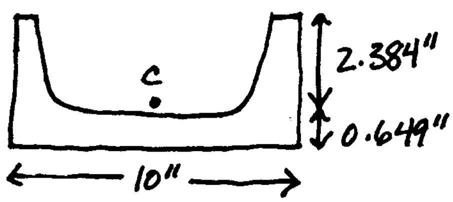
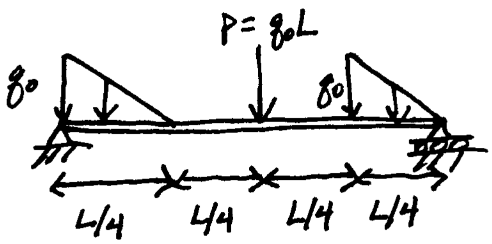
Example: Determine the allowable load intensity q_0 if the allowable stresses in tension and compression are 20 ksi and 11 ksi resplectively. The cross-section is a C 10×30 channel section as shown and $L = 8'$.

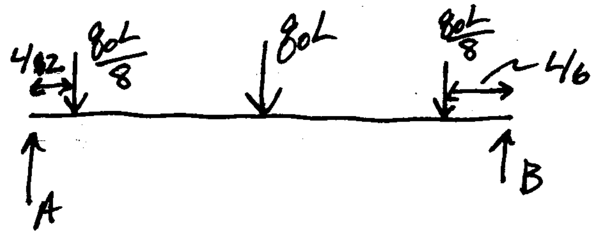


From Appendix E

$I = 3.93\ in^4$ (axis 2-2 in appendix)



Reactions:



$$\Sigma M_z^A = BL - \frac{q_0L}{8}\frac{5L}{6} - q_0L\frac{L}{2} - \frac{q_0L}{8}\frac{L}{12} = 0$$

$$B = \frac{59}{96}q_0L$$

$$\Sigma F_y = A + B - \frac{q_0L}{8} - \frac{q_0L}{8} - q_0L = 0$$

$$A = \frac{61}{96}q_0L$$

[redacted] 

Find M_{max}: $$q(x) = \frac{4q_0}{L}\left\langle \frac{L}{4}-x\right\rangle + q_0 L\left\langle x-\frac{L}{2}\right\rangle^{-1} + q_0\left\langle x-\frac{3L}{4}\right\rangle^{0} - \frac{4q_0}{L}\left\langle x-\frac{3L}{4}\right\rangle$$

$$\frac{dV}{dx} = -q \rightarrow V = +\frac{4q_0}{2L}\left\langle \frac{L}{4}-x\right\rangle^{2} - q_0 L\left\langle x-\frac{L}{2}\right\rangle^{0} - q_0\left\langle x-\frac{3L}{4}\right\rangle^{1} + \frac{4q_0}{2L}\left\langle x-\frac{3L}{4}\right\rangle^{2} + C_1$$

$$V(x=0) = \frac{2q_0}{L}\frac{L^2}{16} - 0 - 0 + 0 + C_1 = \frac{61}{96}q_0 L$$

$$C_1 = \frac{49}{96}q_0 L$$

$$\boxed{V(x) = \frac{2q_0}{L}\left\langle \frac{L}{4}-x\right\rangle^{2} - q_0 L\left\langle x-\frac{L}{2}\right\rangle^{0} - q_0\left\langle x-\frac{3L}{4}\right\rangle + \frac{2q_0}{L}\left\langle x-\frac{3L}{4}\right\rangle^{2} + \frac{49}{96}q_0 L}$$

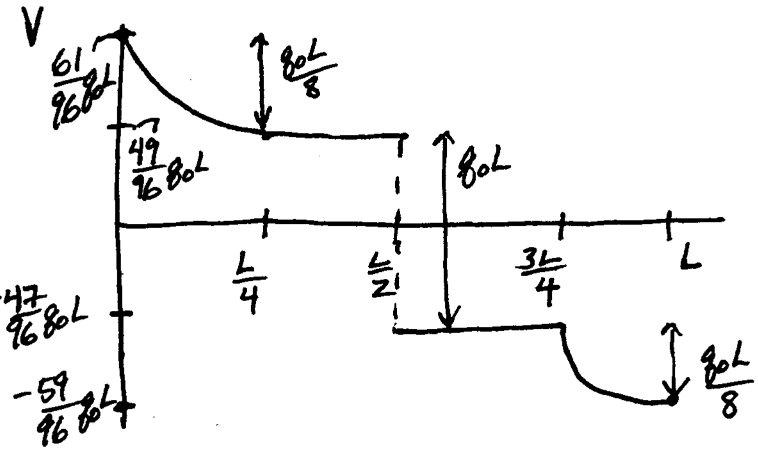
$$\frac{dM}{dx} = V \rightarrow M = -\frac{2q_0}{3L}\left\langle \frac{L}{4}-x\right\rangle^{3} - q_0 L\left\langle x-\frac{L}{2}\right\rangle - \frac{q_0}{2}\left\langle x-\frac{3L}{4}\right\rangle^{2} + \frac{2q_0}{3L}\left\langle x-\frac{3L}{4}\right\rangle^{3} + \frac{49}{96}q_0 L x + C_2$$

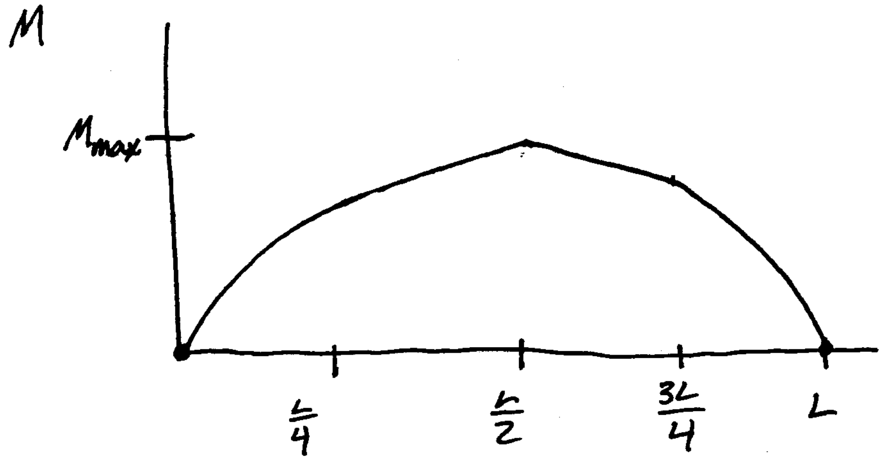
$$M(x=0) = -\frac{2q_0}{3L}\frac{L^3}{64} - 0 - 0 + 0 + 0 + C_2 = 0$$

$$C_2 = \frac{q_0 L^2}{96}$$

$$\boxed{M(x) = -\frac{2q_0}{3L}\left\langle \frac{L}{4}-x\right\rangle^{3} - q_0 L\left\langle x-\frac{L}{2}\right\rangle - \frac{q_0}{2}\left\langle x-\frac{3L}{4}\right\rangle^{2} + \frac{2q_0}{3L}\left\langle x-\frac{3L}{4}\right\rangle^{3} + \frac{49}{96}q_0 L x + \frac{q_0 L^2}{96}}$$

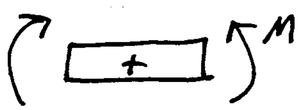
Plots:





M_{max} occurs at $x = \frac{L}{2}$.

$$M\left(x=\frac{L}{2}\right) = 0-0-0+0+\frac{49}{96}q_0L\frac{L}{2}+\frac{q_0L^2}{96} = \underbrace{\frac{51}{192}q_0L^2}_{M_{max}}$$

Recall: 

→ Compression occurs at top for $M>0$
Tension occurs at bottom for $M>0$

$$\sigma = \frac{-My}{I}$$

Therefore the maximum q_0 for tension is

$$\sigma = \frac{51}{192} q_0 \underbrace{L^2}_{(8\cdot 12)^2 in^2} \frac{0.649\ in}{3.93\ in^4} = 20{,}000\ \frac{lbs}{in^2}$$

$$q_0^{max} = 49.5\ \frac{lbs}{in}$$

The maximum q_0 for compression is

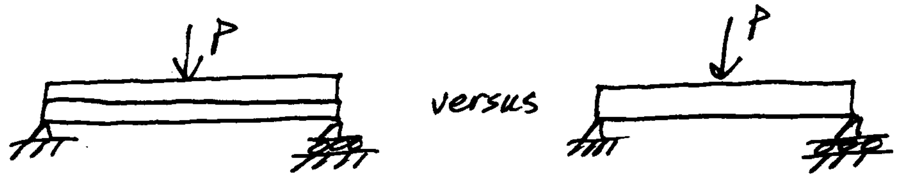
$$\sigma = \frac{51}{192} q_0 \underbrace{L^2}_{(8\cdot 12)^2 in^2} \frac{2.384\ in}{3.93\ in^4} = 11{,}000\ \frac{lbs}{in^2}$$

$$\boxed{q_0^{max} = 7.4\ \frac{lbs}{in}}$$

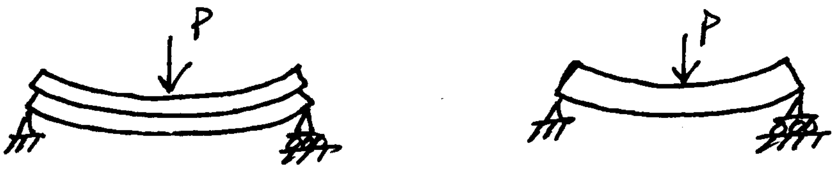
$$\therefore\ q_0^{max} = 7.4\ lbs/in.$$

* Notice that if either the cross-section is asymmetric or if the material strength differs in tension versus compression, then you have to analyze both the top and bottom of the cross-section for the critical condition.

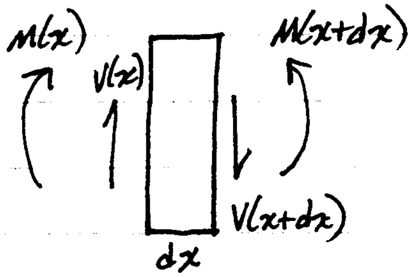
Shear Stresses in Beams

Consider: 

We expect the deformations to differ.



What is the source of the difference? The cut beam is not able to support any shear (at least if the contact is frictionless) along the mid-plane. So far we have only considered the axial stresses in beams. This example demonstrates that there are also shear stresses in beams. The analysis requires us to look at a differential element.

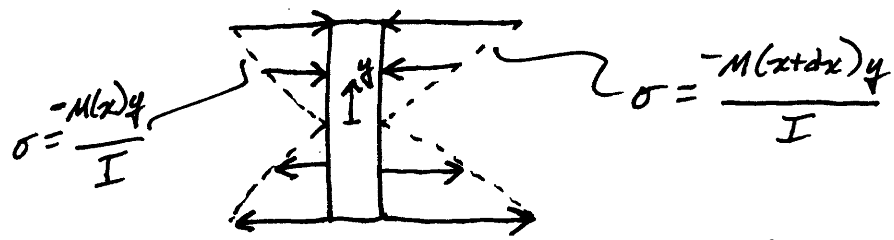


Recall that moment equilibrium gives $\frac{dM}{dx} = V$.

(F_y equilibrium gives $\frac{dV}{dx} = -q$.)

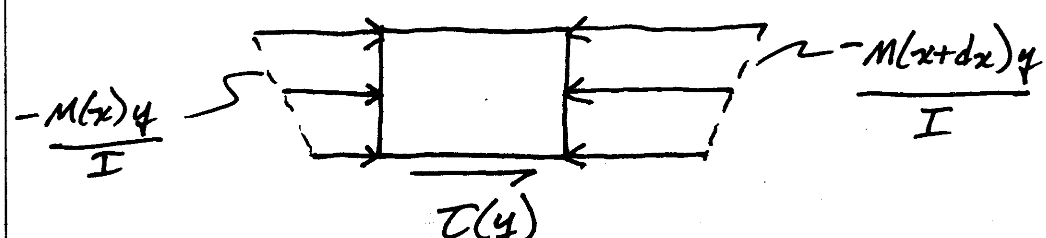
not shown on our FBD

But we need more detail than this. For linear elastic beams we have.



*Shear forces are not shown, but they are still there.

Next, let's cut at some arbitrary y-location.



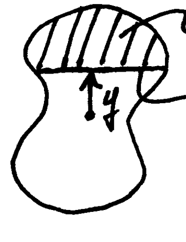
* Again, we are only showing the x-forces/stresses.

$$\text{Equilibrium} \rightarrow \Sigma F_x = -\int_{A(y)} -\frac{M(x)y'}{I}\,dA + \int_{A(y)} \frac{-M(x+dx)y'}{I}\,dA$$

$$+\ \tau(y)\, b(y)\, dx = 0$$

What do we mean by $A(y)$ and $b(y)$?

Cross-section



$A(y)$ = area of shaded region

$b(y)$ = base of shaded region.

$$\Sigma F_x \rightarrow -\left[M(x+dx)-M(x)\right]\frac{1}{I}\underbrace{\int_{A(y)} y'\,dA}_{} + \tau(y)\,b(y)\,dx = 0$$

This is called
$Q(y)$ = first moment of the area of the cross-section above y

Divide by dx and take $\lim_{dx\to 0}$

$$\lim_{dx\to 0}\left[-\frac{M(x+dx)-M(x)}{dx}\,\frac{Q(y)}{I} + \tau(y)\,b(y)\right] = 0$$

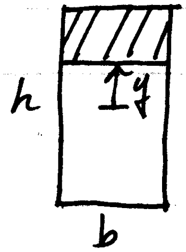
$$\tau(y) = \frac{dM}{dx}\,\frac{Q(y)}{I\,b(y)}$$

but $\dfrac{dM}{dx} = V(x)$

$$\boxed{\tau(x,y) = \frac{V(x)\,Q(y)}{I\,b(y)}} \quad **$$

or simply $\tau = \dfrac{VQ}{Ib}$

Consider a rectangular cross-section.



$b(y) = b$

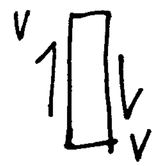
$$Q(y) = \int_y^{h/2} y'\,\underbrace{b\,dy'}_{dA} = \frac{1}{2}b{y'}^2\Big]_y^{h/2} = \frac{bh^2}{8} - \frac{1}{2}by^2$$

The maximum τ occurs where $\frac{Q}{b}$ is a maximum in the cross-section. For the rectangle $\frac{Q}{b} = \frac{h^2}{8} - \frac{1}{2}y^2$, which has a maximum at $y=0$.

$$\therefore \quad \tau_{max} = \frac{12V}{bh^3}\frac{h^2}{8} \quad \text{for the rectangle}$$

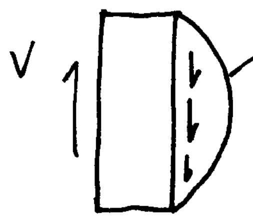
$$\tau_{max} = \frac{3}{2}\frac{V}{bh}$$

What is $\tau_{average}$? $\tau_{average} = \frac{V}{A} = \frac{V}{bh}$



So the maximum is 1.5 times the average.

Let's check this from the distribution.



$$\tau(y) = \frac{12V}{bh^3}\left[\frac{h^2}{8} - \frac{1}{2}y^2\right]$$

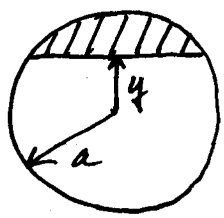
$$\tau(y) = \frac{3}{2}\frac{V}{bh}\left[1 - \left(\frac{2y}{h}\right)^2\right]$$

$$\tau_{average} = \frac{1}{A}\int_A \tau \, dA = \frac{1}{bh}\int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{3}{2}\frac{V}{bh}\left[1-\left(\frac{2y}{h}\right)^2\right] b\,dy$$

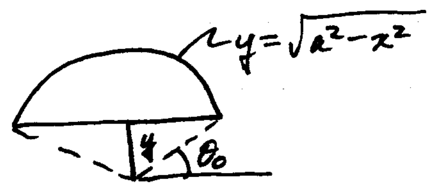
$$= \frac{3}{2}\frac{V}{bh^2}\left[y - \frac{4y^3}{3h^2}\right]_{-h/2}^{h/2}$$

$$= \frac{3}{2}\frac{V}{bh^2}\left[h - \frac{1}{3}h\right] = \frac{V}{bh} \checkmark$$

Consider a circular section.



$$I = \frac{\pi a^4}{4}$$



$$\theta_0(y) = \arcsin\left(\frac{y}{a}\right)$$

$$b(y) = 2a\cos\theta_0(y)$$

$$Q(y) = \int_{-\frac{b(y)}{2}}^{\frac{b(y)}{2}} \int_{y}^{\sqrt{a^2-x^2}} y' \, dy' \, dx$$

$$= \int_{-\frac{b(y)}{2}}^{\frac{b(y)}{2}} \frac{1}{2} y'^2 \Big]_{y}^{\sqrt{a^2-x^2}} dx$$

$$= \int_{-\frac{b}{2}}^{b/2} \frac{1}{2}a^2 - \frac{1}{2}x^2 - \frac{1}{2}y^2 \, dx$$

$$Q(y) = \frac{1}{2}a^2 b(y) - \frac{1}{24} b^3(y) - \frac{1}{2} b(y) y^2$$

$$\frac{Q(y)}{b(y)} = \frac{1}{2}(a^2 - y^2) - \frac{b^2(y)}{24}$$

Again, this is a maximum when $y = 0 \rightarrow \left.\frac{Q}{b}\right|_{max} = \frac{1}{2}a^2 - \frac{a^2}{6}$

$(b(y=0) = 2a)$

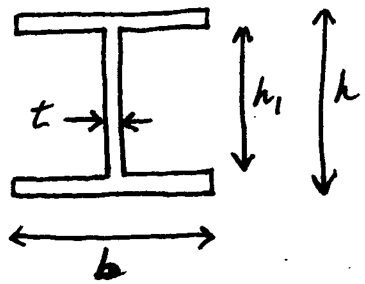
$$\left.\frac{Q}{b}\right|_{max} = \frac{a^2}{3}$$

$$\tau_{max} = \left.\frac{VQ}{Ib}\right|_{max} = \frac{4Va^2}{\pi a^4 3} = \frac{4}{3}\frac{V}{\pi a^2}$$

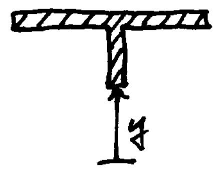
$$\boxed{\tau_{max} = \frac{4}{3}\frac{V}{A}}$$

for a circular cross-section

Consider an I-beam.



Consider y in the web.



$$Q(y) = b\int_{h_1/2}^{h/2} y'\,dy' + t\int_{y}^{h_1/2} y'\,dy'$$

$$Q(y) = \frac{1}{2}b\left(\frac{h^2-h_1^2}{4}\right) + t\left(\frac{h_1^2}{8} - \frac{1}{2}y^2\right)$$

$b(y) = t$ in the web

$$\rightarrow \frac{Q(y)}{b(y)} = \frac{1}{8}\frac{b}{t}\left(h^2-h_1^2\right) + \frac{h_1^2}{8} - \frac{y^2}{2}$$

$$\tau_{web} = \frac{V}{8It}\left[b(h^2-h_1^2) + t(h_1^2 - 4y^2)\right]$$

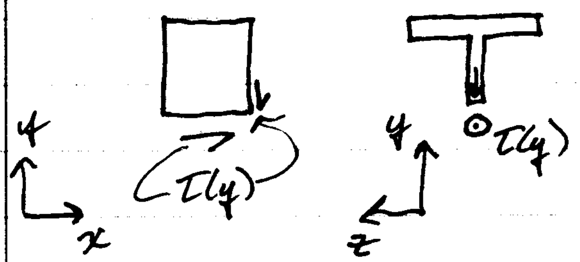
where $I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12}\left(bh^3 - bh_1^3 + th_1^3\right)$

$$\tau_{web}^{max} = \tau_{web}(y=0) = \frac{V}{8It}\left(bh^2 - bh_1^2 + th_1^2\right)$$

$$\tau_{web}^{min} = \tau_{web}\left(y=\frac{h_1}{2}\right) = \frac{V}{8It}\left(bh^2 - bh_1^2\right)$$

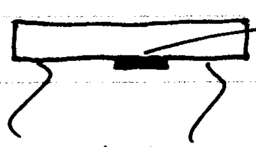
This formula of [redacted] $\tau = \frac{VQ}{Ib}$ is actually approximate. Why?

Consider the I-beam. [redacted] We assumed τ was constant across the length $b(y)$. This is not true in general.



This assumption is pretty good in the web.

However, consider the location where the web and the flange meet.



Here $\tau \neq 0$ because there is material on the other side of the cut applying load.

Here and here τ must be zero because these surfaces are not loaded.

So at this location it is impossible for τ to be uniformly distributed along $b(y)$.

In fact, the only type of section where it is possible for this assumption to hold is a rectangular one.

So, $\tau(x,y) = \dfrac{V(x)\,Q(y)}{I\,b(y)}$ is approximate for non-rectangular sections.