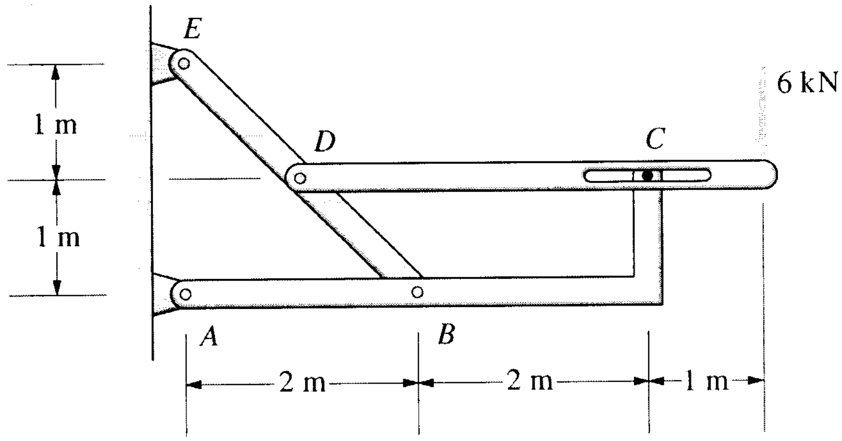
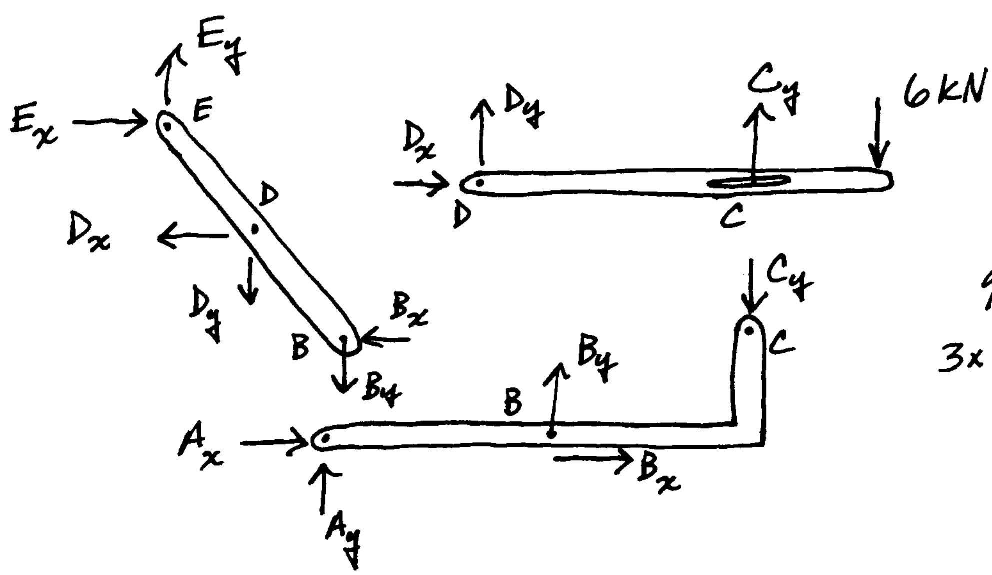


Determine the forces on each of the frame members.

Are any of the frame elements 2-force members? <u>No.</u>

Will an analysis of the entire structure yield significant information? Not really, because there will be 4 support forces but only 3 equilibrium equations.

FBDs of each component:



9 unknowns
3×3 equations

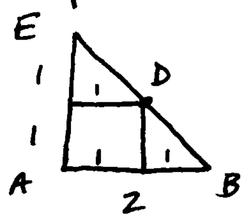
All internal forces appear in equal but opposite pairs.

Which component should we analyze first?
DC because it only has 3 unknown forces.

Equilibrium analysis of DC: $\Sigma F_x = \boxed{D_x = 0}$

$$\Sigma M_z^D = C_y \cdot 3 - 6 \cdot 4 = 0 \rightarrow \boxed{C_y = 8\,kN}$$

$$\Sigma F_y = C_y - 6 + D_y = 0 \rightarrow \boxed{D_y = -2\,kN}$$

Geometry needed: 

Each of the two smaller triangles is similar to EAB.

Next, both EDB and ABC have 4 remaining unknown forces, so we will have to analyze both, and can look at either one first.

ABC:

$$\Sigma F_x = A_x + B_x = 0$$

$$\Sigma M_z^B = -A_y \cdot 2 - C_y \cdot 2 = 0 \rightarrow \boxed{A_y = -8\,kN}$$

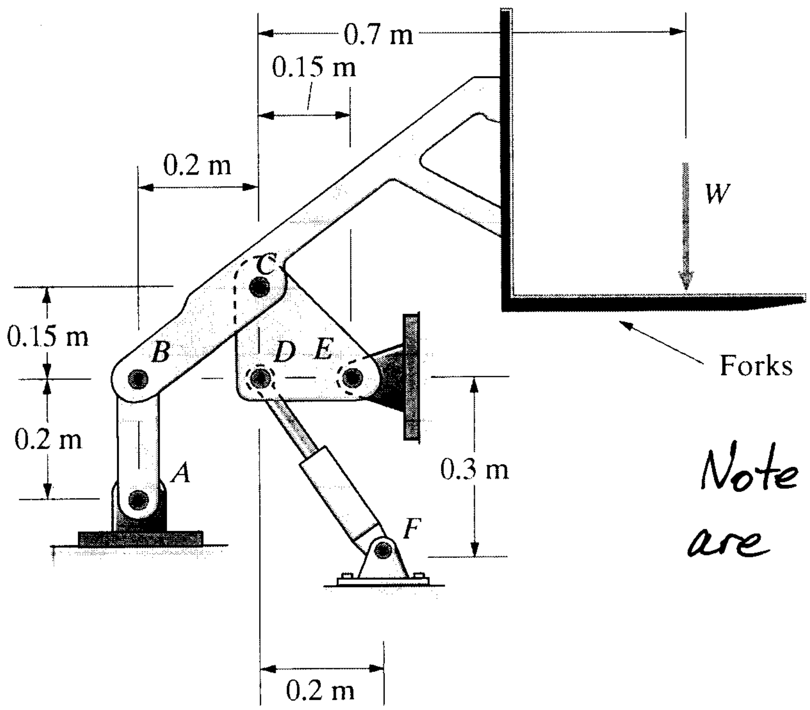
$$\Sigma F_y = A_y + B_y - C_y = 0 \rightarrow \boxed{B_y = 16\,kN}$$

EDB: $\Sigma F_y = E_y - D_y - B_y = 0 \rightarrow \boxed{E_y = 14\,kN}$

$$\Sigma M_z^B = D_x \cdot 1 + D_y \cdot 1 - E_x \cdot 2 - E_y \cdot 2 = 0$$

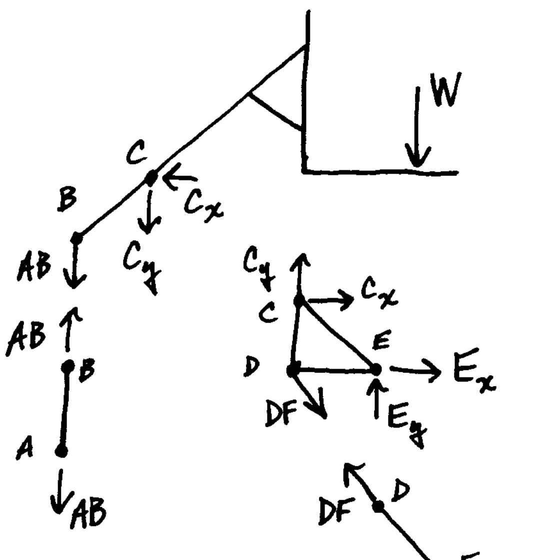
$$\rightarrow E_x = \tfrac{1}{2} D_x + \tfrac{1}{2} D_y - E_y \rightarrow \boxed{E_x = -15\,kN}$$

$$\Sigma F_x = E_x - D_x - B_x = 0 \rightarrow \boxed{B_x = -15\,kN} \rightarrow \boxed{A_x = 15\,kN}$$



Determine all of the forces at the joints.

Note that AB and DF are 2-force members.



6 Unknowns
2×3 equations
(AB and DF do not provide any equations.)

BCW has 3 unknown forces so let's analyze that first.

<u>BCW</u>: $\Sigma F_x = -C_x = 0 \rightarrow \boxed{C_x = 0}$

$$\Sigma M_z^c = -W \cdot 0.7 + AB \cdot 0.2 = 0 \rightarrow \boxed{AB = \tfrac{7}{2}W}$$

$$\Sigma F_y = -AB - W - C_y = 0 \rightarrow \boxed{C_y = \tfrac{-9}{2}W}$$

$$CDE: \quad \Sigma F_x = C_x + E_x + DF \frac{2}{\sqrt{13}} = 0$$

should have done this first

$$\Sigma M_z^E = -\cancel{C_x}^{\,0} \cdot 0.15 - C_y \cdot 0.15 + DF \frac{3}{\sqrt{13}} \cdot 0.15 = 0$$

$$\rightarrow DF = \frac{\sqrt{13}}{3} C_y \rightarrow \boxed{DF = -\frac{3\sqrt{13}}{2} W}$$

$$\Sigma F_x \rightarrow \boxed{E_x = 3W}$$

$$\Sigma F_y = C_y + E_y - DF \frac{3}{\sqrt{13}} = 0$$

$$\rightarrow E_y = -\frac{3\sqrt{13}}{2} W \frac{3}{\sqrt{13}} - \left(-\frac{9}{2} W\right) = \boxed{0 = E_y}$$

Note that DF is in compression. If your intuition has developed then this should make sense to you.

We can also make 2 quick checks (the moment analysis is not so quick) on the equilibrium of the entire structure to see if our answers are possible.

$$\Sigma F_x = DF \frac{2}{\sqrt{13}} + E_x$$
$$= -\frac{3\sqrt{13}}{2} \frac{2}{\sqrt{13}} W + 3W = 0 \checkmark$$

$$\Sigma F_y = -AB - DF \frac{3}{\sqrt{13}} + E_y - W$$
$$= -\frac{7}{2} W + \frac{3\sqrt{13}}{2} \frac{3}{\sqrt{13}} W + 0 - W = 0 \checkmark$$

Distributed Loads

Recall that two force-couple systems are equivalent if

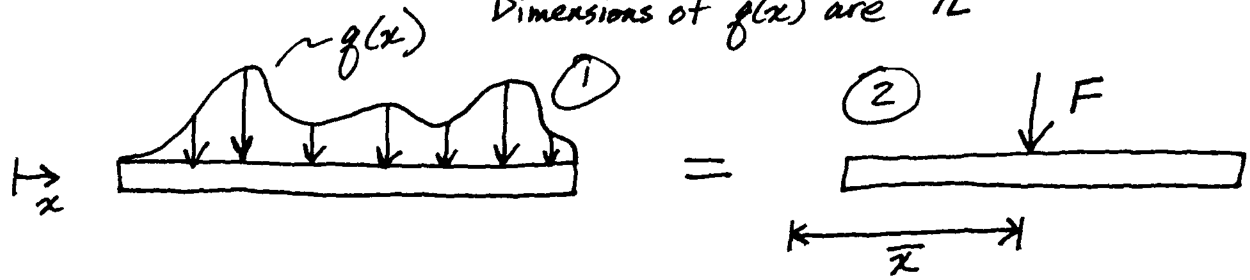
$$\sum \vec{F}^{①} = \sum \vec{F}^{②}$$

and

$$\sum \vec{M}_A^{①} = \sum \vec{M}_A^{②}$$

So far we have only dealt with point forces and couples/moments. However, every type of loading is actually a distributed load (usually over an area for contact forces, or a volume for gravitational/electrical/magnetic forces). For now we will focus on distributions that can be analyzed in one dimension. The extension to higher dimensions is relatively straightforward.

Dimensions of $q(x)$ are F/L

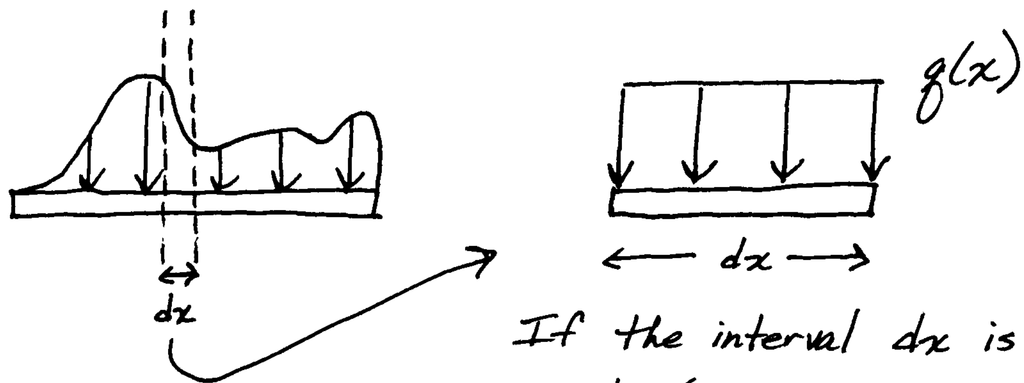


We want to find the point force F and its point of application $\bar{x}$ such that force systems ① and ② are equivalent.

Recall, two force systems are equivalent if the net force is the same and the net moment about any point is the same.

First, $\sum F_y^{①} = \sum F_y^{②}$ (Let's take downward as positive.)

System ② is simple, so let's look at system ①. Consider a small differential element of the beam.



If the interval dx is small enough (i.e. in the limit that $dx \to 0$) then the load on this element of the beam appears to be uniform. Or, to leading order, the distributed load is simply a constant.

Then, the contribution dF to the total force from this element of the beam is

$$dF = \underbrace{q(x)}_{\text{Force per length at the location } x} \; \underbrace{dx}_{\text{length that this force distribution acts over.}}$$

Next we must sum up all of the dF contributions along the beam to get the total force.

$$\Sigma F_y^{(1)} = \Sigma F_y^{(2)}$$

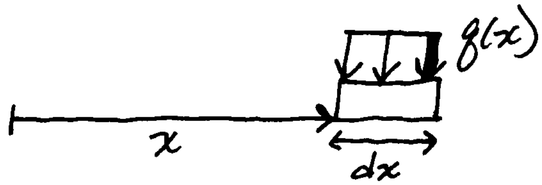
$$\int_L dF = F$$

or $$\int_{x_0}^{x_f} q(x)\,dx = F$$

So now we know what the magnitude F must be, but where do we apply it?

To determine $\bar{x}$ we need to equate the moments about some arbitrary point in each system. To simplify matters (without sacrificing any generality) we can take moments about $x=0$.

Consider our differential element:



$$dM = \underbrace{x}_{\text{moment arm}} \; \underbrace{\overbrace{dF}^{q(x)dx}}_{\text{force}} \quad (+ \text{ into page})$$

Note that in the limit as $dx \to 0$, $x + dx \to x$ and $x + dx/2 \to x$, so the moment arm is just x. (A more rigorous analysis will show that including a dx contribution in the moment arm leads to terms of order dx^2 in dM which are ultimately neglected in the limiting procedure.)

Now we sum up all of our dM contributions and equate to system (2).

$$\sum M_{z,o}^{①} = \sum M_{z,o}^{②}$$

$$\int_L dM = F\bar{x}$$

$$\int_{x_o}^{x_f} x\, q(x)\, dx = F\bar{x}$$

$$\rightarrow \boxed{\bar{x} = \frac{\int_{x_o}^{x_f} x\, q(x)\, dx}{\int_{x_o}^{x_f} q(x)\, dx}}$$

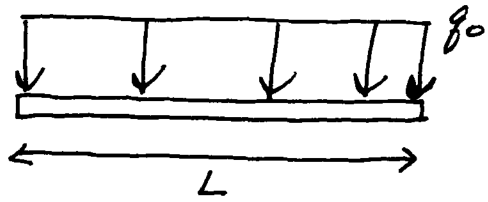
$$\boxed{F = \int_{x_o}^{x_f} q(x)\, dx}$$

Note:
$$\frac{\int_{x_o}^{x_f} x\, q(x)\, dx}{\int_{x_o}^{x_f} q(x)\, dx} \neq \int_{x_o}^{x_f} x\, dx$$

Now we know the force magnitude F and effective point of application $\bar{x}$ which creates an equivalent system to the original distributed load $q(x)$.

Let's take a look at a few examples.

Examples

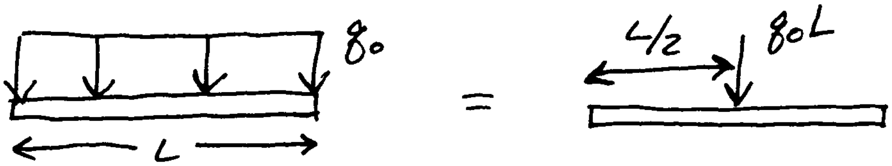


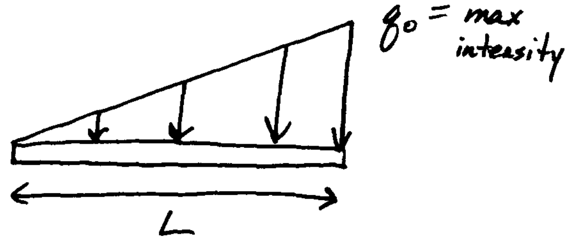
$$q(x) = q_0$$

$$F = \int_0^L q_0 \, dx = q_0 x \Big]_0^L = q_0 L$$

$$M = \int_0^L x q_0 \, dx = \frac{1}{2} q_0 x^2 \Big]_0^L = \frac{1}{2} q_0 L^2$$

$$\bar{x} = \frac{M}{F} = \frac{\frac{1}{2}\cancel{q_0} L^{\cancel{2}}}{\cancel{q_0}\cancel{L}} = \frac{L}{2}$$



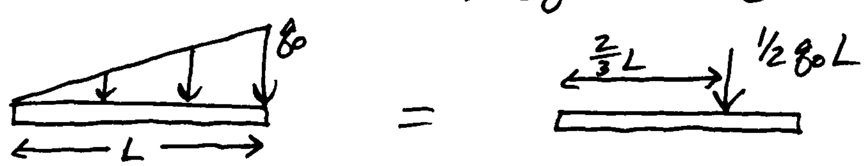


$$q(x) = q_0 \frac{x}{L}$$

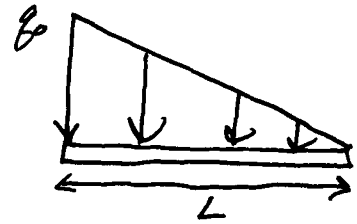
$$F = \int_0^L q_0 \frac{x}{L} \, dx = \frac{1}{2} \frac{x^2}{L} q_0 \Big]_0^L = \underbrace{\frac{1}{2} q_0 L}_{\text{area of the triangle}}$$

$$M = \int_0^L x q_0 \frac{x}{L} \, dx = \frac{1}{3} \frac{x^3}{L} q_0 \Big]_0^L = \frac{1}{3} q_0 L^2$$

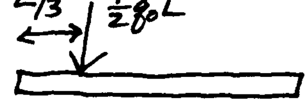
$$\bar{x} = M/F = \frac{1}{3}\cancel{q_0} L^{\cancel{2}} \Big/ \frac{1}{2}\cancel{q_0}\cancel{L} = \frac{2}{3} L$$



How about



From the previous problem it is obvious that the equivalent system is



Or we could do it the long way.

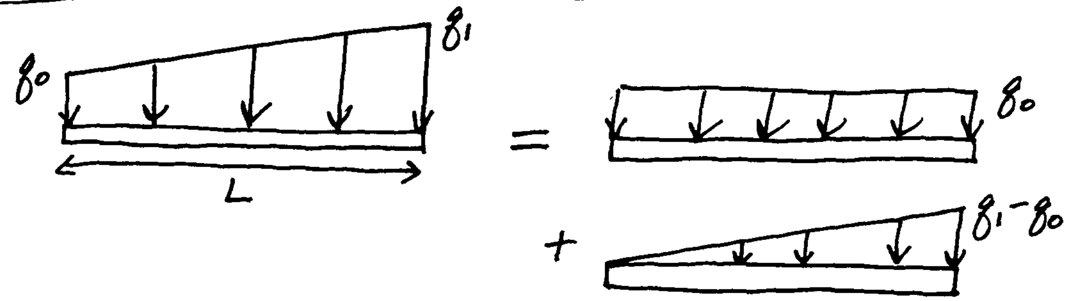
$$q(x) = q_0\left(1 - \frac{x}{L}\right) \rightarrow F = \int_0^L q_0\left(1 - \frac{x}{L}\right) dx = q_0 x - \frac{1}{2} q_0 \frac{x^2}{L}\Big]_0^L$$

$$= q_0 L - \frac{1}{2} q_0 L = \frac{1}{2} q_0 L$$

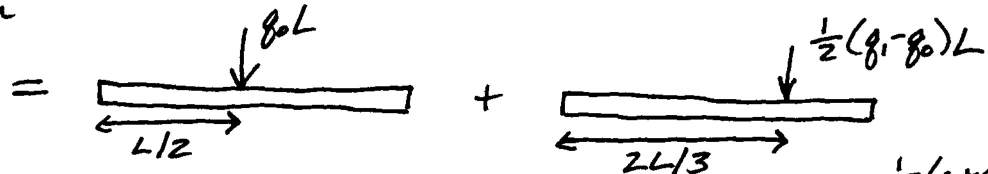
$$M = \int_0^L q_0\left(x - \frac{x^2}{L}\right) dx = \frac{1}{2} q_0 x^2 - \frac{1}{3} q_0 \frac{x^3}{L}\Big]_0^L$$

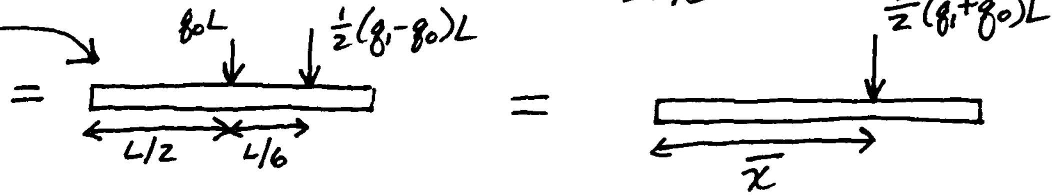
$$= \frac{1}{2} q_0 L^2 - \frac{1}{3} q_0 L^2 = \frac{1}{6} q_0 L^2$$

$$\rightarrow \bar{x} = \frac{M}{F} = \frac{\frac{1}{6} q_0 L^2}{\frac{1}{2} q_0 L} = \frac{1}{3} L$$



For equilibrium analysis this is where we would stop





Determine $\bar{x}$: $q_0 \cancel{4} \frac{L}{2} + \frac{1}{2}(q_1 - q_0)\cancel{4} \frac{2L}{3} = \frac{1}{2}(q_1 + q_0)\cancel{4}\bar{x}$

$$\frac{q_1 L}{3} + \frac{q_0 L}{6} = \frac{1}{2}(q_1 + q_0)\bar{x}$$

$$\bar{x} = \frac{2q_1 + q_0}{3(q_1 + q_0)} L$$

Or, do it the long way

$$q(x) = q_0 + (q_1 - q_0)\frac{x}{L}$$

$$F = \int_0^L q_0 + (q_1 - q_0)\frac{x}{L}\,dx = q_0 x + \frac{1}{2}(q_1 - q_0)\frac{x^2}{L}\Big]_0^L$$

$$= q_0 L + \frac{1}{2}(q_1 - q_0)L$$

$$= \frac{1}{2}(q_0 + q_1)L$$

$$M = \int_0^L q_0 x + (q_1 - q_0)\frac{x^2}{L}\,dx = \frac{1}{2}q_0 x^2 + \frac{1}{3}(q_1 - q_0)\frac{x^3}{L}\Big]_0^L$$

$$= \frac{1}{2}q_0 L^2 + \frac{1}{3}(q_1 - q_0)L^2$$

$$= \frac{q_0 L^2}{6} + \frac{q_1 L^2}{3}$$

$$\rightarrow \bar{x} = \frac{2q_1 + q_0}{3(q_1 + q_0)} L \quad \checkmark$$