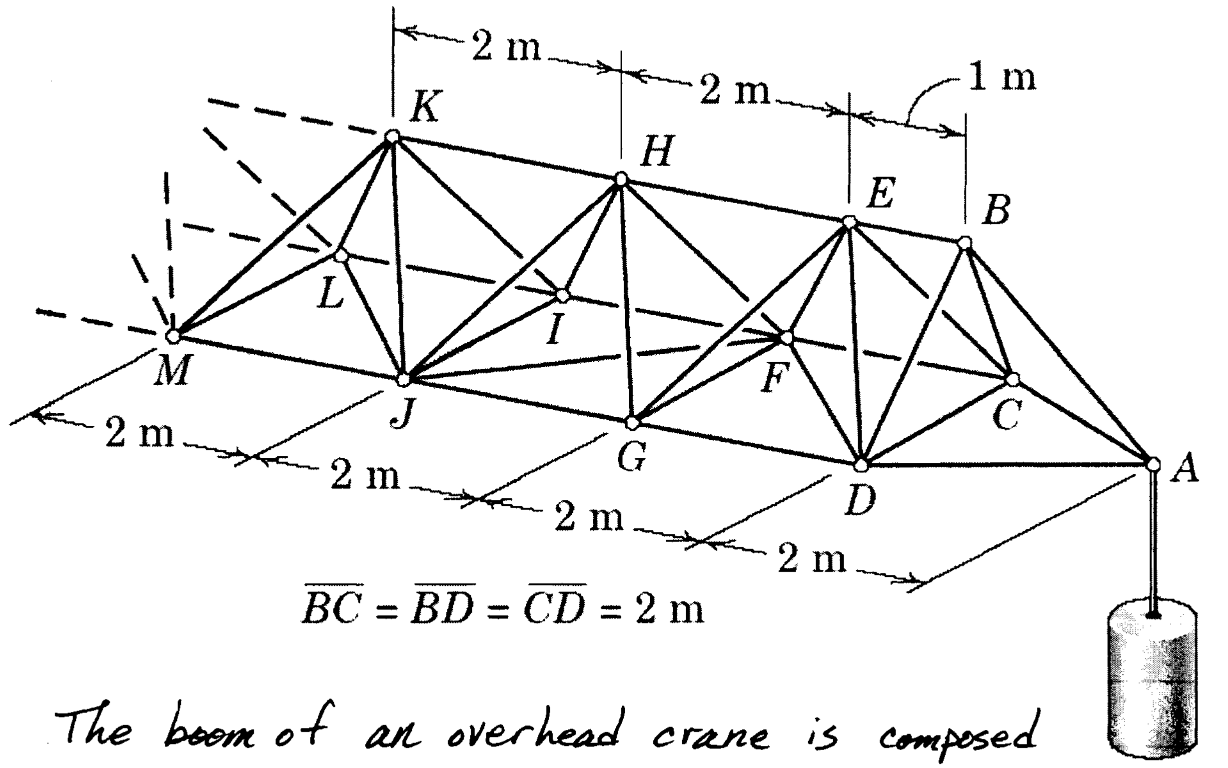
$$\Sigma F_x = A_x + BD + CE + CD \frac{2}{\sqrt{5}} = 0$$
$$\Sigma F_y = A_y + CD \frac{1}{\sqrt{5}} = 0$$
$$\Sigma M_z^C = -A_y \cdot 1 - BD \cdot 1 = 0$$

$$\rightarrow BD = -A_y = -\frac{1}{4}P$$
$$CD = -\sqrt{5}A_y = -\frac{\sqrt{5}}{4}P$$
$$CE = -BD - A_x - \frac{2}{\sqrt{5}}CD = \frac{1}{4}P - 0 + \frac{2}{4}P = \frac{3}{4}P$$

Right side:

$$\Sigma F_x = -BD - CD\frac{2}{\sqrt{5}} - CE = 0$$
$$\Sigma F_y = -P - CD\frac{1}{\sqrt{5}} + F_y = 0$$
$$\Sigma M_z^D = -CE \cdot 1 + F_y \cdot 1 = 0$$

$$\rightarrow CE = F_y = \frac{3}{4}P$$
$$CD = \sqrt{5}(F_y - P) = -\frac{\sqrt{5}}{4}P$$
$$BD = -\left(CE + \frac{2}{\sqrt{5}}CD\right) = -\left(\frac{3}{4}P - \frac{2}{4}P\right) = -\frac{1}{4}P$$

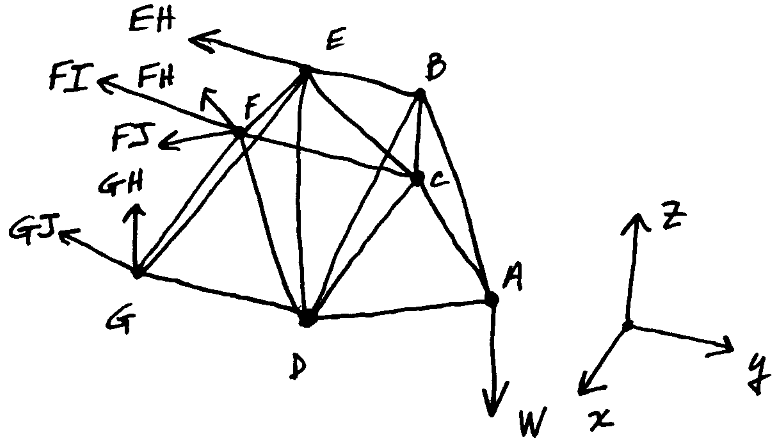


$\overline{BC} = \overline{BD} = \overline{CD} = 2$ m

The boom of an overhead crane is composed of repeated and identical structural units. Use the method of sections to find the forces in members FJ and GJ.

Analysis of the entire structure is not necessary here. Cut through EH, FH, FI, FJ, GH, and GJ.

FBD to the right of the cut.



Forces: $-W\vec{k}$

$-GJ\,\vec{j}$, $GH\left(\dfrac{-\vec{i}-\vec{j}+\sqrt{3}\vec{k}}{\sqrt{[redacted]5}}\right)$

$-FI\,\vec{j}$, $-EH\,\vec{j}$

$FJ\left(\frac{\sqrt{2}}{2}\vec{i}-\frac{\sqrt{2}}{2}\vec{j}\right)$, $FH\left(\dfrac{\vec{i}-\vec{j}+\sqrt{3}\vec{k}}{\sqrt{5}}\right)$

[redacted]

$$
\begin{aligned}
\Sigma\vec{F} = \; & \tfrac{1}{\sqrt{5}}FH\vec{i} - \tfrac{1}{\sqrt{5}}FH\vec{j} + \tfrac{\sqrt{3}}{\sqrt{5}}\text{[redacted]}FH\vec{k} \\
& \tfrac{\sqrt{2}}{2}FJ\vec{i} - \tfrac{\sqrt{2}}{2}FJ\vec{j} \\
& -\tfrac{1}{\sqrt{5}}GH\vec{i} - \tfrac{1}{\sqrt{5}}GH\vec{j} + \tfrac{\sqrt{3}}{\sqrt{5}}\text{[redacted]}GH\vec{k} \\
& - EH\vec{j} \\
& - FI\vec{j} \\
& - GJ\vec{j} \\
& - W\vec{k} \quad \text{[redacted]} \\
\hline
& 0\vec{i} + 0\vec{j} + 0\vec{k}
\end{aligned}
$$

$$\Sigma\vec{M}_H = \vec{r}_{HG}\times\vec{GJ} + \vec{r}_{HF}\times(\vec{FJ}+\vec{FI}) + \vec{r}_{HA}\times\vec{W} = 0$$

Using scalar methods $\rightarrow =$

$$
\begin{aligned}
& GJ\cdot 1(-\vec{k}) + GJ\sqrt{3}(-\vec{i}) \\
& + FI\cdot 1(\vec{k}) + FI\sqrt{3}(-\vec{i}) \\
& + FJ\sqrt{3}\left(-\tfrac{\sqrt{2}}{2}\vec{i} - \tfrac{\sqrt{2}}{2}\vec{j}\right) \\
& + W\cdot 5(-\vec{i}) \;=\; 0\vec{i} + 0\vec{j} + 0\vec{k}
\end{aligned}
$$

$\vec{j} \rightarrow \boxed{FJ = 0}$, $\vec{k} \rightarrow FI = GJ$, $\vec{i} \rightarrow FI = GJ = \dfrac{-5W}{2\sqrt{3}}$

$$\rightarrow \boxed{FI = GJ = -70.8 \text{ kN}}$$

$\Sigma F_x \rightarrow \quad FH = GH$

$$\Sigma F_z \rightarrow \boxed{FH = GH = \frac{\sqrt{5}}{2\sqrt{3}} W = 31.7 \text{ kN}}$$

$$\Sigma F_y \rightarrow -\frac{1}{\sqrt{5}} FH - \frac{\sqrt{2}}{2} FJ - \frac{1}{\sqrt{5}} GH - EH - FI - GJ = 0$$

$$-\frac{1}{2\sqrt{3}} W - 0 - \frac{1}{2\sqrt{3}} W - EH + \frac{+5}{2\sqrt{3}} W + \frac{5}{2\sqrt{3}} W = 0$$

$$\boxed{EH = \frac{4}{\sqrt{3}} W = 113.3 \text{ kN}}$$

To do the moments using vector methods, we need

$$\vec{r}_{HG} = \vec{i} + \vec{j} - \sqrt{3}\,\vec{k}$$
$$\vec{r}_{HF} = -\vec{i} + \vec{j} - \sqrt{3}\,\vec{k}$$
$$\vec{r}_{HA} = 5\vec{j} - \sqrt{3}\,\vec{k}$$

$$\vec{r}_{HG} \times \vec{GJ} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 1 & 1 & -\sqrt{3} \\ 0 & -GJ & 0 \end{vmatrix} = -\sqrt{3}\,GJ\,\vec{i} - GJ\,\vec{k}$$

$$\vec{r}_{HF} \times (\vec{FJ} + \vec{FI}) = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ -1 & 1 & -\sqrt{3} \\ \frac{\sqrt{2}}{2}FJ & -FI - \frac{\sqrt{2}}{2}FJ & 0 \end{vmatrix} = -\sqrt{3}\,FI\,\vec{i} - \frac{\sqrt{6}}{2} FJ\,\vec{i}$$
$$- \frac{\sqrt{6}}{2} FJ\,\vec{j}$$
$$+ FI\,\vec{k} + \frac{\sqrt{2}}{2} FJ - \frac{\sqrt{2}}{2} FJ$$

$$\vec{r}_{HA} \times \vec{W} = \begin{vmatrix} \vec{i} & \vec{j} & \vec{k} \\ 0 & 5 & -\sqrt{3} \\ 0 & 0 & -W \end{vmatrix} = -5W\,\vec{i}$$