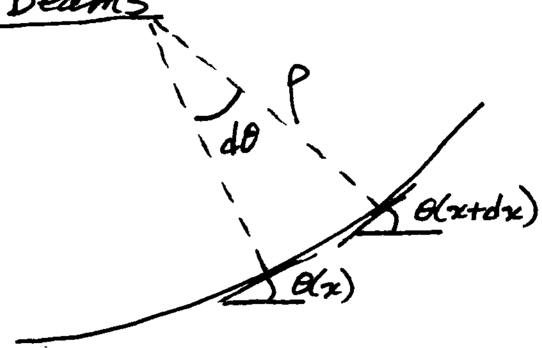
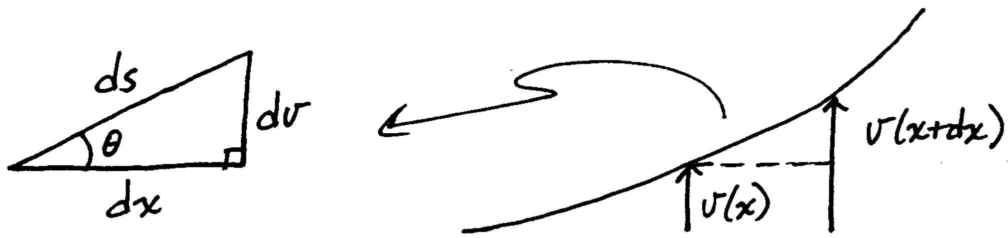
Deflections of Beams





$$ds = \sqrt{dx^2 + dv^2} = dx\sqrt{1+\left(\frac{dv}{dx}\right)^2} \approx dx\left[1+\tfrac{1}{2}\left(\frac{dv}{dx}\right)^2\right]$$

So, for $\frac{dv}{dx} \ll 1 \rightarrow ds \approx dx$

$$ds = \rho\, d\theta \rightarrow dx = \rho\, d\theta \rightarrow \boxed{\frac{d\theta}{dx} = \frac{1}{\rho} = \chi}$$

$\tan\theta = \frac{dv}{dx}$, but if $\frac{dv}{dx} \ll 1$ then $\tan\theta \approx \theta$

$$\rightarrow \boxed{\theta = \frac{dv}{dx} \quad \text{when } \frac{dv}{dx} \ll 1}$$

Finally, $\boxed{\chi = \frac{d\theta}{dx} = \frac{d}{dx}\left(\frac{dv}{dx}\right) = \frac{d^2v}{dx^2}}$

* This is entirely an analysis of kinematics.

Next, recall that analyses of equilibrium and material law lead to:

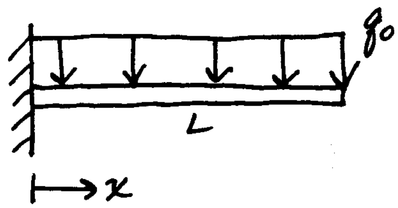
$$M = EI\chi$$

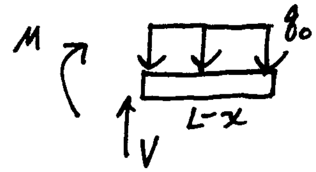
So, $\boxed{\dfrac{M}{EI} = \dfrac{d^2v}{dx^2}}$ ***

or with $\dfrac{dM}{dx} = V \rightarrow \boxed{\dfrac{V}{EI} = \dfrac{d^3v}{dx^3}}$

or with $\dfrac{dV}{dx} = -q \rightarrow \boxed{\dfrac{-q}{EI} = \dfrac{d^4v}{dx^4}}$

<u>Example</u>





$$\Sigma M_z^x = -M - q_0(L-x)\frac{L-x}{2} = 0$$

$$M(x) = -\frac{q_0}{2}(L-x)^2$$

$$\frac{d^2v}{dx^2} = -\frac{q_0}{2EI}(L-x)^2$$

$$\rightarrow \frac{dv}{dx} = \frac{q_0}{6EI}(L-x)^3 + C_1 \qquad \left.\frac{dv}{dx}\right|_{x=0} = 0 \rightarrow C_1 = \frac{-q_0L^3}{6EI}$$

$$\rightarrow v = \frac{-q_0}{24EI}(L-x)^4 - \frac{q_0L^3}{6EI}x + C_2 \;,\; v(0)=0 \rightarrow C_2 = \frac{q_0L^4}{24EI}$$

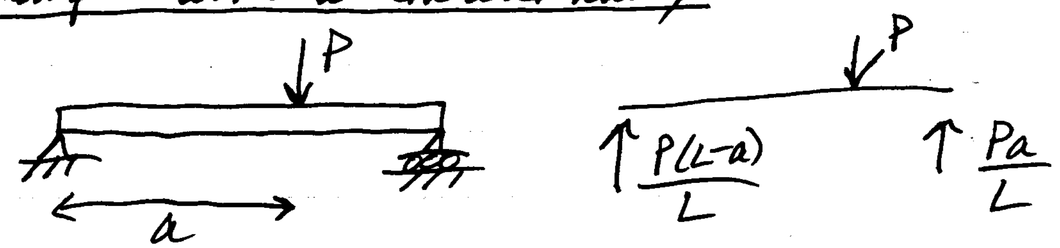
$$\rightarrow \quad v(x) = \frac{-q_0}{24EI}\left[(L-x)^4 - 4L^3x + L^4\right]$$

$$\delta_{max} = -v_{max} = -v(x=L) = q_0L^4/8EI$$

$$\theta_{max} = -\theta(x=L) = \frac{q_0L^3}{6EI}$$

Note that we could have started from $q(x)$ and integrated 4 times, requiring us to find the 4 constants of integration.

<u>Example with a discontinuity</u>



$0 \le x < a$:

$$q(x) = 0$$
$$q(x) = -\frac{dV}{dx} \rightarrow V(x) = c_1$$
$$V(x) = \frac{dM}{dx} \rightarrow M(x) = c_1x + c_2$$
$$M(x) = EI\frac{d\theta}{dx} \rightarrow EI\theta(x) = \tfrac{1}{2}c_1x^2 + c_2x + c_3$$
$$\theta(x) = \frac{dv}{dx} \rightarrow EIv(x) = \tfrac{1}{6}c_1x^3 + \tfrac{1}{2}c_2x^2 + c_3x + c_4$$

Boundary conditions: $V(x=0) = \frac{P(L-a)}{L}$

$$\therefore \quad c_1 = \frac{P(L-a)}{L}$$

$$M(x=0) = 0 = c_1(0) + c_2 \rightarrow c_2 = 0$$
$$v(x=0) = 0 \longrightarrow c_4 = 0$$

We do not know $\theta(x=0)$.

$a < x \le L$: $q(x) = 0$

$$\rightarrow \quad V(x) = d_1$$
$$M(x) = d_1 x + d_2$$
$$EI\theta(x) = \tfrac{1}{2} d_1 x^2 + d_2 x + d_3$$
$$EI v(x) = \tfrac{1}{6} d_1 x^3 + \tfrac{1}{2} d_2 x^2 + d_3 x + d_4$$

Boundary conditions: $V(L) = -\frac{Pa}{L} \rightarrow d_1 = \frac{-Pa}{L}$

$$M(L) = 0 \rightarrow d_2 = Pa$$
$$EI v(L) = 0 \rightarrow -\frac{PaL^2}{6} + \frac{PaL^2}{2} + d_3 L + d_4 = 0$$

Again, we do not know $\theta(L)$. So, now we have d_3, d_4, and c_3 as unknowns, but only one equation for their solution. Now what? Well, we have continuity (or jump) conditions at $x=a$. We already know that,

$$V(x=a^+) - V(x=a^-) = -P$$

and $M(x=a^+) - M(x=a^-) = -M_0 = 0$ in this case

But these are already satisfied: $V(x=a^+) = -\frac{Pa}{L}$

$$V(x=a^-) = \frac{P(L-a)}{L}$$
$$V^+ - V^- = -P \checkmark$$
$$M(x=a^+) = -\frac{Pa}{L} \cdot a + Pa = Pa\left(1 - \frac{a}{L}\right)$$
$$M(x=a^-) = \frac{P(L-a)}{L} \cdot a = Pa\left(1 - \frac{a}{L}\right)$$
$$M^+ - M^- = 0 \checkmark$$

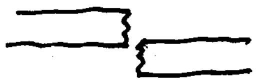
Now, we also need to match $v(x)$ and $\theta(x)$.

If there is no hinge at $x=a$, then $\theta(x=a^+)=\theta(x=a^-)$.

Hinge

If there is a hinge at $x=a$, then $M(x=a^+) = M(x=a^-) = 0$.

If the beam is not broken at $x=a$, then $v(x=a^+)=v(x=a^-)$.



Broken

If the beam is broken at $x=a$, then $V(x=a^+) = V(x=a^-) = 0$.

So, for our problem we have, $\theta(x=a^+)=\theta(x=a^-)$ & $v(x=a^+)=v(x=a^-)$.

$$\theta : -\frac{1}{2}\frac{Pa}{L}a^2 + Pa\cdot a + d_3 = \frac{1}{2}\frac{P(L-a)}{L}a^2 + c_3$$

$$\cancel{-\frac{1}{2}\frac{Pa^3}{L}} + Pa^2 + d_3 = \frac{1}{2}Pa^2 \cancel{-\frac{1}{2}\frac{Pa^3}{L}} + c_3$$

$$v : \cancel{-\frac{1}{6}\frac{Pa}{L}\cdot a^3} + \frac{1}{2}Pa\cdot a^2 + d_3 a + d_4 = \frac{1}{6}\frac{P(L-a)}{L}a^3 + c_3 a$$

$$= \frac{1}{6}Pa^3 \cancel{-\frac{1}{6}\frac{Pa}{L}\cdot a^3} + c_3 a$$

$$① \quad d_3 - c_3 = -\frac{1}{2}Pa^2$$

$$a(d_3 - c_3) + d_4 = -\frac{1}{3}Pa^3$$

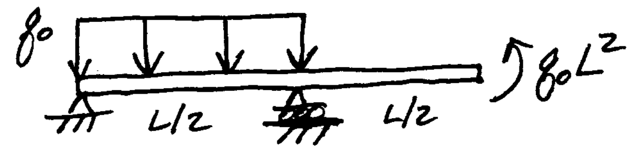
$$-\frac{1}{2}Pa^3 + d_4 = -\frac{1}{3}Pa^3 \rightarrow d_4 = \frac{1}{6}Pa^3$$

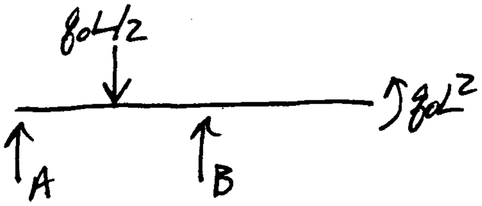
From $v(L)=0 \rightarrow \frac{PaL^2}{3} + d_3 L + \frac{1}{6}Pa^3 = 0 \rightarrow d_3 = -\frac{PaL}{3} - \frac{Pa^3}{6L}$

$$= -Pa^2\left(\frac{L}{3a} + \frac{a}{6L}\right)$$

Then ① $\rightarrow c_3 = \frac{1}{2}Pa^2\left(1 - \frac{2L}{3a} - \frac{a}{3L}\right)$

Example





$$\Sigma M_z^A = q_0L^2 + \frac{BL}{2} - \frac{q_0L^2}{8} = 0$$
$$B = -\frac{7}{4}q_0L$$
$$\Sigma F_y = A + B - \frac{q_0L}{2} = 0 \rightarrow A = \frac{9}{4}q_0L$$

$0 \leq x < \frac{L}{2}$:

$$q(x) = q_0$$
$$\rightarrow V(x) = -q_0x + c_1$$
$$M(x) = -\frac{1}{2}q_0x^2 + c_1x + c_2$$
$$EI\theta(x) = -\frac{1}{6}q_0x^3 + \frac{1}{2}c_1x^2 + c_2x + c_3$$
$$EIv(x) = -\frac{1}{24}q_0x^4 + \frac{1}{6}c_1x^3 + \frac{1}{2}c_2x^2 + c_3x + c_4$$

BCs:
$$v(0) = 0 \rightarrow c_4 = 0$$
$$M(0) = 0 \rightarrow c_2 = 0$$
$$V(0) = A = \frac{9}{4}q_0L \rightarrow c_1 = \frac{9}{4}q_0L$$
$$v\left(\frac{L}{2}\right) = 0 \rightarrow -\frac{1}{24}q_0\left(\frac{L}{2}\right)^4 + \frac{1}{6}\frac{9}{4}q_0L\left(\frac{L}{2}\right)^3 + c_3\frac{L}{2} = 0$$
$$\rightarrow c_3 = -\frac{17}{192}q_0L^3$$

$$\rightarrow EIv(x) = -\frac{1}{24}q_0x^4 + \frac{9}{24}q_0Lx^3 - \frac{17}{192}q_0L^3x$$

$\frac{L}{2} < x \leq L$: $q(x) = 0 \rightarrow V(x) = d_1$

$$M(x) = d_1x + d_2$$
$$EI\theta(x) = \frac{1}{2}d_1x^2 + d_2x + d_3$$
$$EIv(x) = \frac{1}{6}d_1x^3 + \frac{1}{2}d_2x^2 + d_3x + d_4$$

BCs: $V(x=L)=0 \rightarrow d_1=0$
$M(x=L)=q_0L^2 \rightarrow d_2=q_0L^2$ (b/c $d_1=0$)

Matching: $\theta(x=\frac{L}{2}^-)=\theta(x=\frac{L}{2}^+)$

$$-\frac{1}{6}q_0\left(\frac{L}{2}\right)^3+\frac{9}{8}q_0L\left(\frac{L}{2}\right)^2-\frac{17}{192}q_0L^3=q_0L^2\left(\frac{L}{2}\right)+d_3$$

$$\rightarrow d_3=-\frac{21}{64}q_0L^3$$

$$v(x=\tfrac{L}{2}^-)=0=\frac{q_0L^2}{2}\left(\frac{L}{2}\right)^2-\frac{21}{64}q_0L^3\left(\frac{L}{2}\right)+d_4$$

$$\rightarrow d_4=\frac{5}{128}q_0L^4$$

$$\rightarrow EIv(x)=\frac{1}{2}q_0L^2x^2-\frac{21}{64}q_0L^3x+\frac{5}{128}q_0L^4$$

Design specifications could include max/min axial stress, shear stress, and deflections.

Shear stress: $\tau=\dfrac{VQ}{Ib}$ so we need V_{max}.

V_{max} occurs where $q(x)=0$ or at external/internal boundaries.

$q(x)=0$ for $\frac{L}{2}<x<L$, but $V(x)=0$ in this range.
$q(x)\neq 0$ for $0\leq x<\frac{L}{2}$, so $x=0$ and $x=\frac{L}{2}$ are critical points.

$V(x=0)=A=\frac{9}{4}q_0L$
$V(x=\frac{L}{2})=-q_0\frac{L}{2}+\frac{9}{4}q_0L=\frac{7}{4}q_0L=-B$ ✓

so, $V_{max} = \frac{9}{4} q_0 L$

Let's say the cross-section is 

Then, τ_{max} occurs at $y=0$,
with $b(y) = b$, $Q(y) = \frac{bh^2}{8} - \frac{1}{2} b y^2 \rightarrow Q(0) = \frac{bh^2}{8}$
and $I = \frac{1}{12} b h^3$.

$$* \quad \rightarrow \tau_{max} = \frac{9}{4} q_0 L \, \frac{bh^2/8}{\frac{1}{12} b^2 h^3} = \frac{3}{2} \underbrace{\left(\frac{9}{4} q_0 L\right) \frac{1}{bh}}_{V_{max}/A}$$

You can remember $\tau_{max} = \underline{\underline{\frac{3}{2}}} \frac{V}{A}$ for a <u>rectangular</u> section.

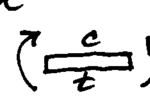
$$\sigma_{max/min} = \pm \frac{M \, y_{max/min}}{I} = \pm |M_{max}| \underbrace{\frac{h/2}{\frac{1}{12} b h^3}}_{6/bh^2}$$

$|M_{max}|$ occurs where $V(x) = 0$ or boundaries.

$V(x) = 0$ for $\frac{L}{2} < x \le L$ and $M = q_0 L^2$

in $0 \le x < \frac{L}{2}$: $V(x) = -q_0 x + \frac{9}{4} q_0 L = 0$
$x = \frac{9}{4} L$ ✗ Not in the range $[0, \frac{L}{2})$

$$** \quad \therefore M_{max} = q_0 L^2 \rightarrow \sigma_{max/min} = \pm 6 q_0 L^2 / bh^2 \quad \text{throughout } (\tfrac{L}{2}, L]$$



$v_{max/min}$ occurs where $\theta = 0$ or boundaries

for $\frac{L}{2} < x \le L$: $EI\theta(x) = q_0 L^2 x - \frac{21}{64} q_0 L^3 = 0$

$\rightarrow x = \frac{21}{64} L$ X Not in $(\frac{L}{2}, L]$

*** So v_{max} is at $x = L$ $\Rightarrow v(L) = \frac{27}{128} \frac{q_0 L^4}{EI}$

$0 \le x < \frac{L}{2}$: $EI\theta(x) = -\frac{1}{6} q_0 x^3 + \frac{9}{8} q_0 L x^2 - \frac{17}{192} q_0 L^3 = 0$

Mathematica: $x = 0.2867L$ is the only root in $[0, \frac{L}{2})$

*** $\rightarrow v_{min} = -0.0139 \frac{q_0 L^2}{EI}$

Note that $|v_{min}| < |v_{max}|$ which should satisfy your intuition.