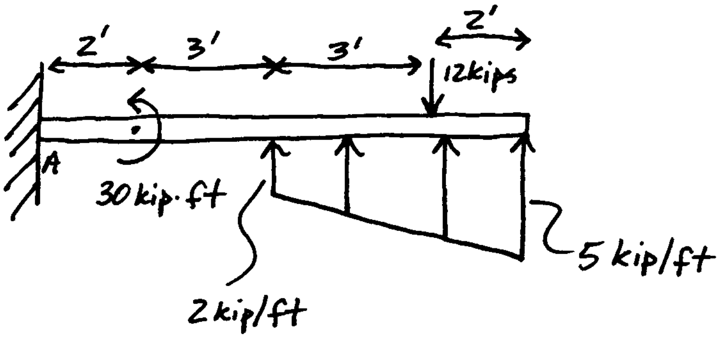
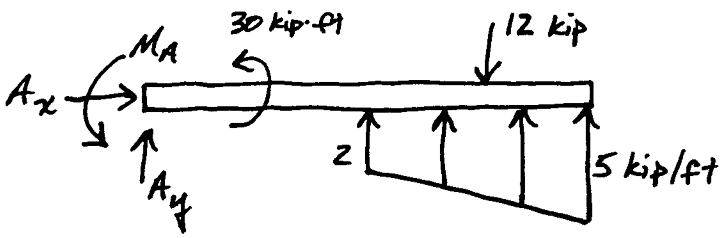
Example:

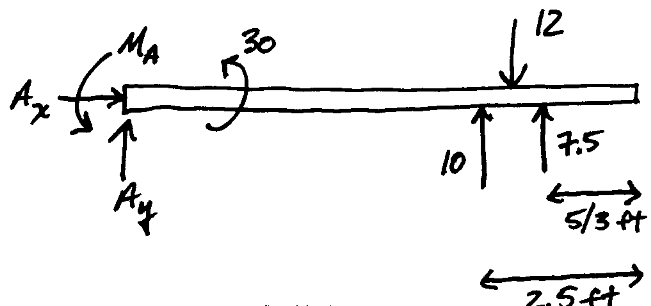


Determine the reactions at A.

FBD



Equivalent System:



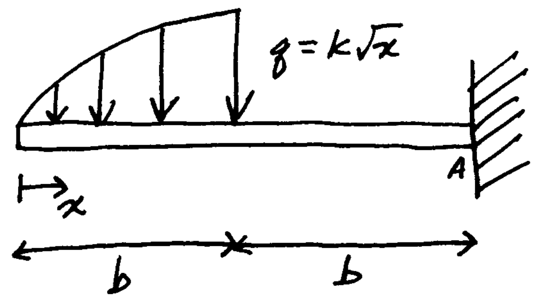
Equilibrium: $\Sigma F_x = \boxed{A_x = 0}$

$$\Sigma F_y = A_y - 12 + 10 + 7.5 = 0 \rightarrow \boxed{A_y = -5.5 \text{ kips}}$$

$$\Sigma M_z^A = M_A + 30 - 12\cdot 8 + 10\cdot 7.5 + 7.5\cdot\left(10-\tfrac{5}{3}\right) = 0$$

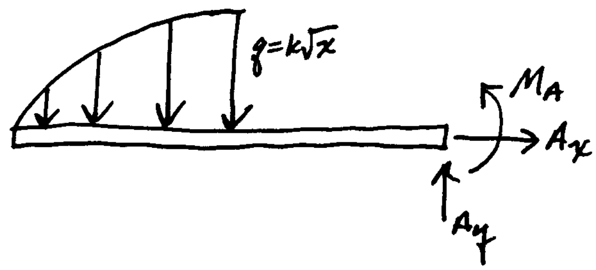
$$\rightarrow \boxed{M_A = -71.5 \text{ kip}\cdot\text{ft}}$$ (− implies CW)

<u>Example</u>:



Determine the reactions at A.

FBD:



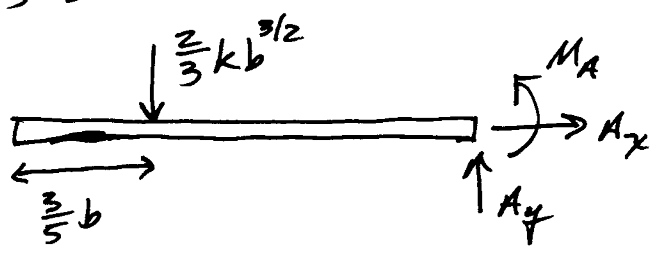
Equivalent force:

$$F = \int_0^b k\sqrt{x}\, dx = \frac{2}{3} k x^{3/2} \Big]_0^b = \frac{2}{3} k b^{3/2}$$

$$M_o = \int_0^b k x^{3/2}\, dx = \frac{2}{5} k x^{5/2} \Big]_0^b = \frac{2}{5} k b^{5/2}$$

$$\bar{x} = \frac{M_o}{F} = \frac{3}{5} b$$

Equivalent system:



Equilibrium: $\Sigma F_x = A_x = 0$

$$\Sigma F_y = -\frac{2}{3} k b^{3/2} + A_y = 0 \rightarrow A_y = \frac{2}{3} k b^{3/2}$$

$$\Sigma M_z^A = M_A + \frac{2}{3} k b^{3/2} \cdot \frac{7}{5} b = 0 \rightarrow M_A = \frac{-14}{15} k b^{5/2}$$

We are also given that $q(x=b) = q_0$.

This means that $k\sqrt{b} = q_0 \rightarrow k = q_0/\sqrt{b}$

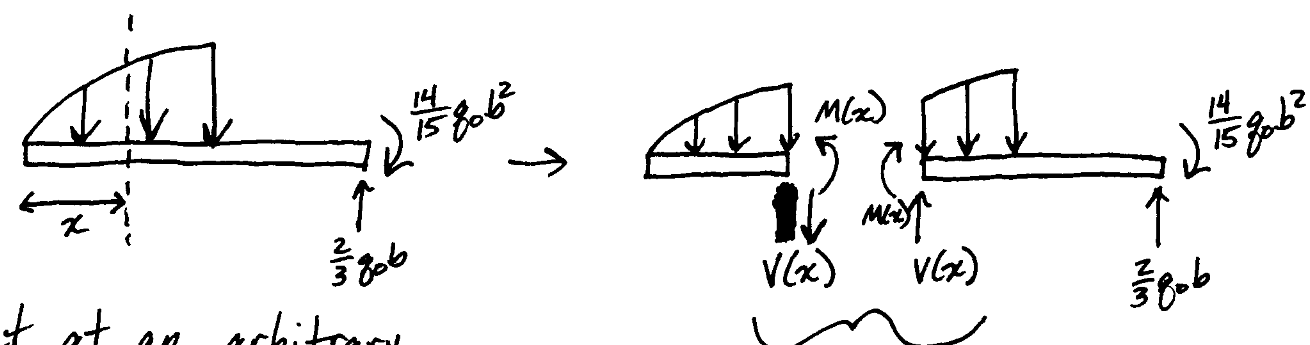
$$\rightarrow \boxed{A_x = 0 \;,\quad A_y = \frac{2}{3} q_0 b \;,\quad M_A = \frac{-14}{15} q_0 b^2}$$

What if we want to know the distribution of internal forces and moments carried by the beam?

To do this we need to cut the beam to expose the internal reactions.

* When making cuts, you must always cut the distributed load before you analyze its equivalent force and point of application.

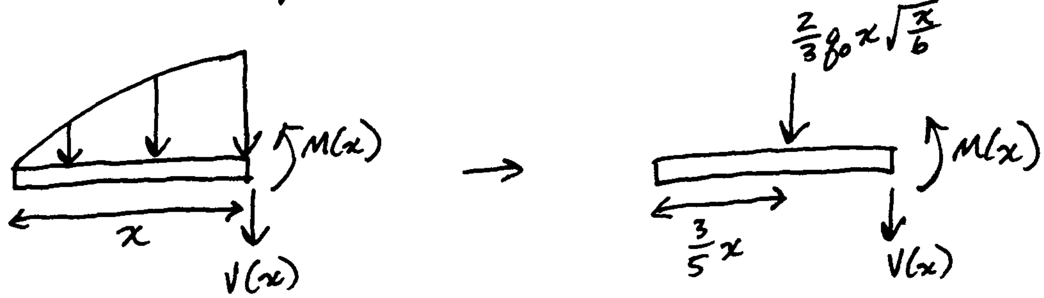
Let's consider our current example.



Cut at an arbitrary value of x, but less than b.

Sign conventions used in the book

We now analyze one of the sides.



$$F = \int_0^x k\sqrt{x}\,dx = \frac{2}{3} k x^{3/2} = \frac{2}{3} q_0 x \sqrt{\frac{x}{b}}$$
$$M_0 = \int_0^x k x^{3/2} dx = \frac{2}{5} k x^{5/2} \rightarrow \bar{x} = \frac{M_0}{F} = \frac{3}{5} x$$

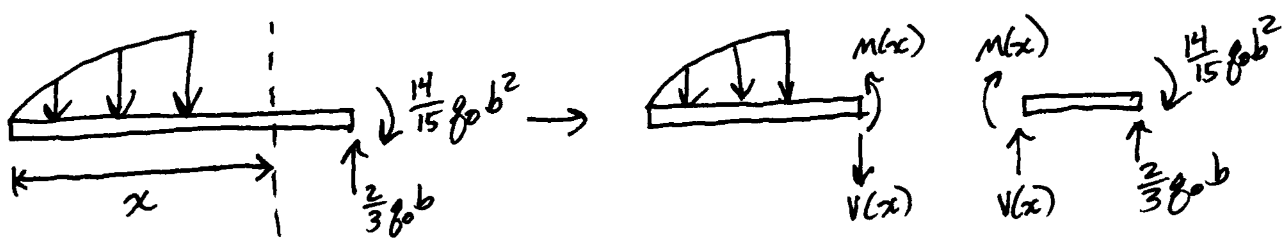
$$\Sigma F_y = -\frac{2}{3} q_0 x \sqrt{\frac{x}{b}} - V(x) = 0 \rightarrow V(x) = -\frac{2}{3} q_0 x \sqrt{\frac{x}{b}}$$

$$\Sigma M_z^x = M(x) + \frac{2}{3} q_0 x \sqrt{\frac{x}{b}}\, \frac{2}{5} x = 0$$

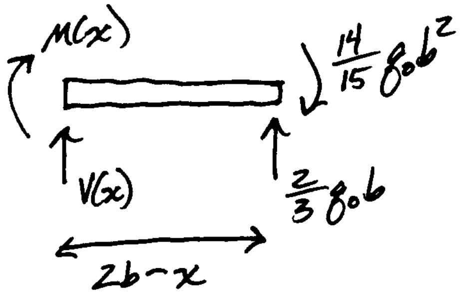
$$\rightarrow M(x) = -\frac{4}{15} q_0 \sqrt{\frac{x}{b}}\, x^2$$

$$\left.\begin{aligned} V(x) &= -\frac{2}{3} q_0 \sqrt{\frac{x}{b}}\, x \\ M(x) &= -\frac{4}{15} q_0 \sqrt{\frac{x}{b}}\, x^2 \end{aligned}\right\} \cancel{\text{for}}\ x \le b$$

Since we have a discontinuous/non-smooth loading, we need to make another cut at an arbitrary point on the other side of the load.



Analyze one of the sides.



$$\Sigma F_y = V(x) + \frac{2}{3} q_0 b = 0$$
$$\rightarrow \quad V(x) = -\frac{2}{3} q_0 b$$

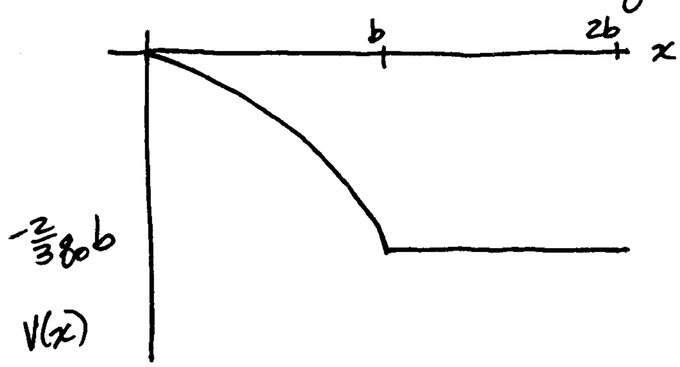
$$\Sigma M_z^x = -M(x) - \frac{14}{15} q_0 b^2 + \underbrace{\frac{2}{3} q_0 b (2b - x)}_{\frac{4}{3} q_0 b^2 - \frac{2}{3} q_0 b x} = 0$$

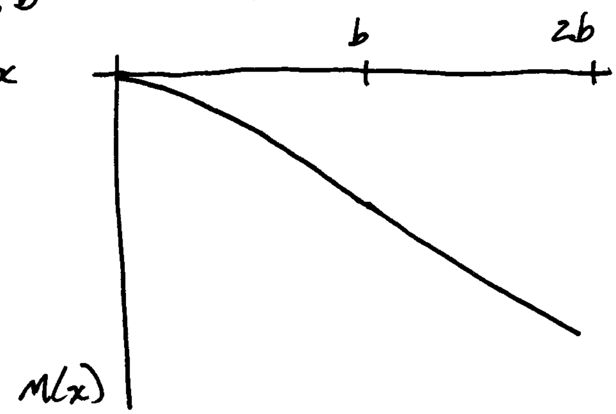
$$\rightarrow M(x) = \frac{2}{5} q_0 b^2 - \frac{2}{3} q_0 b x$$

$$\left.\begin{aligned} V(x) &= -\frac{2}{3} q_0 b \\ M(x) &= \frac{2}{5} q_0 b^2 - \frac{2}{3} q_0 b x \end{aligned}\right\} x \geq b$$

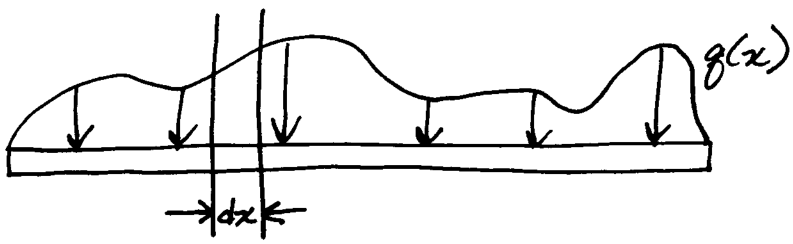
Check:

$V(x=0) = 0$
$M(x=0) = 0$
$V(x=b^-) = -\frac{2}{3} q_0 b$
$V(x=b^+) = -\frac{2}{3} q_0 b$
$M(x=b^-) = -\frac{4}{15} q_0 b^2$
$M(x=b^+) = -\frac{4}{15} q_0 b^2$
$V(x=2b) = -\frac{2}{3} q_0 b$
$M(x=2b) = -\frac{14}{15} q_0 b^2$

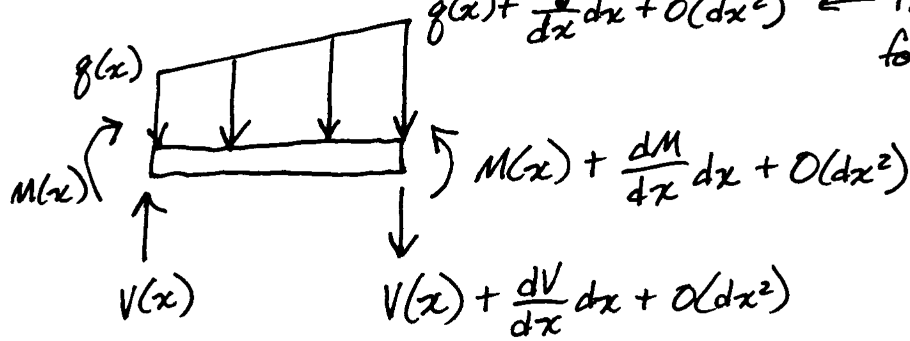




General Relationships for Constructing Shear Force - Bending Moment Diagrams



Consider the small element dx



Note: $O(dx^2)$ means a term with leading "order" of dx times dx. For example, such a term can look like $5dx^2 - 7dx^3 + 9dx^4$, but cannot look like $2dx + 5dx^2 + \ldots$

↑ This is $O(dx)$

Equilibrium: $\Sigma F_y = \cancel{V(x)} - \left[\cancel{V(x)} + \frac{dV}{dx}dx + O(dx^2)\right]$

$$- q(x)dx - \frac{1}{2}\frac{dq}{dx}dx^2 - O(dx^3) = 0$$

Divide by dx → $-\frac{dV}{dx} - q(x) + O(dx) = 0$

Take limit as $dx \to 0$ ⇒ $\boxed{\frac{dV}{dx} = -q(x)}$

In class we neglected the Taylor series expansion of $q(x)$, but we still obtained the same result. This was because we considered all of the terms in $q(x)$ to the appropriate order. However, when doing general derivations it is better to keep the functional analysis as general as possible.

Back to equilibrium:
$$\sum M_z^{x+dx} = M(x) + \frac{dM}{dx}dx + O(dx^2) - M(x) - V(x)\,dx + q(x)\frac{dx^2}{2} + \underbrace{O(dx^3)}_{\substack{\text{moment} \\ \text{from } \frac{dq}{dx}dx \\ \text{and other} \\ \text{higher order terms}}} = 0$$

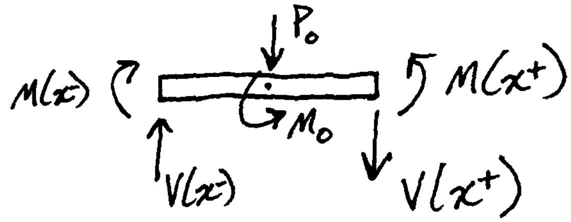
Divide by dx and take $\lim\limits_{dx \to 0}$

$$\Rightarrow \boxed{\frac{dM}{dx} = V(x)}$$

$$\Rightarrow \boxed{\frac{d^2M}{dx^2} = -q(x)}$$

Special Case

We also need to consider points with a point force and/or point moment.



Again, a more rigorous analysis would include $q(x)$ and $O(dx)$ terms in M and V. However, these contributions all go to zero in the limit as $dx \to 0$.

$$\sum F_y = V(x^-) - P_o - V(x^+) + \underbrace{O(dx)}_{\substack{\text{terms like} \\ q(x)dx \text{ that} \\ \text{we did not} \\ \text{draw on the FBD}}} = 0$$

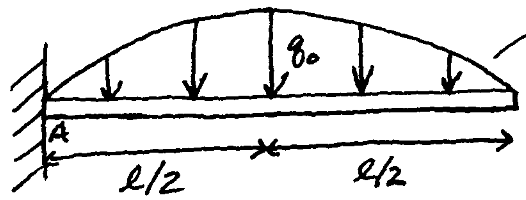
$$\lim_{dx \to 0} \Rightarrow \underbrace{V(x^-) - V(x^+) = P_o}_{\substack{\text{Jump condition on } V \text{ at a} \\ \text{point with a point load.}}}$$

$$\sum M_z^{x+dx} = M(x^+) - M(x^-) + M_o + P_o \frac{dx}{2} - V(x^-)dx + O(dx^2) = 0$$

$$\lim_{dx \to 0} \Rightarrow \underbrace{M(x^-) - M(x^+) = M_o}_{\substack{\text{Jump condition on } M \text{ at} \\ \text{a point with a point couple.}}}$$

Any point without a point force must have continuous shear force. Any point without a point couple must have continuous bending moment. Note that supports can contribute point forces and/or couples.

Example:



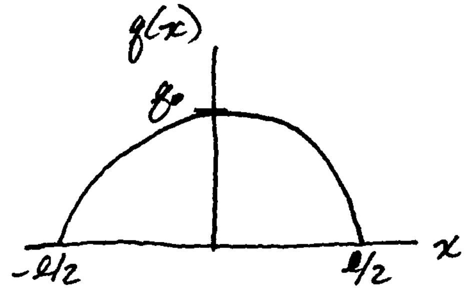
parabolic & symmetric about the center

Determine the shear force and bending moment distributions.

Note that using the wall as the origin for the x-axis is not mandatory. We can choose any point that might be convenient for our origin.

Let's use the center of the beam as the origin for x.

First we need to describe the distributed load.



There are several ways to determine $q(x)$. Here is one:

$$q(x) = a + bx + cx^2$$ ← general equation for a parabola

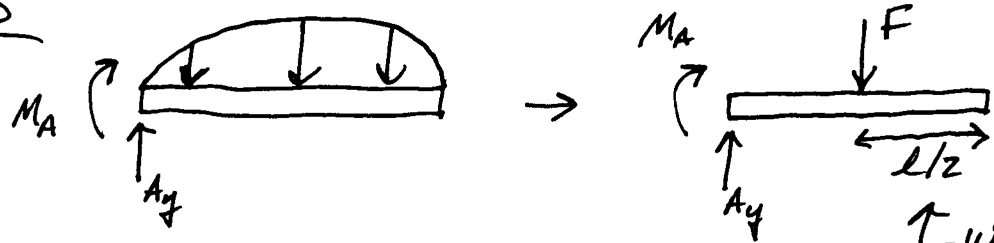
$$q(x=0) = a = q_0$$ value of $q(x)$ at $x=0$

$$\frac{dq}{dx}(x=0) = b = 0$$ $q(x)$ has a maximum at $x=0$

$$q(x=\pm \ell/2) = q_0 + c\,\frac{\ell^2}{4} = 0 \rightarrow c = \frac{-4q_0}{\ell^2}$$ value of $q(x)$ at $x=\pm\ell/2$

$$\Rightarrow q(x) = q_0\left(1 - \frac{4x^2}{\ell^2}\right)$$

FBD



← We know the location of F due to the symmetry of the load.

$$F = \int_{-\ell/2}^{\ell/2} q_0\left(1 - \frac{4x^2}{\ell^2}\right) dx$$

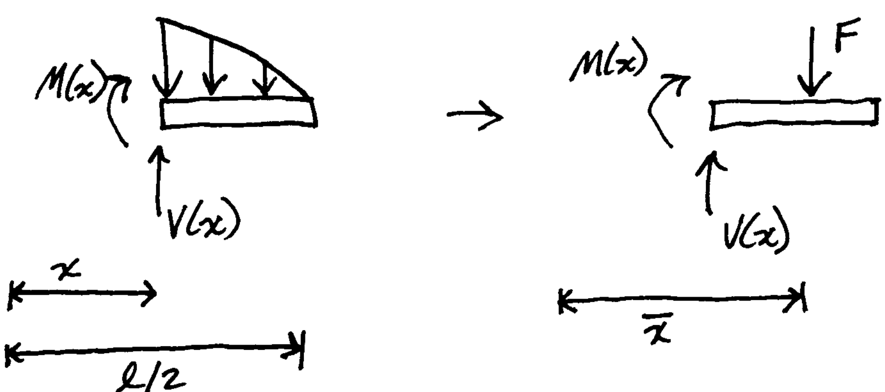
$$= q_0\left(x - \frac{4}{3}\frac{x^3}{\ell^2}\right)\Big]_{-\ell/2}^{\ell/2}$$

$$= q_0\left(\ell - \frac{1}{3}\ell\right) = \frac{2}{3} q_0 \ell$$

$$\rightarrow \quad \Sigma F_y = A_y - \frac{2}{3} q_0 \ell = 0 \rightarrow A_y = \frac{2}{3} q_0 \ell$$

$$\Sigma M_z^A = -M_A - \frac{2}{3} q_0 \ell \frac{\ell}{2} = 0 \rightarrow M_A = -\frac{1}{3} q_0 \ell^2$$

Internal forces and moments



$$F = \int_x^{\ell/2} q_0\left(1 - \frac{4x'^2}{\ell^2}\right) dx' = q_0\left(x' - \frac{4}{3}\frac{x'^3}{\ell^2}\right)_x^{\ell/2}$$

$$= \frac{1}{3} q_0 \ell - q_0 x + \frac{4}{3} q_0 \frac{x^3}{\ell^2}$$

$$M_o = \int_x^{\ell/2} q_o \left(x' - \frac{4x'^3}{\ell^2}\right) dx'$$

$$= q_o \left(\tfrac{1}{2}x'^2 - \frac{x'^4}{\ell^2}\right)\Big]_x^{\ell/2}$$

$$= \frac{1}{16} q_o \ell^2 - \frac{1}{2} q_o x^2 + q_o \frac{x^4}{\ell^2}$$ (Note that this is a CW moment)

$$\bar{x} = \frac{M_o}{F}$$ (but we don't really need this)

$$\Sigma F_y = V(x) - F = 0 \rightarrow \boxed{V(x) = q_o\left(\tfrac{1}{3}\ell - x + \frac{4x^3}{3\ell^2}\right)}$$

M_o was CW

$$\Sigma M_z^o = -M(x) + V(x)\cdot x - M_o = 0$$

$$\rightarrow M(x) = q_o\left(-\frac{\ell^2}{16} + \frac{x^2}{2} - \frac{x^4}{\ell^2} + \frac{\ell x}{3} - x^2 + \frac{4x^4}{3\ell^2}\right)$$

$$\boxed{M(x) = q_o\left(-\frac{\ell^2}{16} + \frac{\ell x}{3} - \frac{x^2}{2} + \frac{x^4}{3\ell^2}\right)}$$

Note that we could also use $\frac{dV}{dx} = -q(x)$ and $\frac{dM}{dx} = V(x)$.

$$q(x) = q_o\left(1 - \frac{4x^2}{\ell^2}\right) \rightarrow V(x) = -q_o\left(x - \frac{4}{3}\frac{x^3}{\ell^2}\right) + c_o$$

$$\rightarrow M(x) = -q_o\left(\tfrac{1}{2}x^2 - \tfrac{1}{3}\frac{x^4}{\ell^2}\right) + c_o x + c_1$$

We know that at $x = \frac{\ell}{2}$ $V(\frac{\ell}{2}) = 0$, $M(\frac{\ell}{2}) = 0$

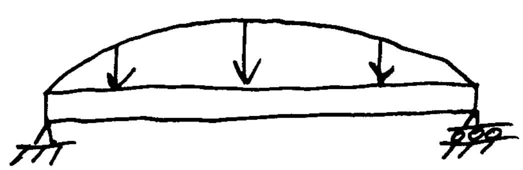
$$V\left(x=\frac{\ell}{2}\right) = -q_0\left(\frac{\ell}{2}-\frac{\ell}{6}\right) + c_0 = 0$$

$$\rightarrow c_0 = q_0\frac{\ell}{3}$$ ✓

$$M\left(x=\frac{\ell}{2}\right) = -q_0\left(\frac{\ell^2}{8}-\frac{\ell^2}{48}\right) + q_0\frac{\ell^2}{6} + c_1 = 0$$

$$\rightarrow c_1 = -q_0\frac{\ell^2}{16}$$ ✓

What if we had different supports?



Now the conditions to fit the unknown coefficients would be:

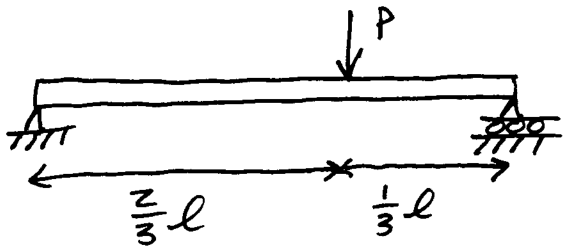
$$M\left(x=-\frac{\ell}{2}\right)=0 \quad \text{and} \quad M\left(x=\frac{\ell}{2}\right)=0$$

$$\rightarrow \underbrace{-q_0\left(\frac{\ell^2}{8}-\frac{\ell^2}{48}\right)}_{-q_0\frac{5\ell^2}{48}} - c_0\frac{\ell}{2} + c_1 = 0$$

and $$-q_0\frac{5\ell^2}{48} + c_0\frac{\ell}{2} + c_1 = 0$$

$$\rightarrow c_1 = \frac{5}{48}q_0\ell^2 \quad \text{and} \quad c_0 = 0$$

An example with jump conditions:



Determine $V(x)$ and $M(x)$.

$0 \le x < \frac{2}{3}\ell$: $q(x) = 0 \rightarrow V(x) = c_0$
$M(x) = c_0 x + c_1$

$\frac{2}{3}\ell < x \le \ell$: $q(x) = 0 \rightarrow V(x) = c_2$
$M(x) = c_2 x + c_3$

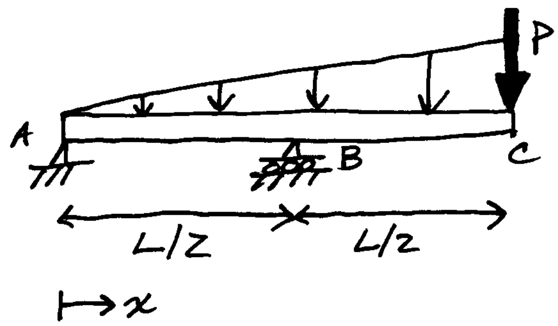
Boundary and jump conditions: $M(x=0) = 0$ (A)
$M(x=\ell) = 0$ (B)
$M(x=\frac{2}{3}\ell^-) - M(x=\frac{2}{3}\ell^+) = 0$ (C)
$V(x=\frac{2}{3}\ell^-) - V(x=\frac{2}{3}\ell^+) = P$ (D)

(A) $\rightarrow$ $c_1 = 0$
(B) $\rightarrow$ $c_2 \ell + c_3 = 0$
(C) $\rightarrow$ $\frac{2}{3}\ell c_0 - \frac{2}{3}\ell c_2 - c_3 = 0$
(D) $\rightarrow$ $c_0 - c_2 = P \rightarrow c_0 = P + c_2$

(C) $\rightarrow \cancel{\frac{2}{3}\ell c_2} + \frac{2}{3}\ell P - \cancel{\frac{2}{3}\ell c_2} - c_3 = 0$

$\rightarrow c_3 = \frac{2}{3}\ell P \rightarrow c_2 = -\frac{2}{3}P \rightarrow c_0 = \frac{1}{3}P$

Another example:



$q(x) = q_0 \frac{x}{L}$

The support at $x = L/2$ creates a point force causing potential discontinuities in $V(x)$ and $M(x)$ so we still need to break up the interval.

$0 \le x < \frac{L}{2}$: $q(x) = q_0 \frac{x}{L} \rightarrow V(x) = -\frac{1}{2} q_0 \frac{x^2}{L} + c_0$

$$M(x) = -\frac{1}{6} q_0 \frac{x^3}{L} + c_0 x + c_1$$

Aside: Note that what we are doing to get $V(x)$ is as follows,

$$\frac{dV}{dx} = -q(x)$$

$$\int \frac{dV}{dx}\, dx = \int -q(x)\, dx$$

$$V(x) = \int -q(x)\, dx + c_0$$

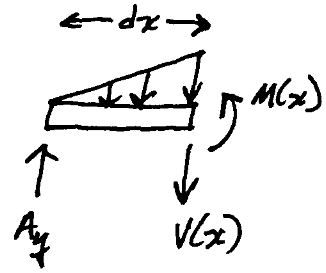
We need this constant of integration when we don't use explicit limits on the integral.

$$\frac{L}{2} < x \le L \quad : \quad q(x) = q_0 \frac{x}{L} \rightarrow V(x) = \frac{-1}{2} q_0 \frac{x^2}{L} + c_2$$

$$M(x) = \frac{-1}{6} q_0 \frac{x^3}{L} + c_2 x + c_3$$

Boundary Conditions:

End A :



$$\Sigma F_y = A_y - V(x) - q_0 \frac{dx^2}{2L} = 0$$

$$\rightarrow V(x=0) = A_y \quad \left(\text{taking} \lim_{dx \to 0} \right)$$

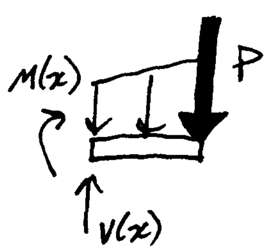
$$\Sigma M_z^{dx} = M(x) - A_y \cdot dx + q_0 \frac{dx^2}{2L} \frac{dx}{2} = 0$$

But we don't know what A_y is.

$$\lim_{dx \to 0} \rightarrow M(x=0) = 0$$

We do know what zero is. This is our useful boundary condition at $x=0$.

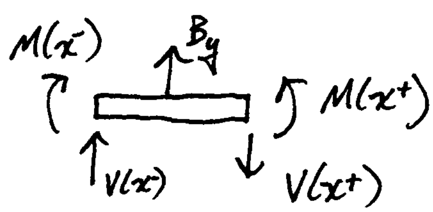
End C :



By a similar limiting procedure at end C we find that,

$$\left.\begin{aligned} V(x=L) &= P \\ M(x=L) &= 0 \end{aligned}\right\} \text{We know both of these.}$$

Point B:



I have not drawn $q(x)$, because we will take $\lim_{dx \to 0}$ at this term will ultimately vanish.

$$\Sigma F_y \rightarrow V(x = \tfrac{L}{2}^-) - V(x = \tfrac{L}{2}^+) = -B_y$$

$$\Sigma M_z \rightarrow M(x = \tfrac{L}{2}^-) - M(x = \tfrac{L}{2}^+) = 0$$

Again, B_y is unknown, so only the moment jump equation is useful here.

Let's list our boundary / jump conditions.

A $\quad M(x=0) = 0 \quad \rightarrow \quad c_1 = 0$

B $\quad V(x=L) = P \quad \rightarrow -\frac{1}{2} q_0 L + c_2 = P \rightarrow c_2 = P + \frac{1}{2} q_0 L$

C $\quad M(x=L) = 0 \quad \rightarrow -\frac{1}{6} q_0 L^2 + c_2 L + c_3 = 0$

D $\quad M(x=\frac{L}{2}^-) - M(x=\frac{L}{2}^+) = 0$

$$\text{C} \rightarrow c_3 = \frac{1}{6} q_0 L^2 - PL - \frac{1}{2} q_0 L^2 = -PL - \frac{1}{3} q_0 L^2$$

$$\text{D} \rightarrow \cancel{-\frac{1}{6} q_0 \frac{L^2}{8}} + c_0 \frac{L}{2} + \overset{0 \text{ from A}}{\cancel{c_1}} = \cancel{-\frac{1}{6} q_0 \frac{L^2}{8}} + c_2 \frac{L}{2} + c_3$$

$$c_0 \frac{L}{2} = \left(P + \frac{1}{2} q_0 L\right) \frac{L}{2} - PL - \frac{1}{3} q_0 L^2$$
$$= -\frac{PL}{2} - \frac{1}{12} q_0 L^2$$

$$\rightarrow c_0 = -P - \frac{1}{6} q_0 L$$

So for, $0 \leq x < \frac{L}{2}$: $\quad V(x) = -\frac{1}{2} q_0 \frac{x^2}{L} - \frac{1}{6} q_0 L - P$

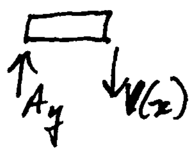
$$M(x) = -\frac{1}{6} q_0 \frac{x^3}{L} - \left(\frac{1}{6} q_0 L + P\right) x$$

$\frac{L}{2} < x \leq L$: $\quad V(x) = -\frac{1}{2} q_0 \frac{x^2}{L} + \frac{1}{2} q_0 L + P$

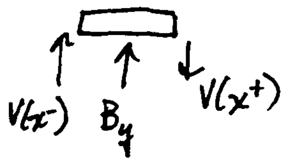
$$M(x) = -\frac{1}{6} q_0 \frac{x^3}{L} + \left(\frac{1}{2} q_0 L + P\right) x - \frac{1}{3} q_0 L^2 - PL$$

* We could obtain the same formulas by 1) analyzing the entire structure to get A_y & B_y and 2) making cuts at arbitrary $x < \frac{L}{2}$ and $x > \frac{L}{2}$.

Note that if you now also want to know the reactions at A and B we have:



$$A_y = V(x=0) = -\frac{1}{6}q_0 L - P$$



$$B_y = V\big(\underset{\frac{L}{2}^+}{\underset{\downarrow}{x^+}}\big) - V\big(\underset{\frac{L}{2}^-}{\underset{\downarrow}{x^-}}\big) = -\frac{1}{2}\cancel{q_0\frac{L}{4}} + \frac{1}{2}q_0 L + P$$

$$+\frac{1}{2}\cancel{q_0\frac{L}{4}} + \frac{1}{6}q_0 L + P$$

$$B_y = \frac{2}{3}q_0 L + 2P$$

Whether you choose to use $\frac{dV}{dx} = -q$ and $\frac{dM}{dx} = V$ and then fit boundary/jump conditions to determine the constants, or you make cuts and arbitrary x positions and perform equilibrium analysis is <u>up to you</u>. Use what you are comfortable with.