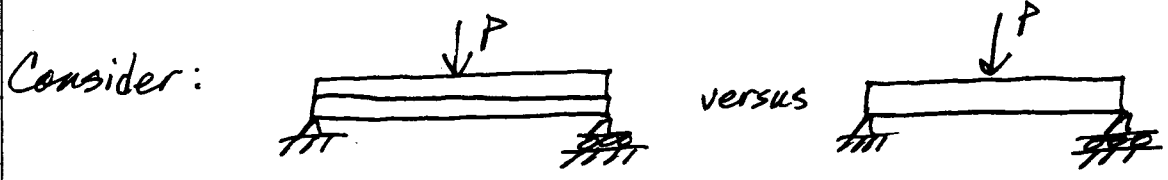


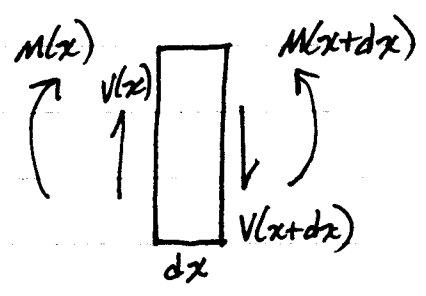
# Shear Stresses in Beams



We expect the deformations to differ.



What is the source of the difference?  
 The cut beam is not able to support any shear (at least if the contact is frictionless) along the mid-plane. So far we have only considered the axial stresses in beams. This example demonstrates that there are also shear stresses in beams. The analysis requires us to look at a differential element.

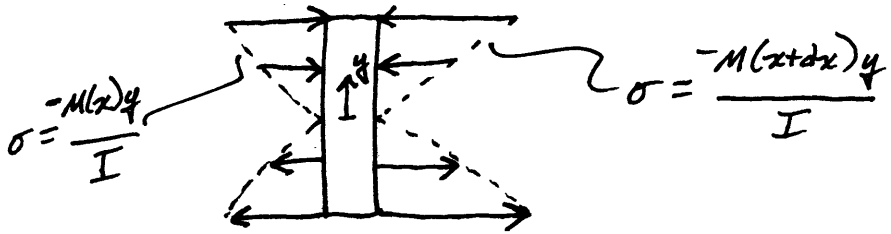


Recall that moment equilibrium gives  $\frac{dM}{dx} = V$ .

( $F_y$  equilibrium gives  $\frac{dV}{dx} = -\rho g$ .)

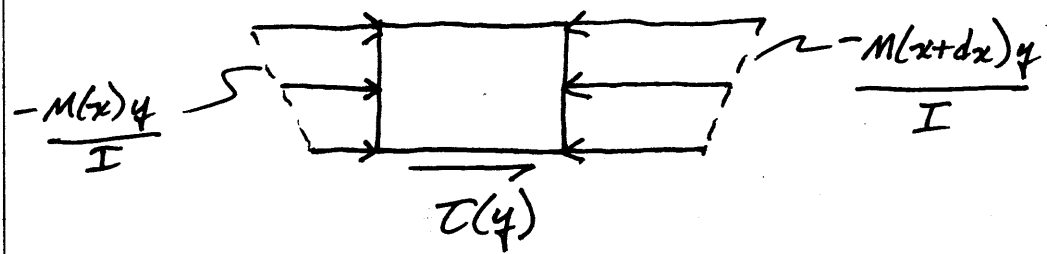
↳ not shown on our FBD

But we need more detail than this. For linear elastic beams we have.



\* Shear forces are not shown, but they are still there.

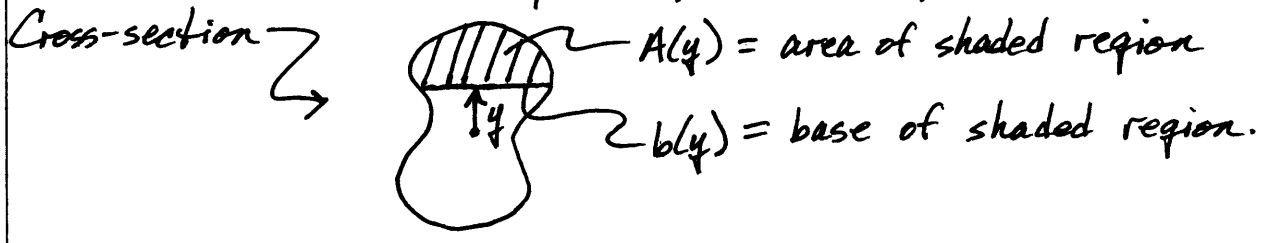
Next, let's cut at some arbitrary y-location.



\* Again, we are only showing the x-forces/stresses.

Equilibrium  $\rightarrow \sum F_x = - \int_{A(y)} \frac{-M(x)y}{I} dA + \int_{A(y)} \frac{-M(x+dx)y}{I} dA + \tau(y) b(y) dx = 0$

What do we mean by  $A(y)$  and  $b(y)$ ?



$$\Sigma F_x \rightarrow -[M(x+dx) - M(x)] \frac{1}{I} \underbrace{\int_{A(y)} y' dA + \tau(y) b(y) dx}_{Q(y)} = 0$$

This is called  
 $Q(y) =$  first moment of the area of the cross-section above  $y$

Divide by  $dx$  and take  $\lim_{dx \rightarrow 0}$

$$\lim_{dx \rightarrow 0} \left[ - \frac{M(x+dx) - M(x)}{dx} \frac{Q(y)}{I} + \tau(y) b(y) \right] = 0$$

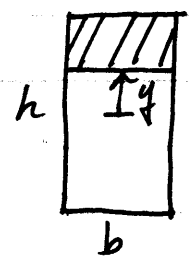
$$\tau(y) = \frac{dM}{dx} \frac{Q(y)}{I b(y)}$$

but  $\frac{dM}{dx} = V(x)$

$$\tau(x, y) = \frac{V(x) Q(y)}{I b(y)} \quad **$$

or simply  $\tau = \frac{VQ}{Ib}$

Consider a rectangular cross-section.



$$b(y) = b$$

$$Q(y) = \int_y^{h/2} y' \underbrace{b dy'}_{dA} = \frac{1}{2} b y'^2 \Big|_y^{h/2} = \frac{bh^2}{8} - \frac{1}{2} b y^2$$

The maximum  $\tau$  occurs where  $\frac{Q}{b}$  is a maximum in the cross-section. For the rectangle  $\frac{Q}{b} = \frac{h^2}{8} - \frac{1}{2}y^2$ , which has a maximum at  $y=0$ .

$$\therefore \tau_{\max} = \frac{12V}{bh^3} \frac{h^2}{8} \text{ for the rectangle}$$

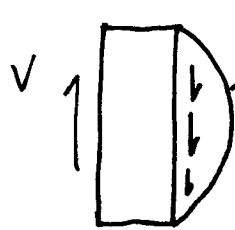
$$\tau_{\max} = \frac{3}{2} \frac{V}{bh}$$

What is  $\tau_{\text{average}}$ ?  $\tau_{\text{average}} = \frac{V}{A} = \frac{V}{bh}$



So the maximum is 1.5 times the average.

Let's check this from the distribution.

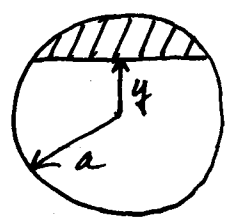


$$\tau(y) = \frac{12V}{bh^3} \left[ \frac{h^2}{8} - \frac{1}{2}y^2 \right]$$

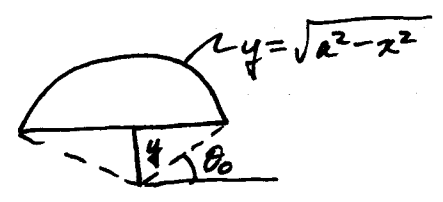
$$\tau(y) = \frac{3}{2} \frac{V}{bh} \left[ 1 - \left( \frac{2y}{h} \right)^2 \right]$$

$$\begin{aligned} \tau_{\text{average}} &= \frac{1}{A} \int_A \tau \, dA = \frac{1}{bh} \int_{-\frac{h}{2}}^{\frac{h}{2}} \frac{3}{2} \frac{V}{bh} \left[ 1 - \left( \frac{2y}{h} \right)^2 \right] b \, dy \\ &= \frac{3}{2} \frac{V}{bh^2} \left[ y - \frac{4y^3}{3h^2} \right]_{-\frac{h}{2}}^{\frac{h}{2}} \\ &= \frac{3}{2} \frac{V}{bh^2} \left[ h - \frac{1}{3}h \right] = \frac{V}{bh} \checkmark \end{aligned}$$

Consider a circular section.



$$I = \frac{\pi a^4}{4}$$



$$\theta_0(y) = \arcsin\left(\frac{y}{a}\right)$$

$$b(y) = 2a \cos \theta_0(y)$$

$$\begin{aligned}
 Q(y) &= \int_{-\frac{b(y)}{2}}^{\frac{b(y)}{2}} \int_y^{\sqrt{a^2-x^2}} y' dy' dx \\
 &= \int_{-\frac{b(y)}{2}}^{\frac{b(y)}{2}} \left[ \frac{1}{2} y'^2 \right]_y^{\sqrt{a^2-x^2}} dx \\
 &= \int_{-\frac{b}{2}}^{\frac{b}{2}} \left( \frac{1}{2} a^2 - \frac{1}{2} x^2 - \frac{1}{2} y^2 \right) dx
 \end{aligned}$$

$$Q(y) = \frac{1}{2} a^2 b(y) - \frac{1}{24} b^3(y) - \frac{1}{2} b(y) y^2$$

$$\frac{Q(y)}{b(y)} = \frac{1}{2} (a^2 - y^2) - \frac{b^2(y)}{24}$$

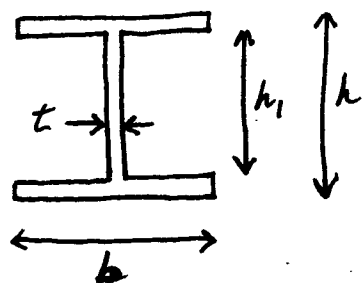
Again, this is a maximum when  $y=0 \rightarrow \frac{Q}{b} \Big|_{\max} = \frac{1}{2} a^2 - \frac{a^2}{6}$   
( $b(y=0) = 2a$ )

$$\frac{Q}{b} \Big|_{\max} = \frac{a^2}{3}$$

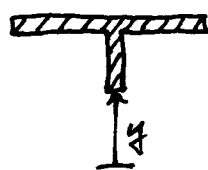
$$I_{\max} = \frac{VQ}{Ib} \Big|_{\max} = \frac{4Va^2}{\pi a^4 3} = \frac{4}{3} \frac{V}{\pi a^2}$$

$I_{\max} = \frac{4}{3} \frac{V}{A}$   
for a circular cross-section

Consider an I-beam.



Consider  $y$  in the web.



$$Q(y) = b \int_{h_1/2}^{h/2} y' dy' + t \int_y^{h_1/2} y' dy'$$

$$Q(y) = \frac{1}{2} b \left( \frac{h^2 - h_1^2}{4} \right) + t \left( \frac{h_1^2}{8} - \frac{1}{2} y^2 \right)$$

$b(y) = t$  in the web

$$\rightarrow \frac{Q(y)}{b(y)} = \frac{1}{8} \frac{b}{t} (h^2 - h_1^2) + \frac{h_1^2}{8} - \frac{y^2}{2}$$

$$\tau_{web} = \frac{V}{8It} \left[ b(h^2 - h_1^2) + t(h_1^2 - 4y^2) \right]$$

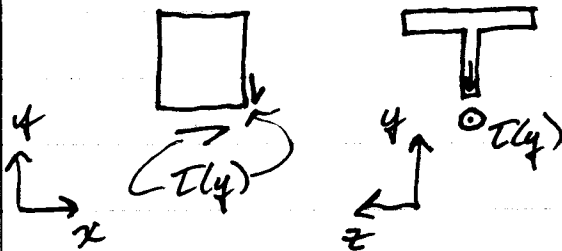
where  $I = \frac{bh^3}{12} - \frac{(b-t)h_1^3}{12} = \frac{1}{12} (bh^3 - bh_1^3 + th^3)$

$$\tau_{web}^{max} = \tau_{web}(y=0) = \frac{V}{8It} (bh^2 - bh_1^2 + th_1^2)$$

$$\tau_{web}^{min} = \tau_{web}(y = \frac{h_1}{2}) = \frac{V}{8It} (bh^2 - bh_1^2)$$

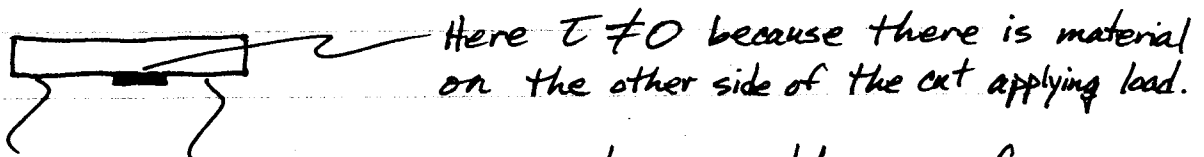
This formula of ~~the~~  $\tau = \frac{VQ}{Ib}$  is actually approximate. Why?

Consider the I-beam. ~~We~~ We assumed  $\tau$  was constant across the length  $b(y)$ . This is not true in general.



This assumption is pretty good in the web.

However, consider the location where the web and the flange meet.



Here and here  $\tau$  must be zero because these surfaces are not loaded.

So at this location it is impossible for  $\tau$  to be uniformly distributed along  $b(y)$ .

In fact, the only type of section where it is possible for this assumption to hold is a rectangular one.

So,  $\tau(x, y) = \frac{V(x)Q(y)}{I b(y)}$  is approximate for non-rectangular sections.