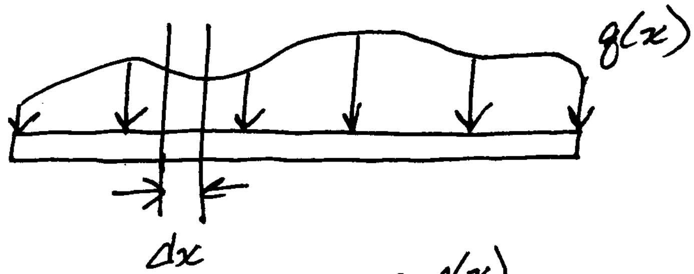
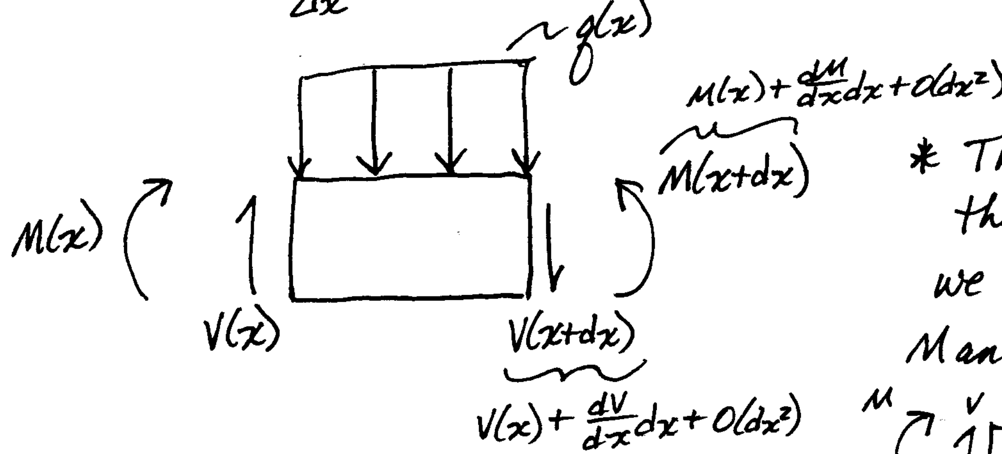
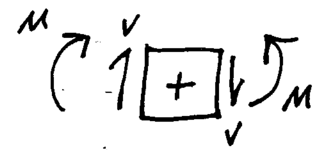
Shear Force and Bending Moment Distributions / Diagrams





* These are the + conventions we will use for M and V



$$\Sigma F_y = \cancel{V(x)} - \left[\cancel{V(x)} + \frac{dV}{dx}dx + O(dx^2)\right] - q(x)dx - O(dx^2) = 0$$

↑ due to variations in q over dx

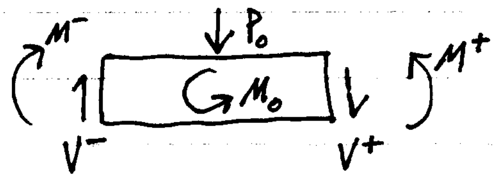
$$\lim_{dx\to 0} \rightarrow \boxed{q(x) = -\frac{dV}{dx}}$$

$$\Sigma M_z^x = -\cancel{M(x)} + \cancel{M(x)} + \frac{dM}{dx}dx + O(dx^2)$$
$$- \left[V(x) + \frac{dV}{dx}dx + O(dx^2)\right]dx$$
$$- q(x)\,dx\,\frac{dx}{2} - O(dx^3) = 0$$

$$\lim_{dx\to 0} \longrightarrow \boxed{V(x) = \frac{dM}{dx}}$$

Combining these two results → $\boxed{q(x) = -\frac{d^2M}{dx^2}}$

"Special" Points :



$$\Sigma F_y = V^- - V^+ - P_o = 0$$

$$\rightarrow \boxed{V^+ - V^- = -P_o}$$

* There is a jump in the internal shear force across a point load. This also applies if there is a $q(x)$ because such a loading would contribute $-q(x)\,dx$ to our ΣF_y equation, which is a higher order term that vanishes in $\lim_{dx \to 0}$.

$$\Sigma M_z^o = M_o + M^+ - M^- - \overbrace{\cancel{(V^+ + V^-)\frac{dx}{2}}}^{\lim_{dx\to 0} \to 0} = 0$$

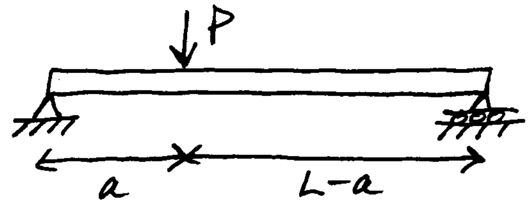
$$\rightarrow \boxed{M^+ - M^- = -M_o}$$

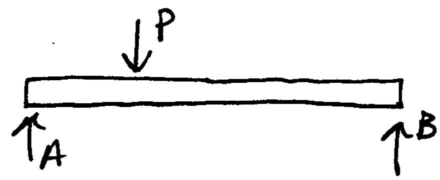
* There is a jump in the internal bending moment across a concentrated moment.

These considerations can be used to calculate shear force and bending moment distributions, or to construct the diagrams directly.

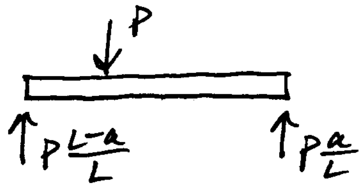
How?

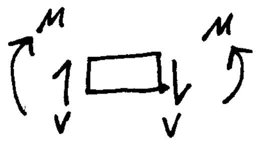
Example :





$$\Sigma M_z^A = BL - Pa = 0$$
$$B = P\frac{a}{L}$$
$$\Sigma F_y = A + B - P = 0$$
$$\rightarrow A = P\frac{L-a}{L}$$



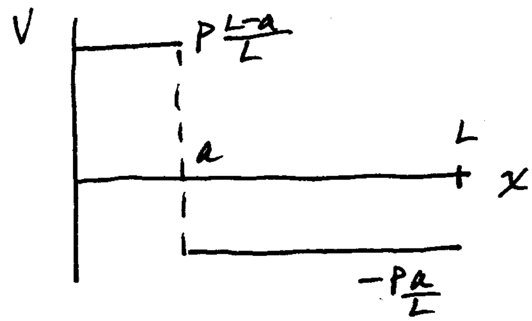


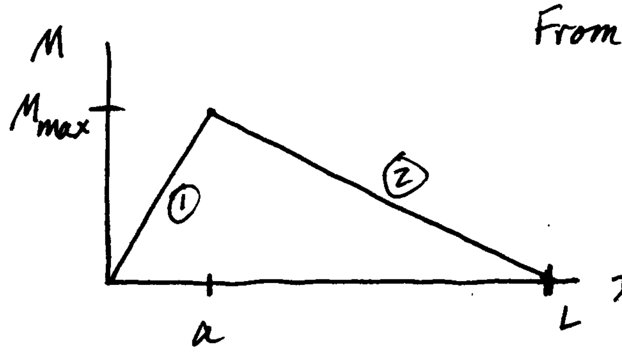
$$V(x=0) = \frac{P(L-a)}{L} \qquad V(x=L) = -\frac{Pa}{L}$$
$$M(x=0) = 0 \qquad M(x=L) = 0$$

From $0 \le x \le a$: $\frac{dV}{dx} = 0$, $\frac{dM}{dx} = \frac{P(L-a)}{L}$

$a \le x \le L$: $\frac{dV}{dx} = 0$, $\frac{dM}{dx} = -\frac{Pa}{L}$

and there is a jump in $V(x)$ of $-P$ at $x=a$.



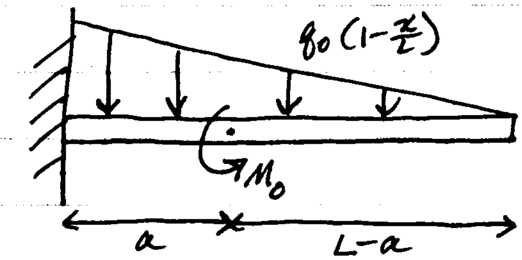


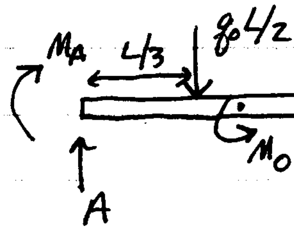
From ① : $M_{max} = \frac{P(L-a)}{L} a$

② : $M_{max} = \frac{Pa}{L}(L-a)$

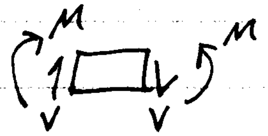
✓ same from either side.

Example:





$$\Sigma F_y = A - \frac{q_0 L}{2} = 0$$
$$A = \frac{q_0 L}{2}$$
$$\Sigma M_z^A = M_0 - M_A - q_0\frac{L^2}{6} = 0$$
$$M_A = M_0 - \frac{q_0 L^2}{6}$$



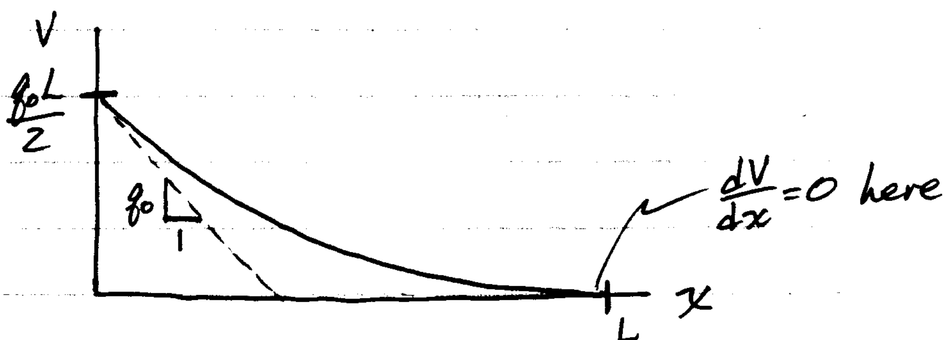
$V(0) = \frac{q_0 L}{2}$ $\quad V(L) = 0$

$M(0) = M_0 - \frac{q_0 L^2}{6}$ $\quad M(L) = 0$

$M(x)$ has a jump of $-M_0$ at $x = a$

$V(x)$ <u>does not</u> have a jump at $x = \frac{L}{3}$. (Look at the original)

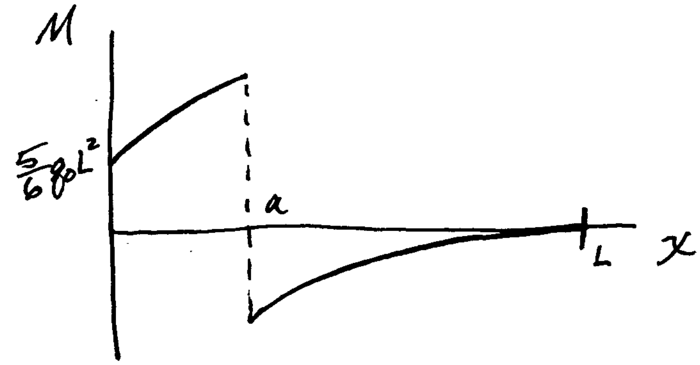
$$\left.\begin{aligned} 0 \le x < a &: \frac{dV}{dx} = -q_0\left(1-\frac{x}{L}\right) \\ a < x < L &: \frac{dV}{dx} = -q_0\left(1-\frac{x}{L}\right) \end{aligned}\right\} \text{ V(x) is smooth at all points}$$



$$\frac{dV}{dx} = -q_0\left(1-\frac{x}{L}\right) \rightarrow V(x) = -q_0\left(x - \frac{x^2}{2L}\right) + \underbrace{\frac{q_0 L}{2}}_{\text{Determined from BC}}$$

Next, consider $M(x)$. $\frac{dM}{dx} = V$ & jump of $-M_0$ at $x=a$.

Let's be more specific with M_0. Let's take $M_0 = q_0 L^2$.



$0 \le x < a$: $\frac{dM}{dx} = -q_0\left(x - \frac{x^2}{2L}\right) + \frac{q_0 L}{2}$

$$\hookrightarrow M(x) = -q_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + \frac{q_0 L x}{2} + \underbrace{\frac{5}{6} q_0 L^2}_{\text{BC at } x=0}$$

$$a < x \le L: \quad M(x) = -q_0\left(\frac{x^2}{2} - \frac{x^3}{6L}\right) + \frac{q_0 L x}{2} \underbrace{- \frac{q_0 L^2}{6}}_{\text{BC at } x=L}$$

Check $M(x = a^+) - M(x = a^-) = - \frac{q_0 L^2}{6} - \frac{5}{6} q_0 L^2 = \underbrace{-q_0 L^2}_{= -M_0 \checkmark}$

other contributions cancel out

Note the decreasing slope of $M(x)$ due to the fact that $V(x)$ decreases with x.

From the diagram it is clear that M_{max} occurs at a^-.

$$M(a^+) = -q_0\left(\frac{a^2}{2} - \frac{a^3}{6L}\right) + q_0 \frac{La}{2} - \frac{q_0 L^2}{6}$$
$$M(a^-) = -q_0\left(\frac{a^2}{2} - \frac{a^3}{6L}\right) + q_0 \frac{La}{2} + \frac{5}{6} q_0 L^2$$

The max can occur at a^+ or 0 if M_0 and/or a change.