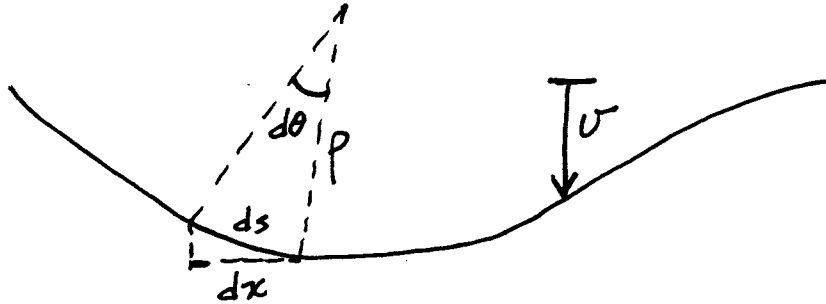


Stresses and Strains in Beams

Kinematics: How do we describe the deformation of a beam?



v = downward deflection/displacement of the beam

ρ = radius of curvature of the deformed beam
(this will vary along the beam)

dx = differential element along the beam

ds = differential arc length of deformed beam

$d\theta$ = differential angle swept out by ds

$$ds = \rho d\theta \quad \text{or} \quad \rho = \frac{1}{\chi}$$

$$\text{and} \quad ds = \frac{1}{\chi} d\theta \quad \chi \equiv \text{curvature}$$

$$\chi = \frac{d\theta}{ds}$$

For beams where the deflection v , and the slope θ is small at all locations, we ~~can~~ can make the approximation that

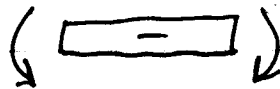
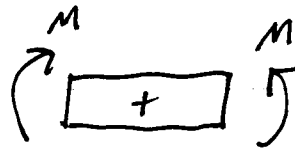
$$\kappa = \frac{d\theta}{ds} \approx \frac{d\theta}{dx} \quad (\text{because } ds \approx dx)$$

Later in the course we will relate θ and κ to the deflection v .

For now we have

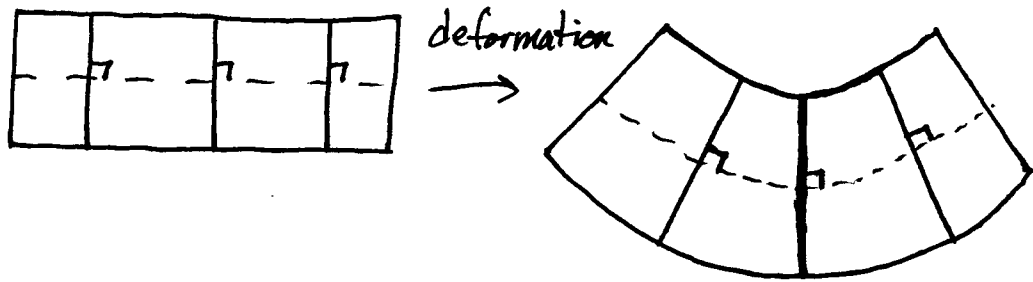
$$\kappa = \frac{1}{\rho} = \frac{d\theta}{dx}$$

Sign conventions:



A kinematic assumption: During the deformation of a beam planar cross sections of the beam that are perpendicular to the longitudinal axis remain planar and perpendicular to the longitudinal axis.

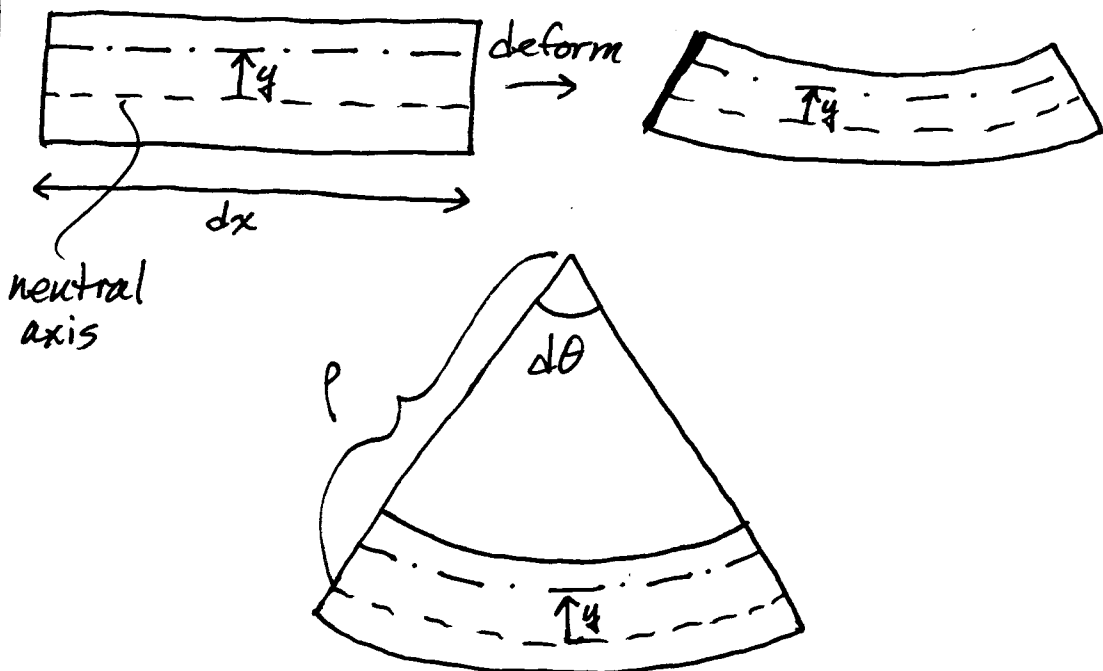
"Planes remain planes" and they remain perpendicular to the longitudinal axis of the beam.



From the "planes remain planes" assumption we can analyze the strain in the beam.

We will measure the coordinate y from a line parallel to the x -direction called the "neutral axis". y will be positive upward (which in my opinion is an unfortunate choice, but this is what the book does).

The neutral axis is the line within the beam that remains unstretched after bending. (No axial force allowed.)

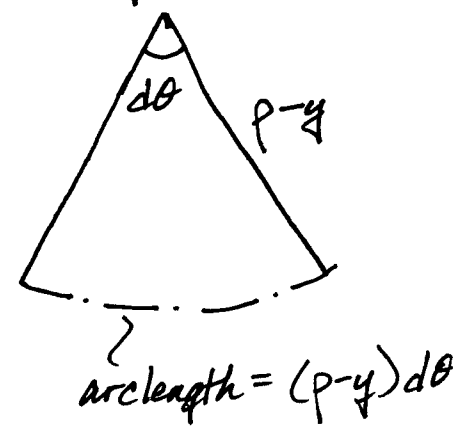


Longitudinal strain, ϵ , of the neutral axis is zero by definition.

What is the strain of the line at y above the neutral axis?

$$\epsilon(y) = \frac{L - L_0}{L_0}$$

$$\epsilon(y) = \frac{(p-y)d\theta - dx}{dx}$$



But, from our definition of the neutral axis we also know $\epsilon(y=0) = 0$.

$$\epsilon(y=0) = \frac{p d\theta - dx}{dx} = 0$$

$$\begin{aligned} \rightarrow p \frac{d\theta}{dx} &= 1 \\ &= x = \frac{1}{p} \checkmark \end{aligned}$$

So: $\epsilon(y) = p \frac{d\theta}{dx} - y \frac{d\theta}{dx} = 1 - y \frac{d\theta}{dx}$

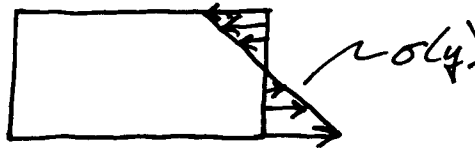
* Important result for kinematics of beams

$$\epsilon(y) = -y \frac{d\theta}{dx} = -x y$$

Now that we have related strain to curvature, we would like to relate stress and eventually moment to curvature.

Material Law: $\sigma = E\varepsilon$ (longitudinal/axial stress in the beam)

$$\sigma = -E\chi y$$

Equilibrium: $M \curvearrowright$ 

$$\sum F_x = \int_A \sigma(y) dA = 0$$

$$\int_A -E\chi y dA = 0$$

$$-E\chi \int_A y dA = 0$$

$$\rightarrow \int_A y dA = 0$$

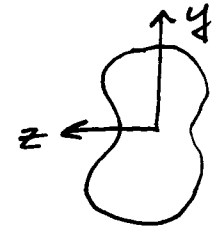
This equation defines the location of the neutral axis.

What this says is that for a homogeneous linear elastic beam the neutral axis passes through the centroid of the cross-sectional area.

$$\sum M_z = -M + \int_A \sigma(y) (-y) dA = 0$$

$$M = \int_A E \chi y^2 dA$$

$$M = E \chi \underbrace{\int_A y^2 dA}$$



I or I_z or I_{zz}
moment of inertia (2nd moment of the area) about the z-axis

$$\boxed{M = EI \chi} \quad **$$

Structural response of a beam in bending.
Compare to:

$$N = EA \frac{\delta}{L} \text{ for a bar}$$

$$T = GI_p \frac{\phi}{L} \text{ for a shaft}$$

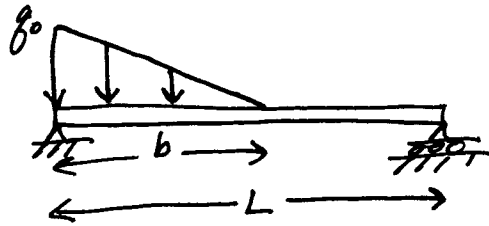
Since we know $\sigma = -E \chi y$ we can relate the stress distribution to the moment.

$$M = EI \chi = - \frac{\sigma I}{y}$$

$$\boxed{\sigma = - \frac{M y}{I}} \quad **$$

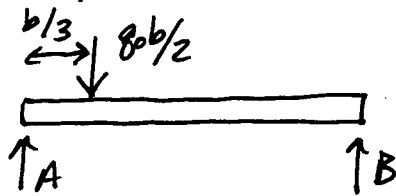
Stress distribution in the cross-section.

Example :



Determine the maximum stress and its location.

First determine $M(x)$.



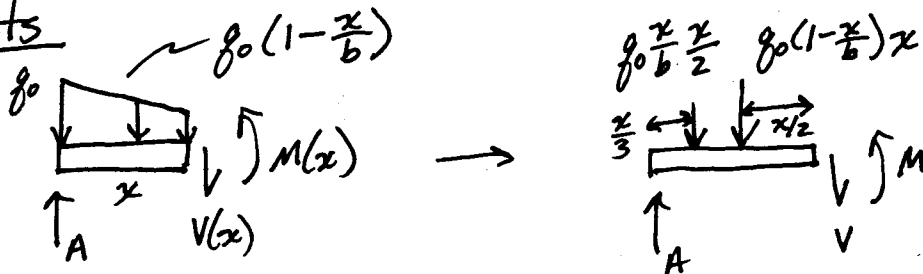
$$\sum M_z^A = BL - \frac{f_0 b^2}{6} = 0$$

$$B = \frac{1}{6} \frac{f_0 b^2}{L}$$

$$\sum F_y = A + B - \frac{f_0 b}{2} = 0$$

$$A = f_0 b \left(\frac{1}{2} - \frac{1}{6} \frac{b}{L} \right)$$

Cuts



$$\sum M_z^x = M - Ax + f_0 \frac{2x^3}{6b} + f_0 \left(1 - \frac{x}{b}\right) \frac{x^2}{2} = 0$$

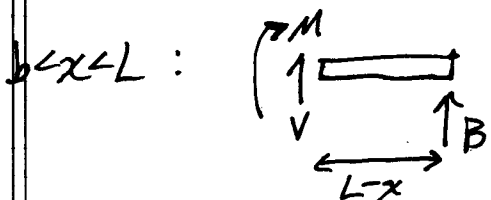
$$M = Ax - f_0 \frac{x^2}{2} + f_0 \frac{x^3}{6b}$$

If you find $M(x)$ first you can always use

$$V(x) = \frac{dM}{dx} = A - f_0 x + f_0 \frac{x^2}{2b} \quad \checkmark \quad \text{(Check that this satisfies } \sum F_y = 0 \text{.)}$$

So, for $0 < x < b$: $M(x) = Ax - g_0 \frac{x^2}{2} + g_0 \frac{x^3}{6b}$

$V(x) = A - g_0 x + g_0 \frac{x^2}{2b}$

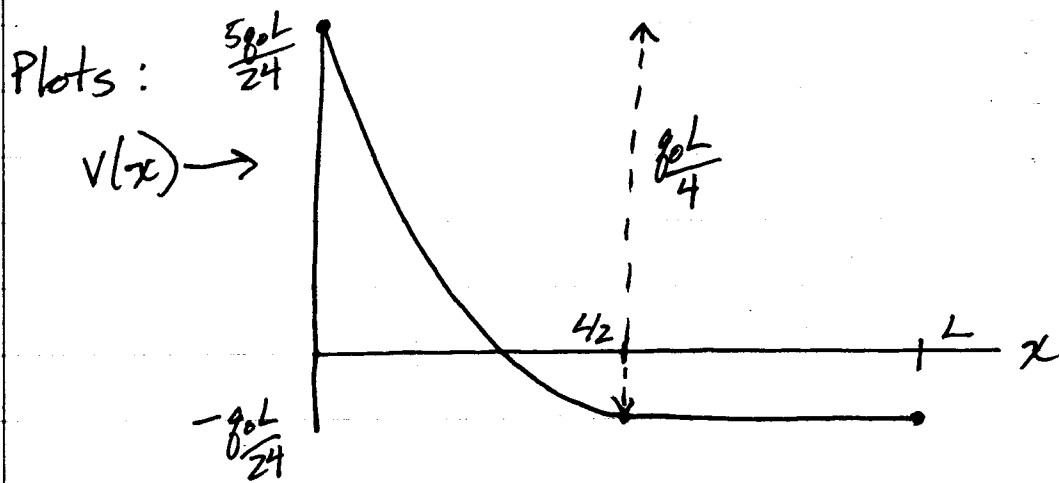


$\sum M_x = -M + B(L-x) = 0$

$M = B(L-x)$

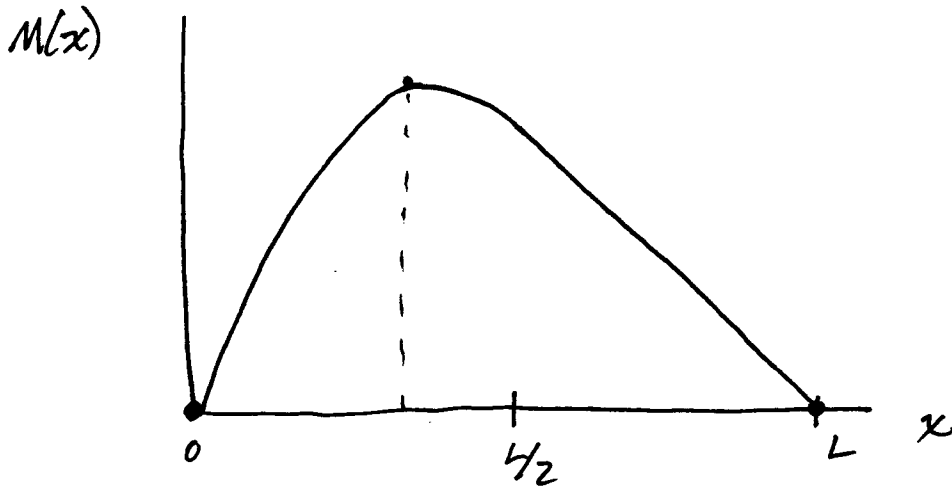
$V = \frac{dM}{dx} = -B \checkmark$

Take $b = \frac{L}{2} \rightarrow$ ~~scribbled out~~, $A = \frac{5}{24} g_0 L$, $B = \frac{1}{24} g_0 L$



$\frac{dV}{dx} = -g \rightarrow$ slope of $V(x) \rightarrow -g_0$ at $x=0$
 $\Rightarrow 0$ at $x=b = \frac{L}{2}$

$V = \int -g dx \rightarrow$ change in $V = -$ area under $g(x)$ up to x
 ΔV at $L/2 = \frac{1}{2} g_0 \left(\frac{L}{2}\right) = g_0 \frac{L}{4}$



$\frac{dM}{dx} = V \rightarrow$ max/min of $M(x)$ occurs where $V(x) = 0$

No point moments $\rightarrow M(x)$ is continuous
 No point forces $\rightarrow M(x)$ is smooth

$$V(x) = \frac{5}{24} \rho_0 L - \rho_0 x + \rho_0 \frac{x^2}{2b} = 0 \quad b = \frac{L}{2}$$

$$x = \frac{\rho_0 \pm \sqrt{\rho_0^2 - 4 \frac{\rho_0^2}{2b} \frac{5}{24} \rho_0 L}}{\rho_0 2/L}$$

$$= \frac{L}{2} \left(1 \pm \sqrt{1 - \frac{5}{6}} \right)$$


$$x_{max} = \frac{L}{2} \left(1 - \sqrt{\frac{1}{6}} \right)$$

$$M_{max} = \frac{5}{24} \rho_0 L \frac{L}{2} \left(1 - \sqrt{\frac{1}{6}} \right) - \frac{\rho_0 L^2}{24} \left(1 - \sqrt{\frac{1}{6}} \right)^2 + \frac{\rho_0}{3L} \frac{L^3}{8} \left(1 - \sqrt{\frac{1}{6}} \right)^3$$

$$M_{max} = 0.0265 \rho_0 L^2$$

$$M_{max} = 0.0265 q_0 L^2$$

$$\sigma_{max} = \frac{M_{max} y_{max}}{I}$$

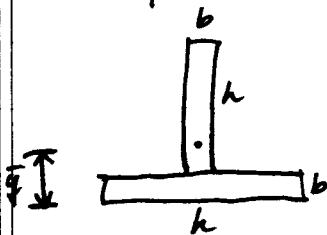
Rectangle: 

$$\frac{y_{max}}{I} = \frac{h/2}{\frac{1}{12} b_0 h^3} = \frac{6}{b_0 h^2} \equiv \frac{1}{S}$$

↑ for a rectangle

$$\sigma_{max} = 6 (0.0265) \frac{q_0 L^2}{b_0 h^2}$$

Non-symmetric section



Find centroid: $\bar{y} = \frac{b/2 hb + (b + \frac{h}{2}) hb}{2hb}$

$$= \frac{3}{4}b + \frac{h}{4} = \frac{1}{4}(3b+h)$$

Find I: $I = \frac{1}{12} hb^3 + bh \left(\frac{b+h}{4}\right)^2$ } bottom

$$+ \frac{1}{12} bh^3 + bh \left(\frac{b+h}{4}\right)^2$$
 } top
$$I = \frac{1}{24} bh (5b^2 + 6bh + 5h^2)$$

If $M > 0 \rightarrow \sigma_{max} = \frac{-M(-\frac{1}{4}(3b+h))}{I}, \sigma_{min} = \frac{-M(\frac{3}{4}h + \frac{1}{4}b)}{I}$