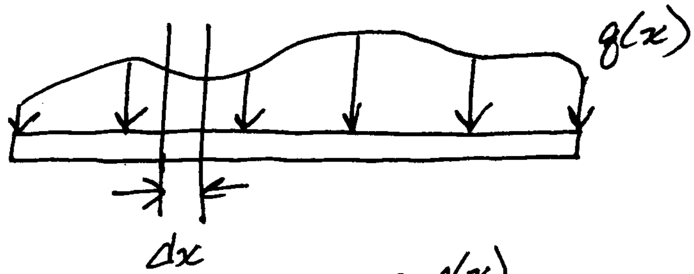
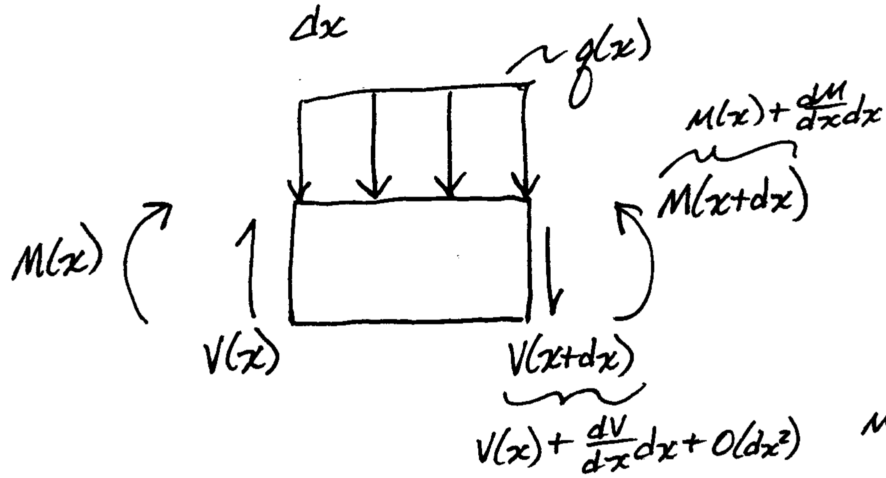
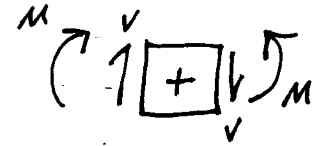
Shear Force and Bending Moment Distributions / Diagrams





* These are the + conventions we will use for M and V



$$\sum F_y = \cancel{V(x)} - \left[\cancel{V(x)} + \frac{dV}{dx}dx + O(dx^2)\right] - q(x)dx - O(dx^2) = 0$$

↑ due to variations in q over dx

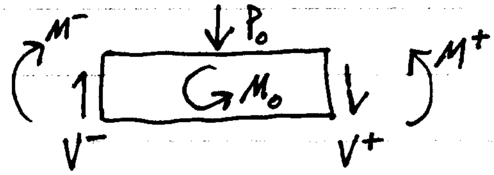
$$\lim_{dx \to 0} \rightarrow \boxed{q(x) = -\frac{dV}{dx}}$$

$$\sum M_z^x = -\cancel{M(x)} + \cancel{M(x)} + \frac{dM}{dx}dx + O(dx^2)$$
$$- \left[V(x) + \frac{dV}{dx}dx + O(dx^2) \right] dx$$
$$- q(x)\,dx\,\frac{dx}{2} - O(dx^3) = 0$$

$$\lim_{dx \to 0} \longrightarrow \boxed{V(x) = \frac{dM}{dx}}$$

Combining these two results $\rightarrow$ $\boxed{q(x) = -\frac{d^2M}{dx^2}}$

"Special" Points :



$$\Sigma F_y = V^- - V^+ - P_0 = 0$$

$$\rightarrow \boxed{V^+ - V^- = -P_0}$$

* **There is a jump in the internal shear force across a point load.** This also applies if there is a $q(x)$ because such a loading would contribute $-q(x)\,dx$ to our ΣF_y equation, which is a higher order term that vanishes in $\lim_{dx \to 0}$.

$$\Sigma M_z^o = M_0 + M^+ - M^- - \underbrace{\cancel{(V^+ + V^-)\frac{dx}{2}}}_{\lim_{dx\to 0} \to 0} = 0$$

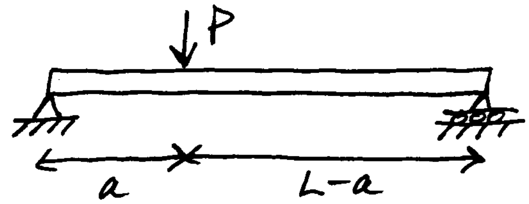
$$\rightarrow \boxed{M^+ - M^- = -M_0}$$

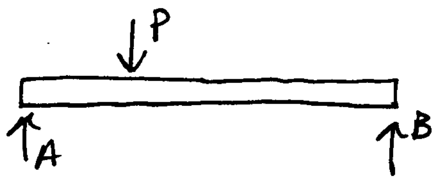
* **There is a jump in the internal bending moment across a concentrated moment.**

These considerations can be used to calculate shear force and bending moment distributions, or to construct the diagrams directly.

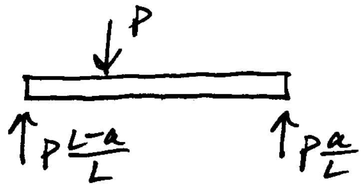
How?

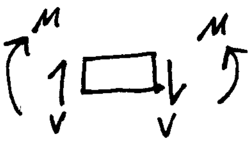
Example :





$$\Sigma M_z^A = BL - Pa = 0$$
$$B = P\frac{a}{L}$$
$$\Sigma F_y = A + B - P = 0$$
$$\rightarrow A = P\frac{L-a}{L}$$



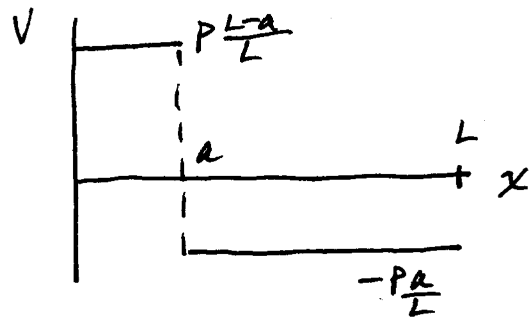


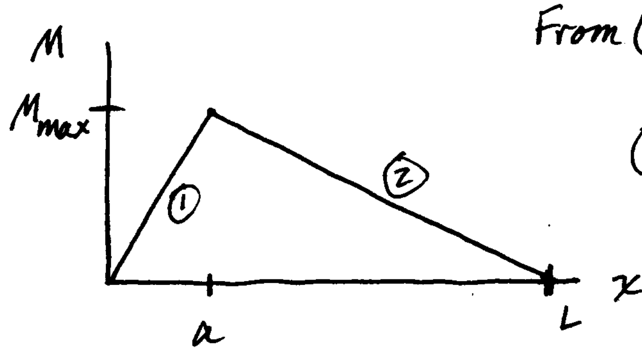
$$V(x=0) = \frac{P(L-a)}{L} \qquad V(x=L) = -\frac{Pa}{L}$$
$$M(x=0) = 0 \qquad M(x=L) = 0$$

From $0 \le x \le a$: $\frac{dV}{dx} = 0$, $\frac{dM}{dx} = \frac{P(L-a)}{L}$

$a \le x \le L$: $\frac{dV}{dx} = 0$, $\frac{dM}{dx} = -\frac{Pa}{L}$

and there is a jump in $V(x)$ of $-P$ at $x=a$.



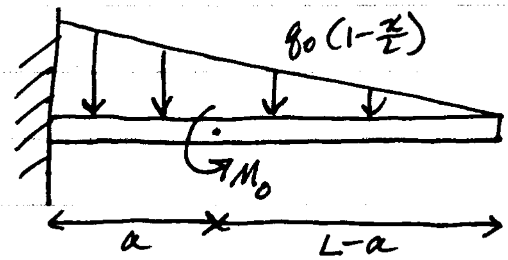


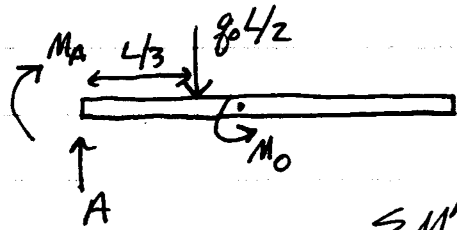
From ① : $M_{max} = \frac{P(L-a)}{L}a$

② : $M_{max} = \frac{Pa}{L}(L-a)$

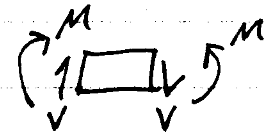
✓ same from either side.

Example:





$$\Sigma F_y = A - \frac{q_0 L}{2} = 0$$
$$A = \frac{q_0 L}{2}$$
$$\Sigma M_z^A = M_0 - M_A - q_0 \frac{L^2}{6} = 0$$
$$M_A = M_0 - \frac{q_0 L^2}{6}$$



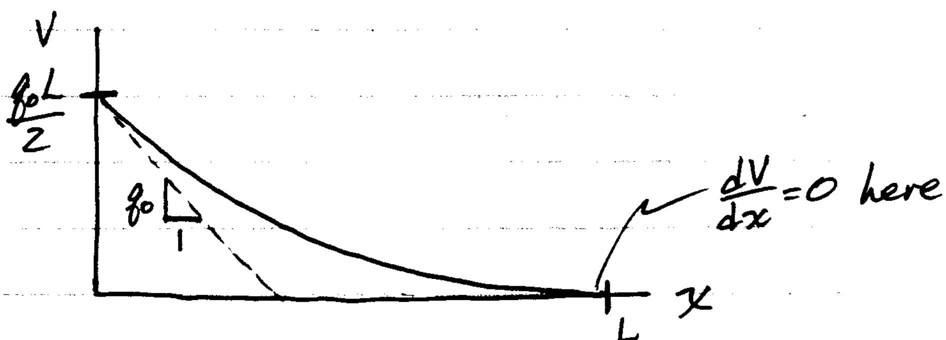
$V(0) = \frac{q_0 L}{2} \quad V(L) = 0$

$M(0) = M_0 - \frac{q_0 L^2}{6} \quad M(L) = 0$

$M(x)$ has a jump of $-M_0$ at $x = a$

$V(x)$ __does not__ have a jump at $x = \frac{L}{3}$. (Look at the original)

$$\left.\begin{aligned} 0 \le x < a &: \frac{dV}{dx} = -q_0\left(1-\frac{x}{L}\right) \\ a < x < L &: \frac{dV}{dx} = -q_0\left(1-\frac{x}{L}\right) \end{aligned}\right\} \text{ V(x) is smooth at all points}$$

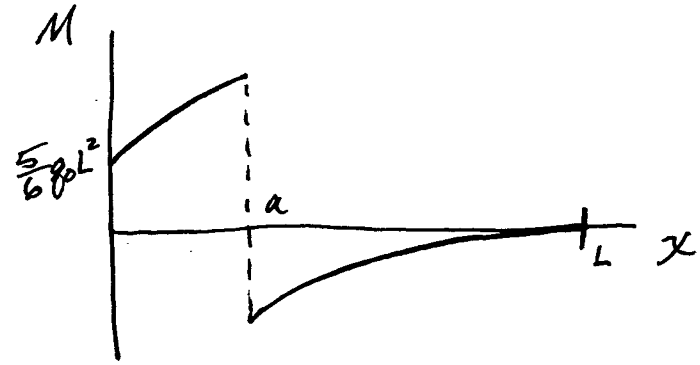


Determined from BC

$$\frac{dV}{dx} = -q_0\left(1-\frac{x}{L}\right) \rightarrow V(x) = -q_0\left(x - \frac{x^2}{2L}\right) + \frac{q_0 L}{2}$$

Next, consider $M(x)$. $\frac{dM}{dx} = V$ & jump of $-M_0$ at $x=a$.

Let's be more specific with M_0. Let's take $M_0 = q_0 L^2$.



$0 \le x < a$: $\frac{dM}{dx} = -q_0\left(x - \frac{x^2}{2L}\right) + \frac{q_0 L}{2}$

$$\hookrightarrow M(x) = -q_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + \frac{q_0 L x}{2} + \overbrace{\frac{5}{6} q_0 L^2}^{\text{BC at } x=0}$$

$a < x \le L$: $M(x) = -q_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + \frac{q_0 L x}{2} \underbrace{- \frac{q_0 L^2}{6}}_{\text{BC at } x=L}$

Check $M(x=a^+) - M(x=a^-) = - \frac{q_0 L^2}{6} - \frac{5}{6} q_0 L^2 = \underbrace{-q_0 L^2}_{= -M_0} \checkmark$

other contributions cancel out

Note the decreasing slope of $M(x)$ due to the fact that $V(x)$ decreases with x.

From the diagram it is clear that M_{max} occurs at a^-.

$$M(a^+) = -q_0\left(\frac{a^2}{2} - \frac{a^3}{6L}\right) + q_0 \frac{La}{2} - \frac{q_0 L^2}{6}$$
$$M(a^-) = -q_0\left(\frac{a^2}{2} - \frac{a^3}{6L}\right) + q_0 \frac{La}{2} + \frac{5}{6} q_0 L^2$$

The max can occur at a^+ or if M_0 and/or a change.

DISCONTINUITY FUNCTIONS

Definition $\langle x-a\rangle = \begin{cases} 0 & \text{for} \quad (x-a)<0 \\ (x-a) & \text{for} \quad (x-a)>0 \end{cases}$

Relationships

$$\frac{d}{dx}\langle x-a\rangle^n = n\langle x-a\rangle^{n-1} \quad (n \neq 0)$$

$$\frac{d}{dx}\langle x-a\rangle^1 = \langle x-a\rangle^0 \quad = \begin{cases} 0 & x<a \\ 1 & x>a \end{cases}$$

$$\frac{d}{dx}\langle x-a\rangle^0 = \langle x-a\rangle^{-1} \quad = \begin{cases} 0 & x<a \;\&\; x>a \\ \infty & x=a \end{cases}$$

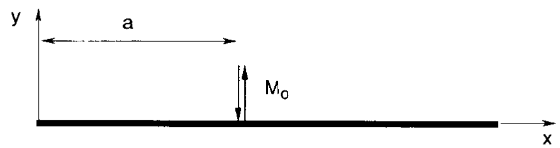
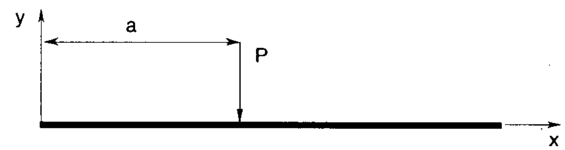
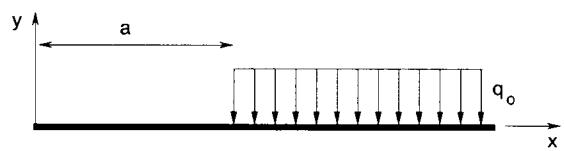
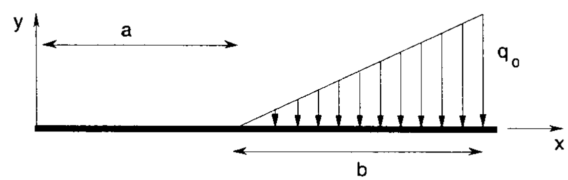
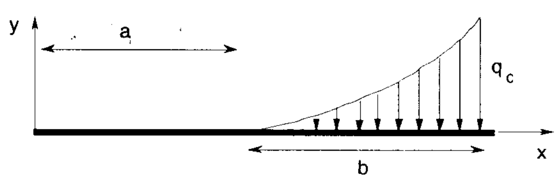
$$\frac{d}{dx}\langle x-a\rangle^{-1} = -\langle x-a\rangle^{-2} \quad = \begin{cases} 0 & x<a \;\&\; x>a \\ \pm\infty & x=a \end{cases}$$

$$\int_0^x \langle x-a\rangle^{n-1} dx = \frac{1}{n}\langle x-d\rangle^n \quad (n \neq 0)$$

$$\int_0^x \langle x-a\rangle^0 dx = \langle x-a\rangle^1$$

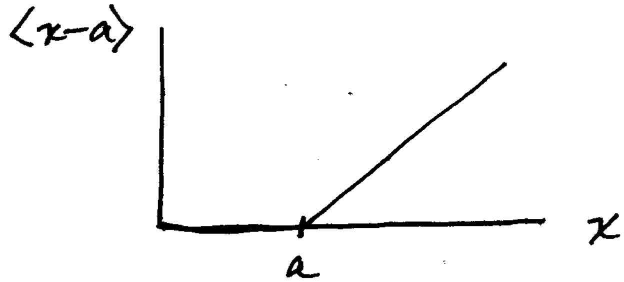
$$\int_0^x \langle x-a\rangle^{-1} dx = \langle x-a\rangle^0$$

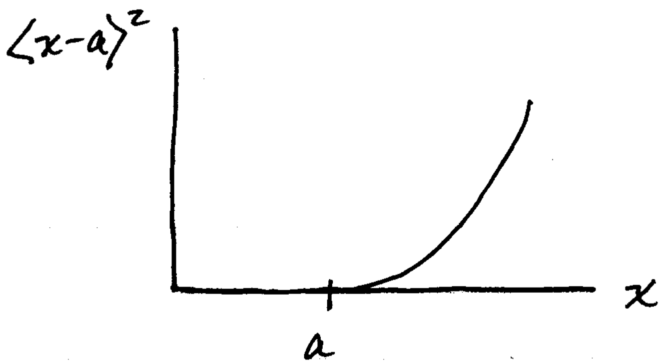
$$\int_0^x \langle x-a\rangle^{-2} dx = -\langle x-a\rangle^{-1}$$

	$M(x)$	$q(x)$
	$-M_o\langle x-a\rangle^0$	$-M_o\langle x-a\rangle^{-2}$
	$-P\langle x-a\rangle$	$P\langle x-a\rangle^{-1}$
	$-\frac{q_o}{2}\langle x-a\rangle^2$	$q_o\langle x-a\rangle^0$
	$-\frac{q_o}{6b}\langle x-a\rangle^3$	$\frac{q_o}{b}\langle x-a\rangle^1$
	$-\frac{q_o}{12b^2}\langle x-a\rangle^4$	$\frac{q_o}{b^2}\langle x-a\rangle^2$

Half-range Functions

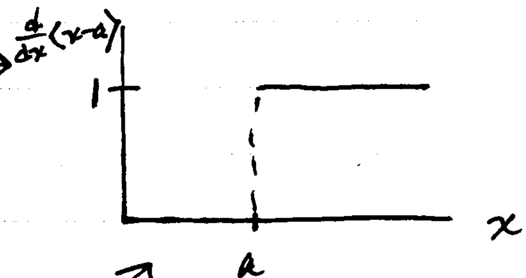
Definition: $\langle x-a \rangle = \begin{cases} 0 & \text{for } x<a \\ x-a & \text{for } x>a \end{cases}$





$\langle x-a \rangle^2 = \begin{cases} 0 & \text{for } x<a \\ (x-a)^2 & \text{for } x>a \end{cases}$

Consider $\frac{d}{dx}\langle x-a \rangle \longrightarrow$

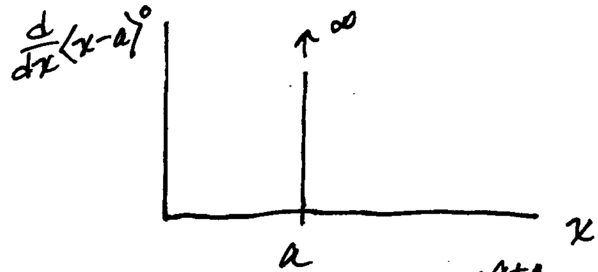


$\frac{d}{dx}\langle x-a \rangle = \langle x-a \rangle^0$

$\langle x-a \rangle^0 = \begin{cases} 0 & \text{for } x<a \\ 1 & \text{for } x>a \end{cases}$

What is $\frac{d}{dx}\langle x-a \rangle^0$?

Looking at the plot it is clear that $\frac{d}{dx}\langle x-a\rangle^0 = 0$ for $x<a$ and $\frac{d}{dx}\langle x-a\rangle^0 = 0$ for $x>a$, but something strange/special happens at $x=a$. Effectively, $\frac{d}{dx}\langle x-a\rangle^0 \to \infty$ at $x=a$.



However note that $\int_{a-\Delta}^{a+\Delta} \frac{d}{dx}\langle x-a\rangle^0 \, dx = \langle x-a\rangle^0 \Big]_{a-\Delta}^{a+\Delta}$

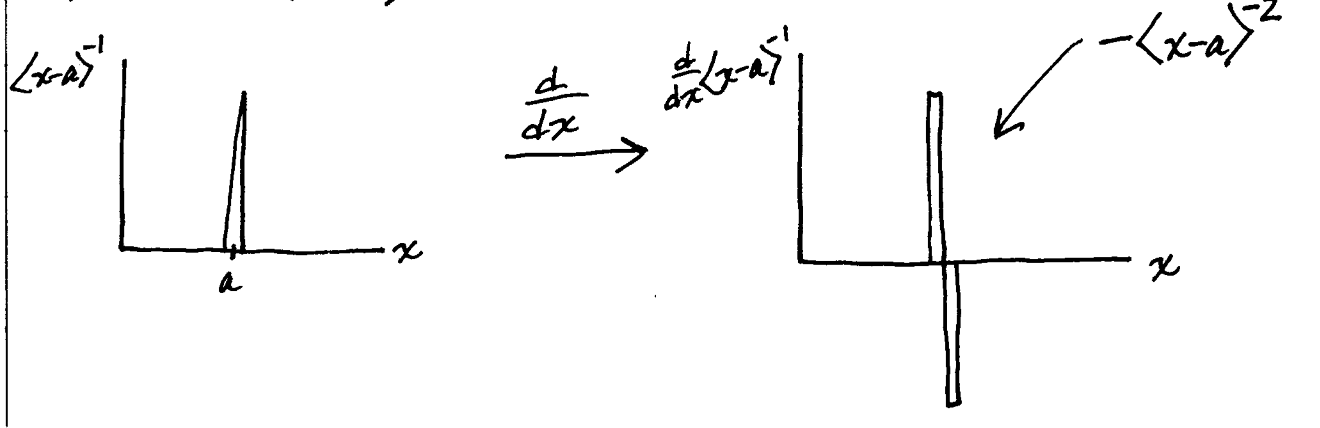
$= 1$ for any $\Delta > 0$

So even though the spike goes to infinity, the area under the spike is finite.

Define $\quad \langle x-a\rangle^{-1} = \frac{d}{dx}\langle x-a\rangle^0 = \begin{cases} 0 & \text{for } x \neq a \\ \infty & \text{for } x = a \end{cases}$

One last case of interest. $\frac{d}{dx}\langle x-a\rangle^{-1}$?

Think of $\langle x-a\rangle^{-1}$ as



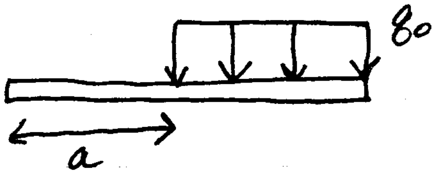
So for $n \neq 0 \quad \frac{d}{dx}\langle x-a\rangle^{n} = n\langle x-a\rangle^{n}$

$n = 0 \quad \frac{d}{dx}\langle x-a\rangle^{0} = \langle x-a\rangle^{-1}$

$$\int_0^x \langle x'-a\rangle^n \, dx' = \frac{1}{n+1}\langle x-a\rangle^{n+1} \quad \text{for } n \neq -1, a > 0$$

$$\int_0^x \langle x'-a\rangle^{-1} \, dx' = \langle x-a\rangle^{0} \quad \text{for } a > 0$$

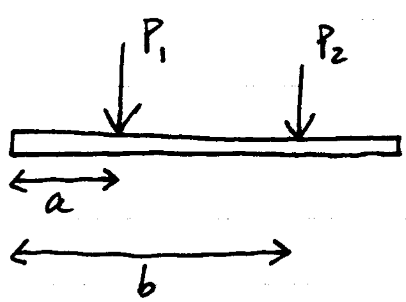
How are these useful?



$$q = q_0\langle x-a\rangle^0$$

$$V = \int -q(x)\,dx = -q_0\langle x-a\rangle^1 + c_1$$

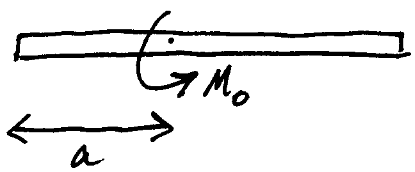
$$M = \int V\,dx = -\frac{1}{2}q_0\langle x-a\rangle^2 + c_1 x + c_2$$



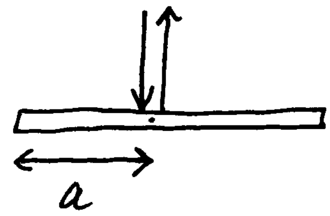
$$q = P_1\langle x-a\rangle^{-1} + P_2\langle x-b\rangle^{-1}$$

$$V = \int -q(x)\,dx = -P_1\langle x-a\rangle^0 + -P_2\langle x-b\rangle^0 + c_1$$

$$M = \int V(x)\,dx = -P_1\langle x-a\rangle - P_2\langle x-b\rangle + c_1 x + c_2$$



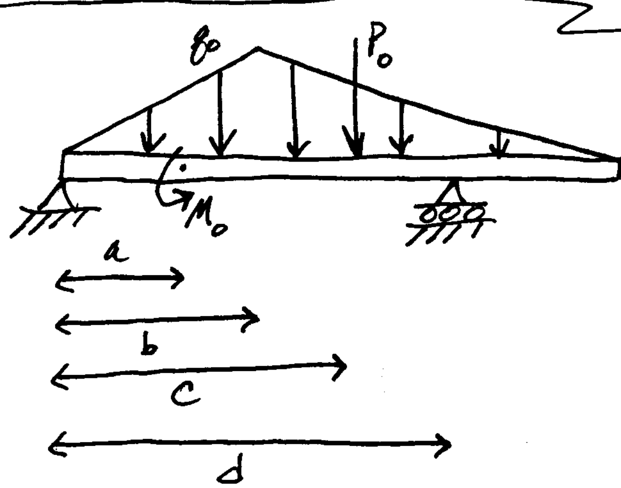
$q(x)$? →



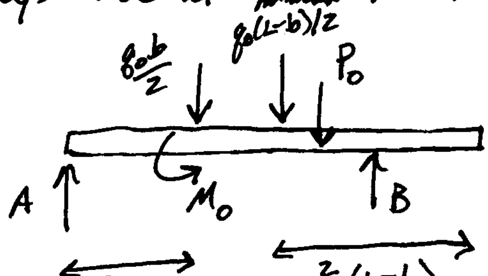
$$q(x) = -M_o \langle x-a \rangle^{-2}$$

$$V = \int -q(x)\,dx = -M_o \langle x-a \rangle^{-1} + C_1$$

$$M = \int V(x)\,dx = -M_o \langle x-a \rangle^{0} + C_1 x + C_2$$



Always useful [redacted] to determine reactions.



$$\Sigma M_z^A = M_o - P_o c - \frac{1}{3} q_o b^2 - \frac{q_o(L-b)}{2}\left(\frac{L}{3} + \frac{2}{3}b\right) + Bd = 0$$

$$\boxed{B = P_o \frac{c}{d} + \frac{1}{6} q_o \frac{L^2}{d} + \frac{1}{6} q_o \frac{Lb}{d} - \frac{M_o}{d}}$$

$$\Sigma F_y = A + B - \frac{q_o b}{2} - \frac{q_o(L-b)}{2} - P_o = 0$$

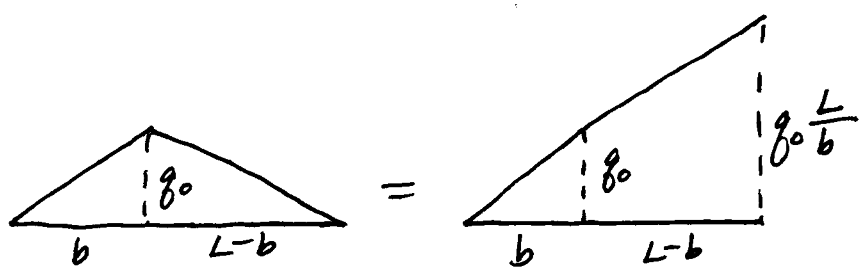
→ $\boxed{A = P_o \frac{d-c}{d} + \text{[redacted]}\, \frac{q_o L}{2} - \frac{1}{6} q_o \frac{L^2}{d} - \frac{1}{6} q_o \frac{Lb}{d} + \frac{M_o}{d}}$

What is $q(x)$?

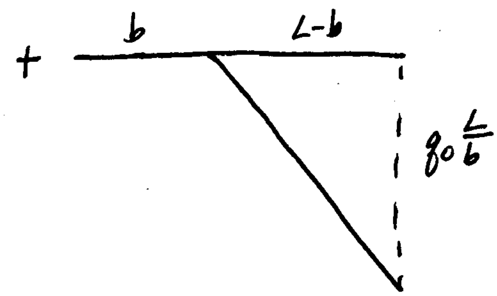
known

$$q(x) = -M_0\langle x-a\rangle^{-2} + P_0\langle x-c\rangle^{-1} - B\langle x-d\rangle^{-1}$$

$$\underbrace{+\,q_0/b\;x - \frac{q_0 L}{b(L-b)}\langle x-b\rangle}_{\text{where did this come from?}}$$



slope $= \frac{q_0}{b}$



slope $= \dfrac{-q_0 \frac{L}{b}}{L-b}$

$= \dfrac{-q_0 L}{b(L-b)}$

$$\frac{dV}{dx} = -q \rightarrow V = -M_0\langle x-a\rangle^{-1} - P_0\langle x-c\rangle^{0} + B\langle x-d\rangle^{0}$$

$$-\frac{1}{2}\frac{q_0}{b}x^2 + \frac{q_0 L}{2b(L-b)}\langle x-b\rangle^2 + c_1$$

c_1? $V(x=0) = A = 0+0+0+0+0+c_1 \rightarrow \boxed{c_1 = A}$

$$\frac{dM}{dx} = V$$

$$M(x) = -M_0\langle x-a\rangle^0 - P_0\langle x-c\rangle + B\langle x-d\rangle$$

$$-\frac{1}{6}\frac{g_0}{b}x^3 + \frac{g_0 L}{6b(L-b)}\langle x-b\rangle^3 + Ax + C_2$$

$$M(x=0) = 0 = 0+0+0+0+0+0+C_2 \rightarrow \boxed{C_2 = 0}$$

$$\boxed{\begin{aligned} V(x) &= -M_0\langle x-a\rangle^{-1} - P_0\langle x-c\rangle^0 + B\langle x-d\rangle^0 - \frac{1}{2}\frac{g_0}{b}x^2 + \frac{g_0 L}{2b(L-b)}\langle x-b\rangle^2 \\ &\quad + A \\ M(x) &= -M_0\langle x-a\rangle^0 - P_0\langle x-c\rangle^1 + B\langle x-d\rangle^1 - \frac{1}{6}\frac{g_0}{b}x^3 \\ &\quad + \frac{g_0 L}{6b(L-b)}\langle x-b\rangle^3 + Ax \end{aligned}}$$