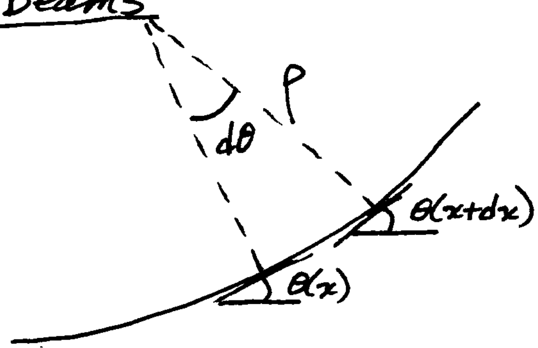
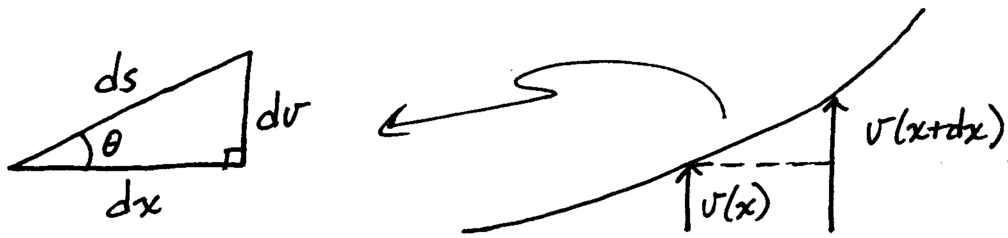
Deflections of Beams





$$ds = \sqrt{dx^2+dv^2} = dx\sqrt{1+\left(\frac{dv}{dx}\right)^2} \approx dx\left[1+\frac{1}{2}\left(\frac{dv}{dx}\right)^2\right]$$

So, for $\frac{dv}{dx} \ll 1 \rightarrow ds \approx dx$

$$ds = \rho\, d\theta \rightarrow dx = \rho\, d\theta \rightarrow \boxed{\frac{d\theta}{dx} = \frac{1}{\rho} = \chi}$$

$\tan\theta = \frac{dv}{dx}$, but if $\frac{dv}{dx} \ll 1$ then $\tan\theta \approx \theta$

$$\rightarrow \boxed{\theta = \frac{dv}{dx} \quad \text{when } \frac{dv}{dx} \ll 1}$$

Finally, $\boxed{\chi = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d^2v}{dx^2}}$

* This is entirely an analysis of kinematics.

Next, recall that analyses of equilibrium and material law lead to:

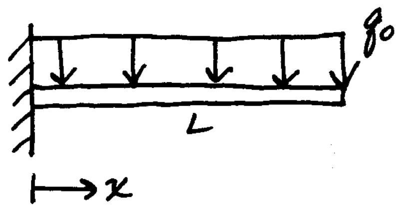
$$M = EI\chi$$

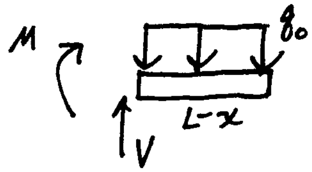
So,
$$\frac{M}{EI} = \frac{d^2v}{dx^2} \qquad ***$$

or with $\frac{dM}{dx} = V \rightarrow$
$$\frac{V}{EI} = \frac{d^3v}{dx^3}$$

or with $\frac{dV}{dx} = -q \rightarrow$
$$\frac{-q}{EI} = \frac{d^4v}{dx^4}$$

<u>Example</u>





$$\Sigma M_z^x = -M - q_0(L-x)\frac{L-x}{2} = 0$$

$$M(x) = -\frac{q_0}{2}(L-x)^2$$

$$\frac{d^2v}{dx^2} = -\frac{q_0}{2EI}(L-x)^2$$

$$\rightarrow \frac{dv}{dx} = \frac{q_0}{6EI}(L-x)^3 + C_1 \qquad \left.\frac{dv}{dx}\right|_{x=0} = 0 \rightarrow C_1 = \frac{-q_0L^3}{6EI}$$

$$\rightarrow v = \frac{-q_0}{24EI}(L-x)^4 - \frac{q_0L^3}{6EI}x + C_2, \quad v(0) = 0 \rightarrow C_2 = \frac{q_0L^4}{24EI}$$

$$\rightarrow v(x) = \frac{-q_0}{24EI}\left[(L-x)^4 - 4L^3x + L^4\right]$$

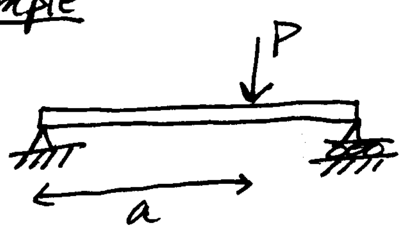
$$\delta_{max} = -v_{max} = -v(x=L) = \frac{q_0 L^4}{8EI}$$

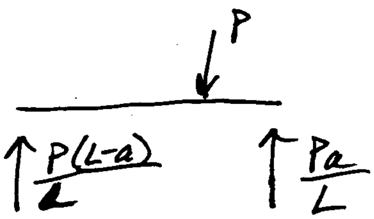
$$\theta_{max} = -\theta(x=L) = \frac{q_0 L^3}{6EI}$$

Note that we could have started from $q(x)$ and integrated 4 times, requiring us to find the 4 constants of integration.

There are cases where it is useful to begin with q and integrate using half-range functions.

Example





$$q(x) = P\langle x-a\rangle^{-1}$$
$$V(x) = -P\langle x-a\rangle^{0} + C_1, \quad V(0) = \frac{P(L-a)}{L} \rightarrow C_1 = \frac{P(L-a)}{L}$$
$$M(x) = -P\langle x-a\rangle^{1} + \frac{P(L-a)}{L}x + C_2, \quad M(0)=0 \rightarrow C_2 = 0$$

$$\theta(x) = \frac{-P}{2EI}\langle x-a\rangle^{2} + \frac{P(L-a)}{2EIL}x^2 + C_3$$

$$v(x) = \frac{-P}{6EI}\langle x-a\rangle^{3} + \frac{P(L-a)}{6EIL}x^3 + C_3 x + C_4$$

Boundary conditions: $v(x=0) = v(x=L) = 0$

$$v(x=0)=0 \rightarrow c_4 = 0$$

$$v(x=L)=0 = \frac{-P}{6EI}\underbrace{(L-a)^3}_{\cancel{L^3}-3L^2a+3a^2L-a^3} + \frac{P}{6EI}\underbrace{L^2(L-a)}_{\cancel{L^3}-L^2a} + c_3 L$$

$$c_3 = \frac{P}{6EIL}\left(-2L^2a + 3a^2L - a^3\right)$$

$$v(x) = -\frac{P}{6EI}\left[\langle x-a\rangle^3 - \frac{L-a}{L}x^3 - \frac{x}{L}\left(-2aL^2+3a^2L-a^3\right)\right]$$

v_{max} occurs where $\theta = 0$. This occurs in the range $x < a$ if $a > \frac{L}{2}$.

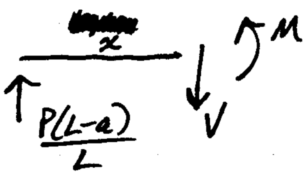
$$\text{if } a > \frac{L}{2} \rightarrow \theta_c = \frac{\cancel{P}(L-a)}{2\cancel{EIL}}x_c^2 + \frac{\cancel{P}}{\cancel{6EIL}}\underbrace{\left(-2L^2a+3a^2L-a^3\right)}_{(L-a)(a^2-2aL)} = 0$$

$$x_c = \sqrt{\frac{a(2L-a)}{3}}$$

after some algebra $\delta_{max} = \dfrac{P(L-a)\left[a(2L-a)\right]^{3/2}}{9\sqrt{3}\,LEI}$

How do we do this without half-range functions?

$0 < x < a$

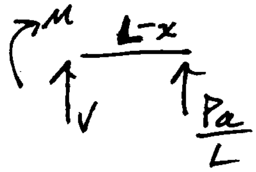


$$M(x) = \frac{P(L-a)x}{L}$$

$$\rightarrow \theta(x) = \frac{P(L-a)x^2}{2LEI} + c_1, \quad v(x) = \frac{P(L-a)x^3}{6LEI} + c_1 x + \cancel{c_2}^{\,0}$$

$v(0)=0$

$a < x < L$



$$M(x) = \frac{Pa(L-x)}{L}$$ (check $M(a)$ is continuous ✓)

$$\rightarrow \theta(x) = \frac{-Pa(L-x)^2}{2LEI} + c_3$$

$$v(x) = \frac{Pa(L-x)^3}{6LEI} + c_3 x + c_4$$

Matching conditions: $\theta(x=a^-) = \theta(x=a^+)$

$$\rightarrow \frac{P(L-a)a^2}{2LEI} + c_1 = \frac{-Pa(L-a)^2}{2LEI} + c_3 \quad (1)$$

$$v(x=a^-) = v(x=a^+)$$

$$\rightarrow \frac{P(L-a)a^3}{6LEI} + c_1 a = \frac{Pa(L-a)^3}{6LEI} + c_3 a + c_4 \quad (2)$$

Boundary conditions: $v(0)=0 \rightarrow c_2 = 0$ (already used)

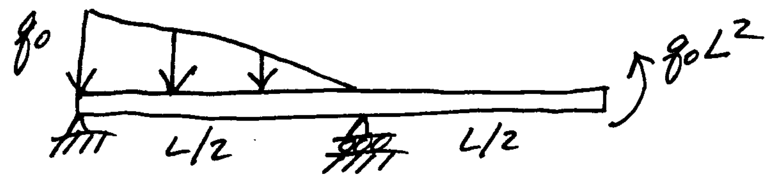
$$v(L)=0$$

$$\rightarrow c_3 L + c_4 = 0 \quad (3)$$

Solve (1), (2) & (3) for c_1, c_3, c_4.

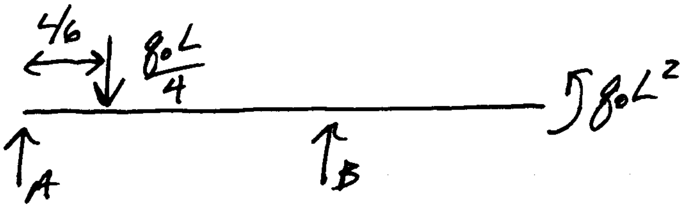
* Note that c's are constants so they can <u>not</u> depend on x.

Example



Design specifications could limit the max/min axial stress, the max shear stress, and/or the max/min deflections. Let's compute each of these.

First, let's determine the reactions.



$$\sum M_z^A = B\frac{L}{2} + q_0L^2 - \frac{q_0L}{4}\frac{L}{6} = 0 \rightarrow B = -\frac{23}{12}q_0L$$

$$\sum F_y = A + B - \frac{q_0L}{4} = 0 \rightarrow A = \frac{13}{6}q_0L$$

Next, let's determine $q(x)$.

$$q(x) = \underbrace{\frac{23}{12}q_0L\left\langle x-\frac{L}{2}\right\rangle^{-1}}_{q(x)\text{ due to }B} + \underbrace{\frac{2q_0}{L}\left\langle \frac{L}{2}-x\right\rangle}_{q(x)\text{ due to ramp}}$$

$= q_0$ at $x=0$
$= 0$ at $x=\frac{L}{2}$
$= 0$ for $x>\frac{L}{2}$

Now, let's determine $V(x)$ and $M(x)$.

$$\frac{dV}{dx} = -q \rightarrow V(x) = -\frac{23}{12} q_0 L \langle x - \tfrac{L}{2} \rangle^0 + \frac{q_0}{L} \langle \tfrac{L}{2} - x \rangle^2 + C_1$$

Boundary condition: $V(x=0) = \frac{13}{6} q_0 L = 0 + \frac{q_0}{L}\frac{L^2}{4} + C_1$

$$C_1 = \frac{23}{12} q_0 L$$

$$V(x) = \frac{q_0}{L} \langle \tfrac{L}{2} - x \rangle^2 - \frac{23}{12} q_0 L \langle x - \tfrac{L}{2} \rangle^0 + \frac{23}{12} q_0 L$$

check $V(x=L) = 0 - \frac{23}{12} q_0 L + \frac{23}{12} q_0 L = 0$ ✓

* V_{max} occurs where $q = 0$ or at any point loads/reactions

$$V(0) = \frac{13}{6} q_0 L \qquad V(\tfrac{L}{2}) = 0 - 0 + \frac{23}{12} q_0 L$$

$$\rightarrow V_{max} = \frac{13}{6} q_0 L \rightarrow \tau_{max} = \frac{V_{max}}{I} \left(\frac{Q}{b}\right)_{max}$$

$$\frac{dM}{dx} = V \rightarrow M(x) = \frac{-q_0}{3L} \langle \tfrac{L}{2} - x \rangle^3 - \frac{23}{12} q_0 L \langle x - \tfrac{L}{2} \rangle + \frac{23}{12} q_0 L x + C_2$$

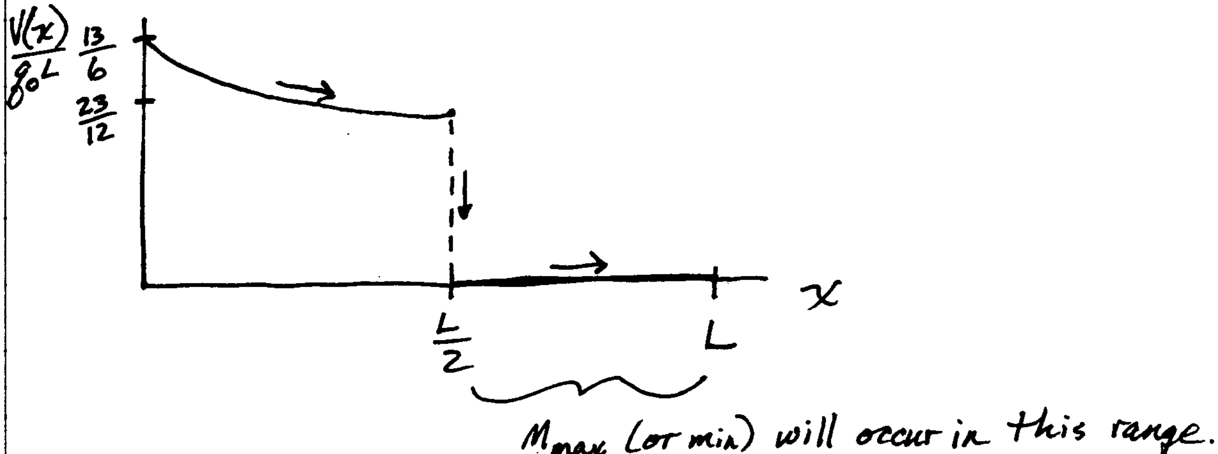
$$M(x=0) = 0 = \frac{-q_0}{3L}\frac{L^3}{8} - 0 + 0 + C_2 \rightarrow C_2 = \frac{q_0 L^2}{24}$$

$$M(x) = \frac{-q_0}{3L} \langle \tfrac{L}{2} - x \rangle^3 - \frac{23}{12} q_0 L \langle x - \tfrac{L}{2} \rangle + \frac{23}{12} q_0 L x + \frac{q_0 L^2}{24}$$

check $M(x=L) = 0 - \frac{23}{12} q_0 L \frac{L}{2} + \frac{23}{12} q_0 L^2 + \frac{q_0 L^2}{24} = q_0 L^2$ ✓

M_{max} occurs where $V = 0$ or at point loads/moments

Here it is useful to plot $V(x)$.



$$M(x=\tfrac{L^+}{2}) = 0 - 0 + \frac{23}{12} q_0 \frac{L^2}{2} + \frac{q_0 L^2}{24} = q_0 L^2 \quad \checkmark \text{ (same as } M(L)\text{)}$$

** $$\sigma_{max} = -\frac{q_0 L^2}{I}(y_{min}) \quad \text{because } \sigma = \frac{-My}{I}$$

$$\sigma_{min} = -\frac{q_0 L^2}{I}(y_{max})$$

$$\frac{d\theta}{dx} = \frac{M}{EI} \rightarrow \theta(x) = \frac{+q_0}{12EIL}\left\langle \frac{L}{2} - x \right\rangle^4 - \frac{23 q_0 L}{24EI}\left\langle x - \frac{L}{2}\right\rangle^2 + \frac{23 q_0 L}{24EI}x^2 + \frac{q_0 L^2}{24EI}x + C_3$$

There are no boundary conditions on θ.

$$\frac{dv}{dx} = \theta \rightarrow v(x) = \frac{-q_0}{60EIL}\left\langle \frac{L}{2} - x\right\rangle^5 - \frac{23 q_0 L}{72EI}\left\langle x - \frac{L}{2}\right\rangle^3 + \frac{23 q_0 L}{72EI}x^3 + \frac{q_0 L^2}{48EI}x^2 + C_3 x + C_4$$

Boundary conditions: $v(x=0) = 0$
$v(x=\tfrac{L}{2}) = 0$

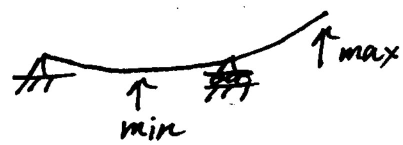
$$v(x=0)=0=\frac{-q_0}{60EIL}\frac{L^5}{32}-0+0+0+0+C_4 \rightarrow C_4=\frac{q_0L^4}{1920EI}$$

$$v\left(x=\frac{L}{2}\right)=0=0-0+\frac{23q_0L}{72EI}\frac{L^3}{8}+\frac{q_0L^2}{48EI}\frac{L^2}{4}+C_3\frac{L}{2}+\frac{q_0L^4}{1920EI}$$

$$C_3=-\frac{263}{2880}\frac{q_0L^3}{EI}$$

$$v(x)=\frac{-q_0}{60EIL}\left\langle\frac{L}{2}-x\right\rangle^5-\frac{23q_0L}{72EI}\left\langle x-\frac{L}{2}\right\rangle^3+\frac{23q_0L}{72EI}x^3+\frac{q_0L^2}{48EI}x^2-\frac{263}{2880}\frac{q_0L^3}{EI}x+\frac{q_0L^4}{1920EI}$$

Max/min for v? $\rightarrow$ Intuition



*** $v(x=L)=$ [redacted] $\frac{q_0L^4}{EI}\underbrace{\left[0-\frac{23}{72}\frac{1}{8}+\frac{23}{72}+\frac{1}{48}-\frac{263}{2880}+\frac{1}{1920}\right]}_{\frac{1207}{5760}=0.21}=v_{max}$

To find the minimum we need to find where $\theta=0$.

For $0<x<\frac{L}{2}$: $\theta(x)=\frac{q_0}{12EIL}\left(\frac{L}{2}-x\right)^4+\frac{23q_0L}{24EI}x^2+\frac{q_0L^2}{24EI}x-\frac{263}{2880}\frac{q_0L^3}{EI}$

Set $\theta(x)=0$ and solve for x.

Mathematica $\rightarrow$ $x=0.287L$

✓ Note that the answer should be between 0 & $\frac{L}{2}$.

**** $v(x=0.287L)=-0.016\frac{q_0L^4}{EI}=v_{min}$

✓ It makes sense intuitively that this magnitude is smaller than that of v_{max}.