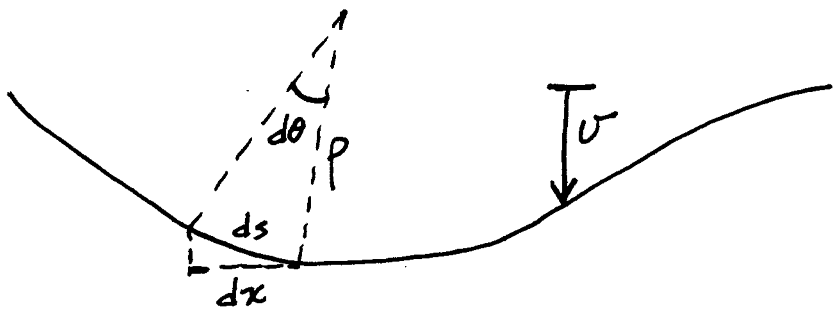
Stresses and Strains in Beams

Kinematics: How do we describe the deformation of a beam?



v = downward deflection/displacement of the beam

ρ = radius of curvature of the deformed beam (this will vary along the beam)

dx = differential element along the beam

ds = differential arc length of deformed beam

$d\theta$ = differential angle swept out by ds

$$ds = \rho\, d\theta \quad \text{or} \quad \rho = \frac{1}{\chi}$$

and $\quad ds = \frac{1}{\chi} d\theta \qquad \chi \equiv$ curvature

$$\chi = \frac{d\theta}{ds}$$

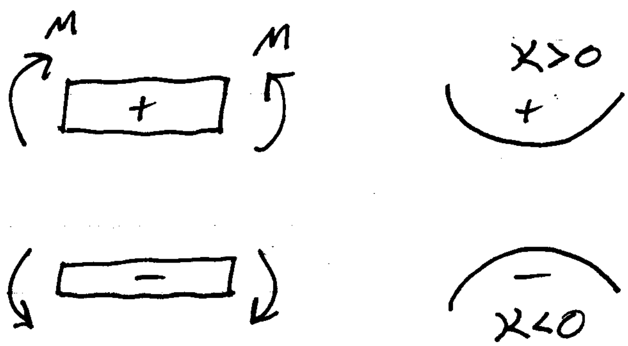
For beams where the deflection v, and the slope θ is small at all locations, we can make the approximation that

$$\kappa = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \quad (\text{because } ds \approx dx)$$

Later in the course we will relate θ and κ to the deflection v.

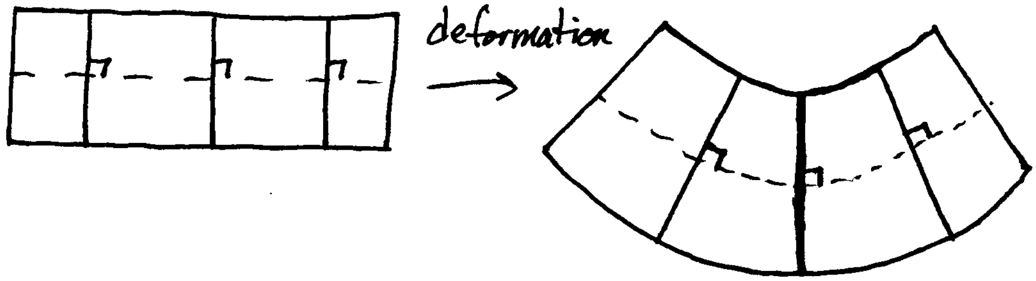
For now we have $\boxed{\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}}$

Sign conventions:



<u>A kinematic assumption</u>: During the deformation of a beam planar cross sections of the beam that are perpendicular to the longitudinal axis remain planar and perpendicular to the longitudinal axis.

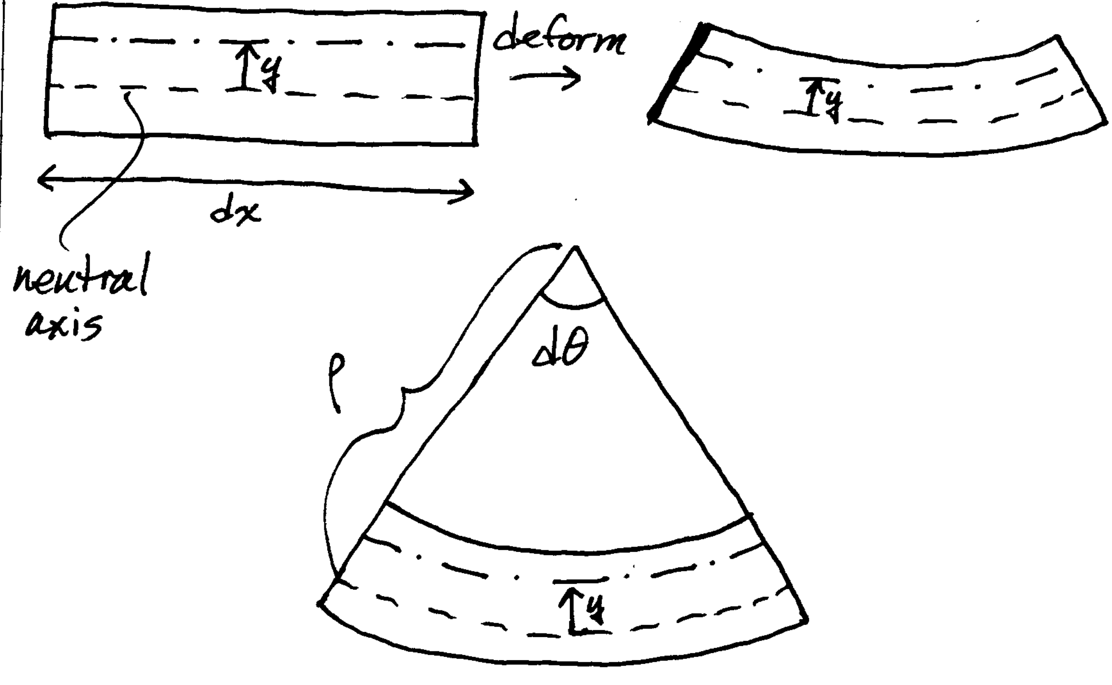
"Planes remain planes" and they remain perpendicular to the longitudinal axis of the beam.



From the "planes remain planes" assumption we can analyze the strain in the beam.

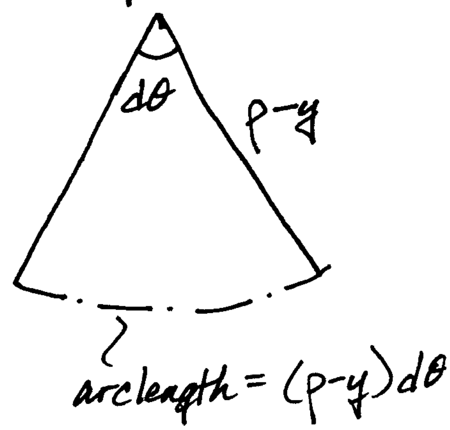
We will measure the coordinate y from a line parallel to the x-direction called the "neutral axis". y will be positive upward (which in my opinion is an unfortunate choice, but this is what the book does).

The neutral axis is the line within the beam that remains unstretched after bending. (No axial force allowed.)



Longitudinal strain, ε, of the neutral axis is zero <u>by definition</u>.

What is the strain of the line at y above the neutral axis?



$$\varepsilon(y) = \frac{L - L_0}{L_0}$$

$$\varepsilon(y) = \frac{(\rho - y)\,d\theta - dx}{dx}$$

But, from our definition of the neutral axis we also know $\varepsilon(y=0) = 0$.

$$\varepsilon(y=0) = \frac{\rho\, d\theta - dx}{dx} = 0$$

$$\rightarrow \rho \underbrace{\frac{d\theta}{dx}}_{} = 1$$

$$= \chi = \frac{1}{\rho}$$

So: $\varepsilon(y) = \cancel{\rho \frac{d\theta}{dx}} - y\frac{d\theta}{dx} \cancel{-1}$

* Important result for kinematics of beams

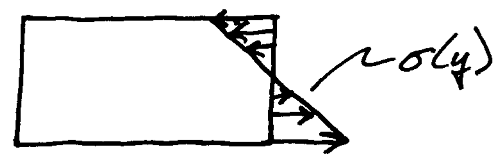
$$\boxed{\varepsilon(y) = -y \underbrace{\frac{d\theta}{dx}}_{\chi} = -\chi y}$$

Now that we have related strain to curvature, we would like to relate stress and eventually moment to curvature.

<u>Material Law</u>: $\sigma = E\varepsilon$ (longitudinal / axial stress in the beam)

$$\boxed{\sigma = -E\chi y}$$

<u>Equilibrium</u>: M



$$\Sigma F_x = \int_A \sigma(y)\, dA = 0$$

$$\int_A -E\chi y\, dA = 0$$

$$-E\chi \int_A y\, dA = 0$$

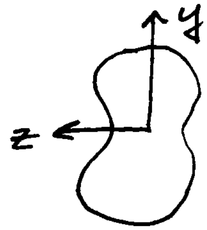
$$\rightarrow \boxed{\int_A y\, dA = 0}$$

This equation defines the location of the neutral axis.

What this says is that for a homogeneous linear elastic beam the neutral axis passes through the centroid of the cross-sectional area.

$$\Sigma M_z = -M + \int_A \sigma(y)(-y)\,dA = 0$$

$$M = \int_A E\kappa y^2\,dA$$



$$M = E\kappa \underbrace{\int_A y^2\,dA}$$

I or $I_{\cancel{z}}$ or $I_{\cancel{zz}}$
moment of inertia (2nd moment of the area)
of the cross-section
about the ~~z~~-axis

$$\boxed{M = EI\kappa} \quad **$$

Structural response of a beam in bending.
Compare to:

$$N = EA\frac{\delta}{L} \text{ for a bar}$$

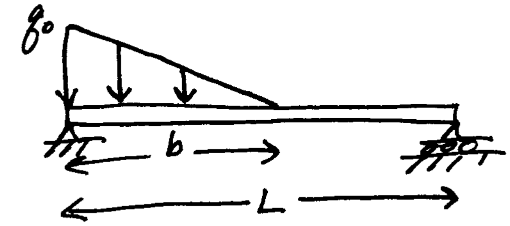
$$T = GI_p\frac{\phi}{L} \text{ for a shaft}$$

Since we know $\sigma = -E\kappa y$ we can relate the stress distribution to the moment.

$$M = EI\kappa = -\frac{\sigma I}{y}$$

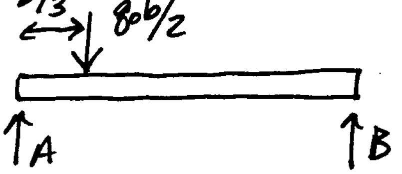
$$\boxed{\sigma = -\frac{My}{I}} \quad **$$ Stress distribution in the cross-section.

Example :



Determine the maximum stress and its location.

First determine $M(x)$.



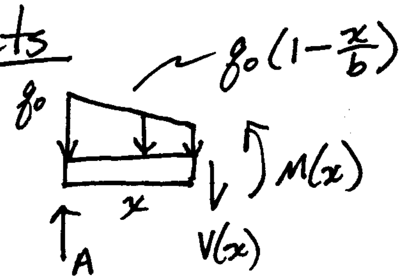
$$\Sigma M_z^A = BL - q_0 b^2/6 = 0$$

$$B = \frac{1}{6}\frac{q_0 b^2}{L}$$

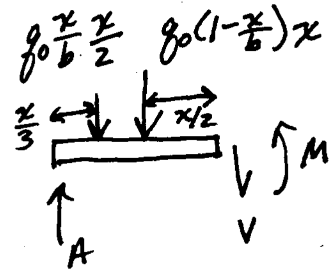
$$\Sigma F_y = A + B - \frac{q_0 b}{2} = 0$$

$$A = q_0 b\left(\frac{1}{2} - \frac{1}{6}\frac{b}{L}\right)$$

Cuts



$\longrightarrow$



$$\Sigma M_z^x = M - Ax + q_0\frac{2x^3}{6b} + q_0\left(1-\frac{x}{b}\right)\frac{x^2}{2} = 0$$

$$M = Ax - q_0\frac{x^2}{2} + q_0\frac{x^3}{6b}$$

If you find $M(x)$ first you can always use

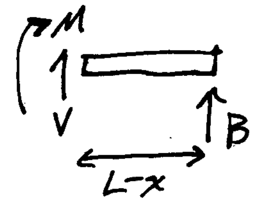
$$V(x) = \frac{dM}{dx} = A - q_0 x + q_0\frac{x^2}{2b} \checkmark$$

(Check that this satisfies $\Sigma F_y = 0$.)

So, for $0 < x < b$: $M(x) = Ax - q_0 \frac{x^2}{2} + q_0 \frac{x^3}{6b}$

$V(x) = A - q_0 x + q_0 \frac{x^2}{2b}$

$b < x < L$:



$\Sigma M_z^x = -M + B(L-x) = 0$

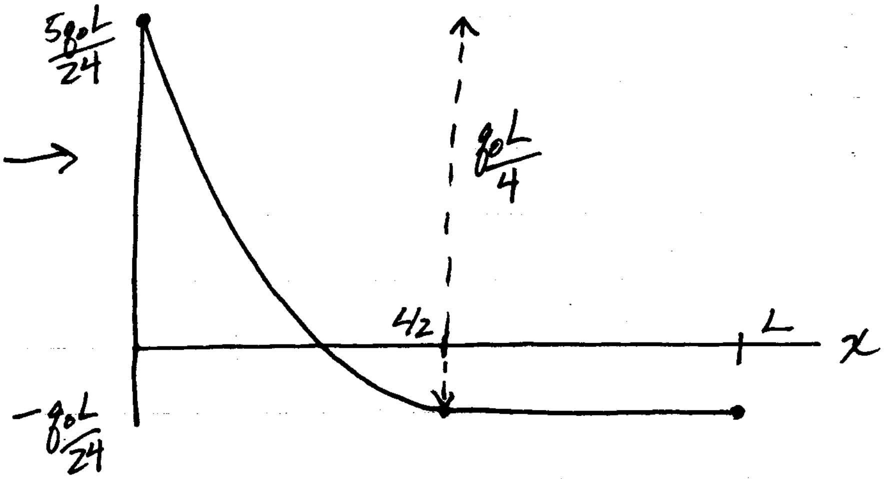
$M = B(L-x)$

$V = \frac{dM}{dx} = -B$ ✓

Take $b = \frac{L}{2} \rightarrow$, $A = \frac{5}{24} q_0 L$, $B = \frac{1}{24} q_0 L$

Plots :

$V(x) \rightarrow$

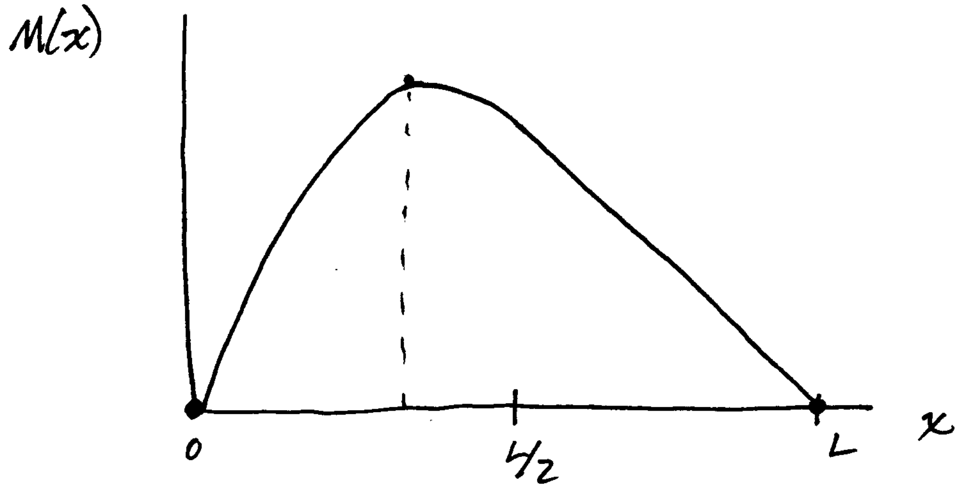


$\frac{dV}{dx} = -q \rightarrow$ slope of $V(x) \Rightarrow -q_0$ at $x=0$

$\Rightarrow 0$ at $x = b = \frac{L}{2}$

$V = \int -q\,dx \rightarrow$ change in $V = -$area under $q(x)$ up to x

ΔV at $L/2 = \frac{1}{2} q_0 \left(\frac{L}{2}\right) = q_0 \frac{L}{4}$



$\frac{dM}{dx} = V \rightarrow$ max/min of $M(x)$ occurs where $V(x) = 0$

No point moments $\rightarrow$ $M(x)$ is continuous
No point forces $\rightarrow$ $M(x)$ is smooth

$$V(x) = \frac{5}{24} q_0 L - q_0 x + q_0 \frac{x^2}{2b} = 0 \qquad b = \frac{L}{2}$$

$$x = \frac{q_0 \pm \sqrt{q_0^2 - 4 \frac{q_0 2}{2b} \frac{5}{24} q_0 L}}{q_0 2/L}$$

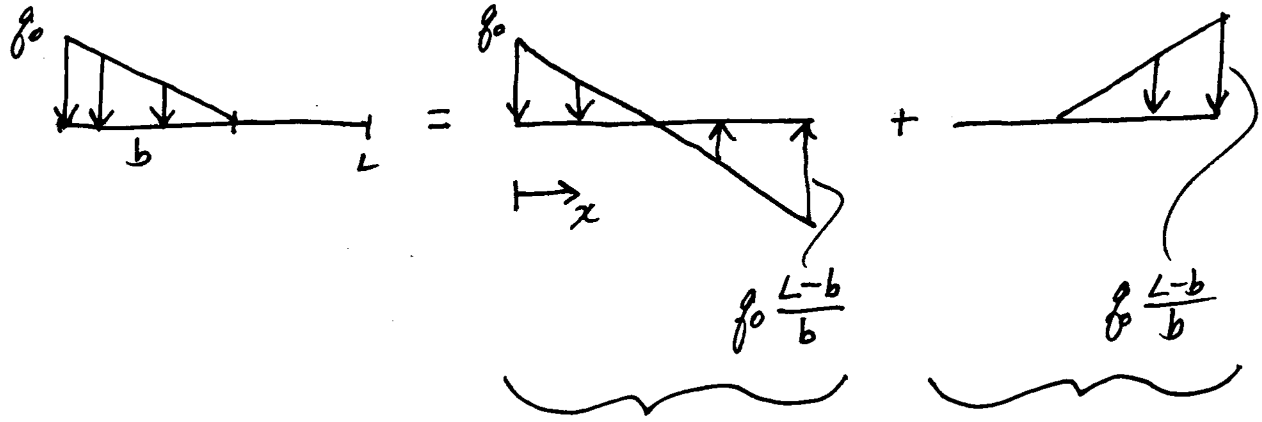
$$= \frac{L}{2}\left(1 \pm \sqrt{1 - \frac{5}{6}}\right)$$

$$x_{max} = \frac{L}{2}\left(1 - \sqrt{\frac{1}{6}}\right)$$

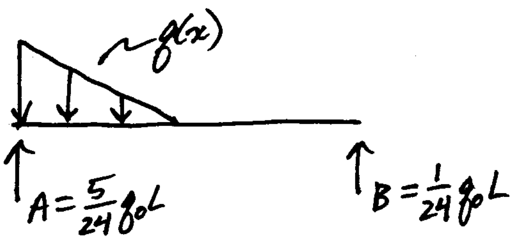
$$M_{max} = \frac{5}{24} q_0 L \frac{L}{2}\left(1 - \sqrt{\frac{1}{6}}\right) - \frac{q_0 L^2}{2 \cdot 4}\left(1 - \sqrt{\frac{1}{6}}\right)^2 + \frac{q_0}{3L} \frac{L^3}{8}\left(1 - \sqrt{\frac{1}{6}}\right)^3$$

$$M_{max} = 0.0265\, q_0 L^2$$

Half-range Functions



$$q_0\left(1-\frac{x}{b}\right)+\frac{q_0}{b}\langle x-b\rangle = \quad q_0 - q_0\frac{x}{b} \quad + \quad \frac{q_0}{b}\langle x-b\rangle$$



Can go to $M(x)$ directly off of chart.

$$\uparrow_{A=\frac{5}{24}q_0L} \longrightarrow M_A(x) = \frac{5}{24}q_0L\langle x-0\rangle = \frac{5}{24}q_0Lx$$

$$q_0 \longrightarrow M_q(x) = -\frac{q_0}{2}x^2$$

$$-q_0\frac{x}{b} \longrightarrow M_{q\frac{x}{b}}(x) = \frac{q_0}{6b}x^3 = \frac{q_0}{3L}x^3$$

$$\frac{q_0}{b}\langle x-b\rangle \rightarrow M_{\langle\rangle}(x) = -\frac{q_0}{6b}\langle x-b\rangle^3 = \frac{-q_0}{3L}\left\langle x-\frac{L}{2}\right\rangle^3$$

$$M(x) = \frac{5}{24}q_0Lx - \frac{q_0}{2}x^2 + \frac{q_0}{3L}x^3 - \frac{q_0}{3L}\left\langle x-\frac{L}{2}\right\rangle^3$$

$$V(x) = \frac{dM}{dx} = \frac{5}{24} q_0 L - q_0 x + \frac{q_0}{L} x^2 - \frac{q_0}{L} \langle x - \tfrac{L}{2} \rangle^2$$

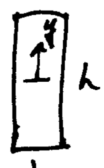
check $\quad V(x=L) = \frac{5}{24} q_0 L - \cancel{q_0 L} + \cancel{q_0 L} - \frac{q_0}{L}\frac{L^2}{4} = -\frac{q_0 L}{24}$ ✓

Plotting follows the same steps and <u>always</u> helps when determining a max/min.

$$M_{max} = 0.0265\, q_0 L^2$$

$$\sigma_{max} = \frac{M_{max}\, y_{max}}{I}$$

<u>Rectangle</u>:

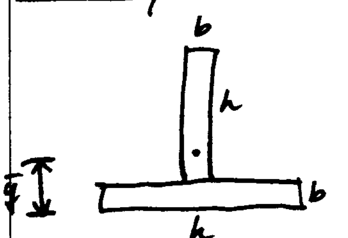


$$\frac{y_{max}}{I} = \frac{h/2}{\frac{1}{12} b_0 h^3} = \frac{6}{b_0 h^2} \equiv \frac{1}{S}$$

for a rectangle

$$\sigma_{max} = 6\,(0.0265)\,\frac{q_0 L^2}{b_0 h^2}$$

<u>Non-symmetric section</u>



Find centroid: $\bar{y} = \dfrac{b/2\, \cancel{hb} + (b + \frac{h}{2})\cancel{hb}}{2\cancel{hb}}$

$= \frac{3}{4} b + \frac{h}{4} = \frac{1}{4}(3b + h)$

Find I: $\quad I = \frac{1}{12} h b^3 + bh\left(\frac{b+h}{4}\right)^2$ } bottom

$\quad + \frac{1}{12} b h^3 + bh\left(\frac{b+h}{4}\right)^2$ } top

$$I = \frac{1}{24} bh\,(5b^2 + 6bh + 5h^2)$$

If $M > 0 \rightarrow \sigma_{max} = \dfrac{-M\left(-\frac{1}{4}(3b+h)\right)}{I}$, $\sigma_{min} = \dfrac{-M\left(\frac{3}{4}h + \frac{1}{4}b\right)}{I}$

Yet another approach with the half range ~~functions~~.
We came up with:

$$q(x) = q_0 - q_0 \frac{x}{b} + \frac{q_0}{b}\langle x-b\rangle$$

To some of you it may be obvious that we can write:

$$q(x) = \frac{q_0}{b}\langle b-x\rangle$$

Check: for $x<b \rightarrow b-x>0 \rightarrow \langle b-x\rangle = b-x$

$$\rightarrow q(x) = \frac{q_0}{b}(b-x) = q_0 - q_0\frac{x}{b} \quad \text{for } x<b$$

for $x>b \rightarrow b-x<0 \rightarrow \langle b-x\rangle = 0$

$$\rightarrow q(x) = 0 \quad \text{for } x>b$$

Let's find $V(x)$ and $M(x)$.

$$\frac{dV}{dx} = -q \rightarrow V(x) = \int -\frac{q_0}{b}\langle b-x\rangle\,dx + c_1$$
$$= \frac{-q_0}{2b}\langle b-x\rangle^2(-1) + c_1$$

This comes from

$$V(x) = \frac{q_0}{2b}\langle b-x\rangle^2 + c_1$$
$$V(x=0) = \frac{q_0 b}{2} + c_1 = A$$
$$c_1 = A - \frac{q_0 b}{2} \quad * \text{ Here } c_1 \neq A$$

$$\frac{dM}{dx} = V \rightarrow M(x) = \frac{-q_0}{6b}\langle b-x\rangle^3 + c_1 x + c_2, \quad M(0)=0 \xrightarrow{**} c_2 = \frac{q_0 b^2}{6}$$