Let's lay out the two approaches to solving for the internal shear force and internal bending moment.

Solution method using Explicit Cuts

1) Draw an FBD of the entire structure
2) Use equilibrium equations to solve for support reactions
3) Determine how many cuts are needed and where they must be placed. Obviously, you always start out needing at least one cut. Then, anytime there is a change in the functional representation of the distributed load, or an internal point load or moment, or an internal support reaction, an additional cut must be made.
4) Make cuts between all such "discontinuities". Be sure to cut distributed loads first and analyze them during the equilibrium analysis. These cuts must be made at arbitrary x locations. Draw FBDs of your cuts.
5) Analyze the equilibrium of the cut FBDs to determine $V(x)$ and $M(x)$.

Solution Method using Differential Equations

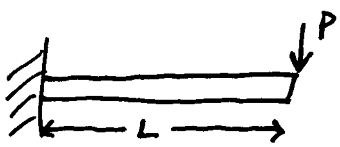
1) Begin with $\frac{dV}{dx} = -q(x)$ (Differential equation for ΣF_y equilibrium)

$\frac{dM}{dx} = V(x)$ (Differential equation for ΣM_z equilibrium)

2) Perform step 3 of the previous procedure on page 131.

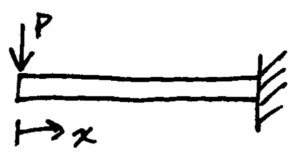
3) Solve for the general forms for $V(x)$ and $M(x)$ by integrating the equations in step 1. This procedure will introduce 2 unknown constants for each additional cut/"discontinuity".

4) Apply boundary and jump conditions to determine the unknown constants. Some examples of boundary conditions and jump conditions are as follows.



: $M(x=0)$, $V(x=0)$ are not known

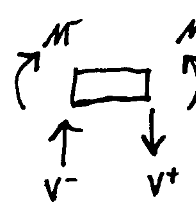
$M(x=L) = 0$, $V(x=L) = P$

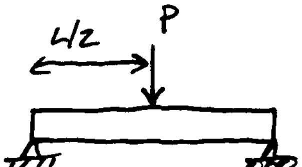


: $M(x=L)$, $V(x=L)$ unknown

$M(x=0)=0$, $V(x=0) = -P$

* Difference in sign is due to convention





: $M(x=0) = M(x=L) = 0$

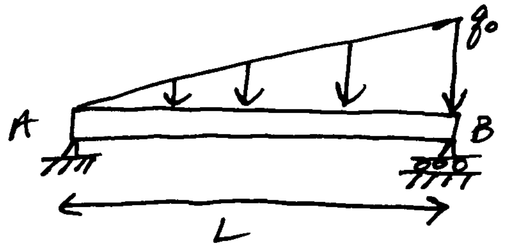
$V(x=0)$, $V(x=L)$ unknown

$M(x=\frac{L}{2}^-) - M(x=\frac{L}{2}^+) = 0$

$V(x=\frac{L}{2}^-) - V(x=\frac{L}{2}^+) = P$

Now, let's use these two procedures for the following problem.

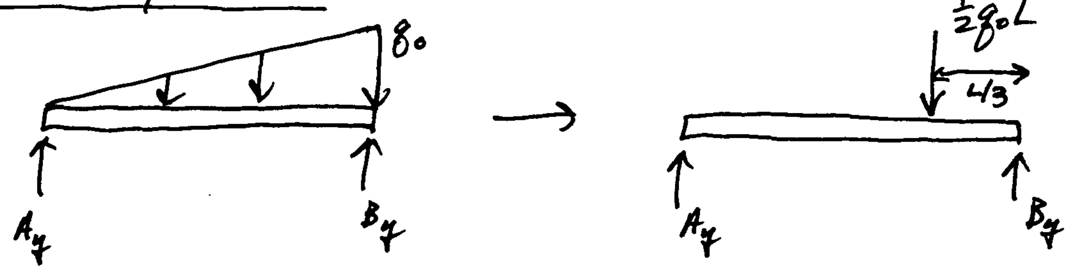
$q(x) = q_0 \frac{x}{L}$



Determine $V(x)$ and $M(x)$, and the maximum $|M(x)|$.

Solution by cuts

1)



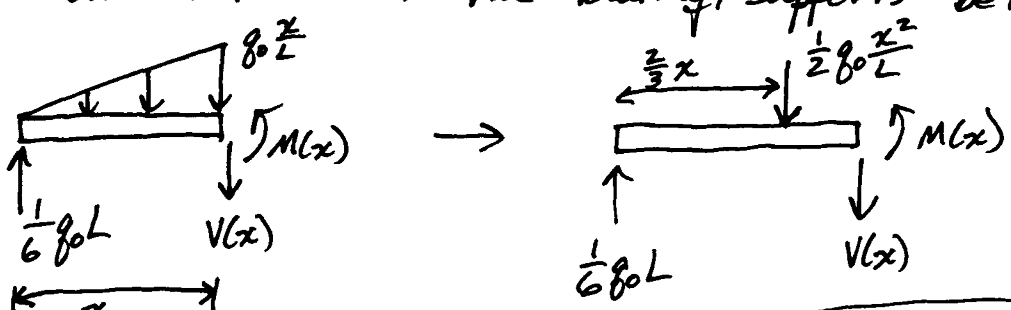
2) $\Sigma F_y = A_y + B_y - \frac{1}{2}q_0 L = 0$

$\Sigma M_z^A = B_y L - \frac{1}{2}q_0 L \frac{2L}{3} = 0 \rightarrow B_y = \frac{1}{3}q_0 L$

$\rightarrow A_y = \frac{1}{6}q_0 L$

3) We only need one cut because there are no "discontinuities" in the loading/supports between A & B.

4)



5) $\Sigma F_y = \frac{1}{6}q_0 L - V(x) - \frac{1}{2}q_0 \frac{x^2}{L} = 0 \rightarrow \boxed{V(x) = \frac{1}{6}q_0 L - \frac{1}{2}q_0 \frac{x^2}{L}}$

$\Sigma M_z^{@x} = M(x) - \frac{1}{6}q_0 L x + \frac{1}{2}q_0 \frac{x^2}{L}\frac{x}{3} = 0 \rightarrow \boxed{M(x) = \frac{1}{6}q_0 L x - \frac{1}{6}q_0 \frac{x^3}{L}}$

Before we determine max $|M(x)|$, let's use the other method.

Solution by differential equations

1) $\frac{dV}{dx} = -q_0 \frac{x}{L}$, $\frac{dM}{dx} = V(x)$

2) Only one cut is needed → 2 unknown coefficients

3) $V(x) = -\frac{1}{2} q_0 \frac{x^2}{L} + c_0$

$M(x) = -\frac{1}{6} q_0 \frac{x^3}{L} + c_0 x + c_1$

Check that $\frac{dM}{dx} = -\frac{3}{6} q_0 \frac{x^2}{L} + c_0 = V(x)$ ✓

and $\frac{dV}{dx} = -\frac{2}{2} q_0 \frac{x}{L} = -q(x)$ ✓

4) $M(x=0) = 0 \rightarrow c_1 = 0$

$M(x=L) = 0 \rightarrow -\frac{1}{6} q_0 L^2 + c_0 L = 0$

$\rightarrow c_0 = \frac{1}{6} q_0 L$

$$\rightarrow \boxed{\begin{aligned} V(x) &= \frac{1}{6} q_0 L - \frac{1}{2} q_0 \frac{x^2}{L} \\ M(x) &= \frac{1}{6} q_0 L x - \frac{1}{6} q_0 \frac{x^3}{L} \end{aligned}}$$

Max $|M(x)| \rightarrow \frac{dM}{dx} = 0$ (Note: this is where $V(x)=0$)

$\rightarrow \frac{1}{6} q_0 L - \frac{1}{2} q_0 \frac{x^2}{L} = 0 \rightarrow x^2 = \frac{1}{3} L^2$

$\rightarrow x = \frac{L}{\sqrt{3}}$

$\rightarrow M_{max} = \frac{1}{6} q_0 \frac{L^2}{\sqrt{3}} - \frac{1}{6} q_0 \frac{L^2}{3\sqrt{3}} = 0.064\, q_0 L^2$

Centroids

The centroid of a curve, area or volume represents the "weighted average" of position with that object.

Curve: $\bar{x} = \dfrac{\int_L x\,dL}{L}$, $\bar{y} = \dfrac{\int_L y\,dL}{L}$, $\bar{z} = \dfrac{\int_L z\,dL}{L}$

$$L = \int_L dL = \text{arclength}$$

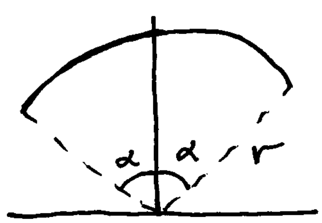
Area: $\bar{x} = \dfrac{\int_A x\,dA}{A}$, $\bar{y} = \dfrac{\int_A y\,dA}{A}$, $\bar{z} = \dfrac{\int_A z\,dA}{A}$

$$A = \int_A dA = \text{area}$$

Volume: $\bar{x} = \dfrac{\int_V x\,dV}{V}$, $\bar{y} = \dfrac{\int_V y\,dV}{V}$, $\bar{z} = \dfrac{\int_V z\,dV}{V}$

$$V = \int_V dV = \text{volume}$$

Examples:



$dL = r\,d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

$\frac{\pi}{2} - \alpha \leq \theta \leq \frac{\pi}{2} + \alpha$

[redacted] $L = \displaystyle\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} r\,d\theta = 2r\alpha$ (α in radians)

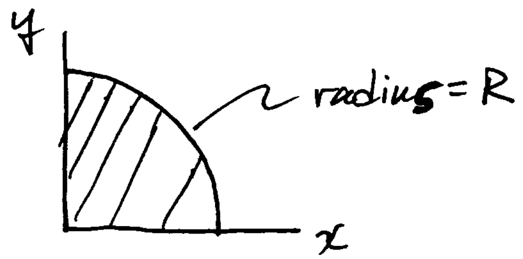
$$\bar{x} = \frac{\int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} r^2\cos\theta\,d\theta}{2r\alpha} = \frac{r^2}{2r\alpha}\sin\theta\Big]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} = 0 \checkmark$$

$$\bar{y}L = \int_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha} r^2 \sin\theta \, d\theta = -r^2 \cos\theta \Big]_{\frac{\pi}{2}-\alpha}^{\frac{\pi}{2}+\alpha}$$

$$= -r^2 \Big[\underbrace{\cos\left(\tfrac{\pi}{2}+\alpha\right)}_{-\sin\alpha} - \underbrace{\cos\left(\tfrac{\pi}{2}-\alpha\right)}_{\sin\alpha} \Big]$$

$$\bar{y}L = 2r^2 \sin\alpha$$

$$\bar{y} = \frac{r\sin\alpha}{\alpha} \quad , \quad \bar{x} = 0$$



Method 1 : $dA = r\,dr\,d\theta$

$x = r\cos\theta$

$y = r\sin\theta$

$0 \le \theta \le \frac{\pi}{2} \quad , \quad 0 \le r \le R$

$$A = \int_0^{\frac{\pi}{2}} \int_0^R r\,dr\,d\theta = \frac{\pi}{2} \int_0^R r\,dr = \frac{\pi}{2}\,\frac{1}{2} r^2 \Big]_0^R = \frac{\pi R^2}{4}$$

$$\bar{x}A = \int_0^{\pi/2} \int_0^R r^2 \cos\theta \, dr\,d\theta$$

$$= \sin\theta \Big]_0^{\pi/2} \frac{1}{3} r^3 \Big]_0^R = \frac{1}{3} R^3 \rightarrow \bar{x} = \frac{4}{3\pi} R$$

$$\bar{y}A = \int_0^{\frac{\pi}{2}} \int_0^R r^2 \sin\theta \, dr\,d\theta$$

$$= -\cos\theta \Big]_0^{\pi/2} \frac{1}{3} r^3 \Big]_0^R = \frac{1}{3} R^3 \rightarrow \bar{y} = \frac{4}{3\pi} R$$

Method 2: $dA = dx\,dy$

Boundary: $x = \sqrt{R^2 - y^2}$

or $y = \sqrt{R^2 - x^2}$

$$\bar{x}\,A = \int_0^R \left(\int_0^{\sqrt{R^2-x^2}} x \; dy \right) dx$$

$$= \int_0^R x y \Big]_0^{\sqrt{R^2-x^2}} dx$$

$$= \int_0^R (R^2x^2 - x^4)^{1/2} - 0 \;\; dx$$

$$= -\frac{1}{3}\left(R^2 - x^2\right)^{3/2} \Big]_0^R = 0 - \left(-\frac{1}{3}R^3\right) = \frac{1}{3}R^3$$

$$\rightarrow \bar{x} = \frac{4}{3\pi}R$$

$$\bar{y}\,A = \int_0^R \left[\int_0^{\sqrt{R^2-x^2}} y \; dy \right] dx$$

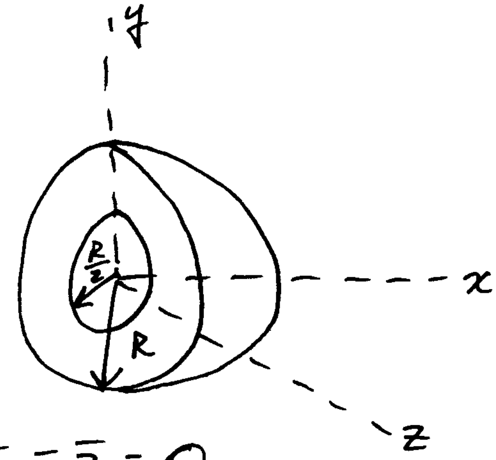
$$= \int_0^R \frac{1}{2}y^2 \Big]_0^{\sqrt{R^2-x^2}} dx$$

$$= \int_0^R \frac{1}{2}\left(R^2 - x^2\right) dx$$

$$= \frac{1}{2}R^2x - \frac{1}{6}x^3 \Big]_0^R = \frac{1}{2}R^3 - \frac{1}{6}R^3 = \frac{1}{3}R^3$$

$$\rightarrow \bar{y} = \frac{4}{3\pi}R$$

Volume example:

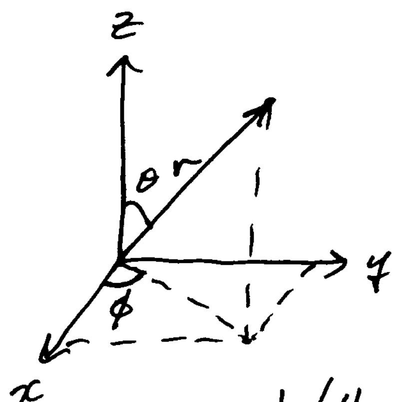


Due to symmetry $\bar{y} = \bar{z} = 0$.

Spherical coordinates: $z = r\cos\theta$

$x = r\sin\theta\cos\phi$

$y = r\sin\theta\sin\phi$



$$dV = r^2 \sin\theta \, dr \, d\theta \, d\phi$$

$$V = \frac{1}{2}\left(\frac{4}{3}\pi R^3 - \frac{4}{3}\pi \frac{R^3}{8}\right) = \frac{7}{12}\pi R^3$$

$$\bar{x} V = \int_V x \, dV$$

$$= \int_{-\pi/2}^{\frac{\pi}{2}} \int_0^{\pi} \int_{R/2}^{R} r^3 \sin^2\theta \cos\phi \, dr \, d\theta \, d\phi$$

$$= \int_{-\pi/2}^{\pi/2} \int_0^{\pi} \frac{1}{4} r^4 \Big]_{R/2}^{R} \sin^2\theta \cos\phi \, d\theta \, d\phi$$

$$= \int_0^{\pi} \frac{15}{64} R^4 \sin\phi \Big]_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \sin^2\theta \, d\theta$$

$$= \frac{15}{32} R^4 \int_0^{\pi} \sin^2\theta \, d\theta$$

$$= \frac{15}{32} R^4 \int_0^{\pi} \frac{1}{2} - \frac{1}{2}\cos 2\theta \, d\theta$$

$$= \frac{15}{32} R^4 \left(\frac{\theta}{2} - \frac{1}{4}\sin 2\theta \right) \Big]_0^{\pi}$$

$$\bar{x} V = \frac{15}{64} \pi R^4$$

$$\bar{x} = \frac{15}{64} \cancel{\pi} R^{\cancel{4}} \frac{12}{7} \frac{1}{\cancel{\pi R^3}} = \boxed{\frac{45}{112} R = \bar{x}}$$

<u>Another Approach</u>

What is $\bar{x}$ for a full hemisphere?

Instead of $\int_{R/2}^{R} dr$ we need $\int_0^R dr$

$$\int_{R/2}^{R} dr \rightarrow \frac{15}{64} R^4 \qquad \int_0^R dr \rightarrow \frac{1}{4} R^4$$

$$\rightarrow \bar{x} V = \frac{1}{4} \pi R^4 \quad \text{with} \quad V = \frac{1}{2}\frac{4}{3} \pi R^3 = \frac{2}{3} \pi R^3$$

$$\rightarrow \bar{x} = \frac{3}{8} R$$

So what? How can we use this to arrive at our previous result?

Well the hemi-melon is just a hemisphere of radius R <u>minus</u> a hemisphere of radius $R/2$. However $\bar{x}$ represents a <u>weighted average</u> of position.

$$V = V_R - V_{R/2} = \frac{2}{3}\pi R^3 - \frac{2}{3}\pi\left(\frac{R}{2}\right)^3 = \frac{7}{12}\pi R^3$$

What about $\bar{x}$?

$$\bar{x} = \frac{\bar{x}_R V_R - \bar{x}_{R/2} V_{R/2}}{V}$$

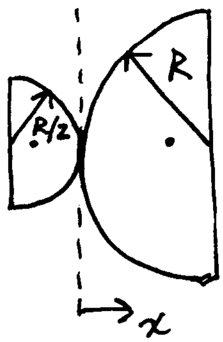
$$= \frac{\frac{3}{8}R\,\frac{2}{3}\cancel{\pi}\cancel{R^3} - \frac{3}{8}\left(\frac{R}{2}\right)\frac{2}{24}\cancel{\pi}\cancel{R^3}}{\frac{7}{12}\cancel{\pi}\cancel{R^3}}$$

$$\bar{x} = \frac{\frac{1}{4}R - \frac{1}{64}R}{\frac{7}{12}} = \frac{15}{64}\,\frac{12}{7}R = \frac{45}{112}R \checkmark$$

What do you need to know?

1) What the centroid means.
2) How to set up integrals to compute centroids for relatively simple objects.
*3) How to look up centroids for simple objects and use this information to compute centroids for <u>composite</u> objects.

Another composite object.



Both are hemispheres.

$$V_R = \frac{2}{3}\pi R^3 \quad , \quad \bar{x}_R = R - \frac{3}{8}R = \frac{5}{8}R$$

$$V_{R/2} = \frac{2}{3}\pi\left(\frac{R}{2}\right)^3 \quad , \quad \bar{x}_{R/2} = -\left(\frac{R}{2} - \frac{3}{8}\frac{R}{2}\right) = -\frac{5}{16}R$$

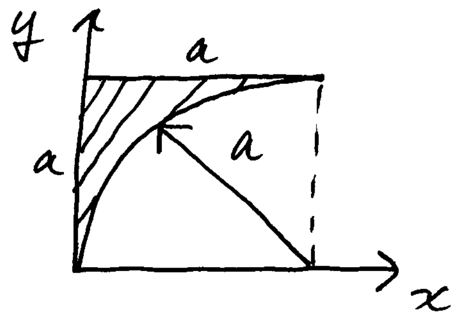
$$\bar{x} = \frac{\bar{x}_R V_R + \bar{x}_{R/2} V_{R/2}}{V_R + V_{R/2}}$$

$$= \frac{\frac{5}{8}R\,\frac{2}{3}\cancel{\pi}\cancel{R^3} - \frac{5}{16}R\,\frac{\cancel{\pi}}{12}\cancel{R^3}}{\frac{2}{3}\cancel{\pi}\cancel{R^3} + \frac{1}{12}\cancel{\pi}\cancel{R^3}}$$

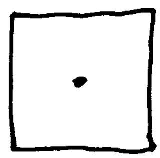
$$= \frac{\frac{5}{12} - \frac{5}{16\cdot 12}}{\frac{3}{4}}R = \left(\frac{5}{9} - \frac{5}{144}\right)R$$

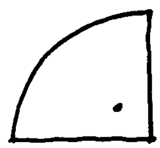
$$\boxed{\bar{x} = \frac{75}{144}R}$$

Example:



Square minus quarter circle.





$$\bar{x}_s = \frac{a}{2} \qquad \bar{x}_c = a - \frac{4}{3\pi}a$$

$$\bar{y}_s = \frac{a}{2} \qquad \bar{y}_c = \frac{4}{3\pi}a$$

$$A_s = a^2 \qquad A_c = \frac{\pi a^2}{4}$$

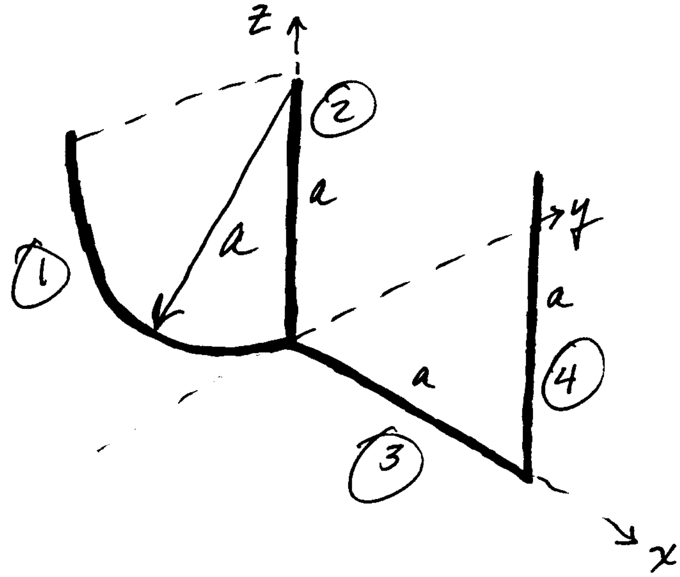
$$A = A_s - A_c = \left(1-\frac{\pi}{4}\right)a^2$$

$$\bar{x} = \frac{\bar{x}_s A_s - \bar{x}_c A_c}{A} = \frac{\frac{a}{2}\cancel{a^2} - \left(1-\frac{4}{3\pi}\right)a\,\frac{\pi \cancel{a^2}}{4}}{\left(1-\frac{\pi}{4}\right)\cancel{a^2}} = \frac{\frac{5}{6}-\frac{\pi}{4}}{1-\frac{\pi}{4}}a$$

$$\bar{y} = \frac{\bar{y}_s A_s - \bar{y}_c A_c}{A} = \frac{\frac{a}{2}\cancel{a^2} - \frac{4}{3\pi}a\,\frac{\pi \cancel{a^2}}{4}}{\left(1-\frac{\pi}{4}\right)\cancel{a^2}}$$

$$\bar{y} = \frac{a}{6\left(1-\frac{\pi}{4}\right)}$$

Example:



(2) $L_2 = a$ $\qquad \bar{x}_2 = 0$

$\bar{y}_2 = 0$

$\bar{z}_2 = a/2$

(3) $L_3 = a$ $\qquad \bar{x}_3 = a/2$

$\bar{y}_3 = 0$

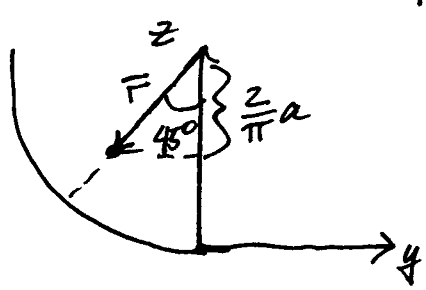
$\bar{z}_3 = 0$

(4) $L_4 = a$ $\qquad \bar{x}_4 = a$

$\bar{y}_4 = 0$

$\bar{z}_4 = a/2$

(1)



$$\bar{r} = \frac{a \sin 45^\circ}{\pi/4} = \frac{4}{\pi}\frac{\sqrt{2}}{2}a$$

$\bar{z}_1 = \left(1 - \frac{2}{\pi}\right)a$

$\bar{y}_1 = -\frac{2}{\pi}a$

$\bar{x}_1 = 0$

$L_1 = \frac{\pi}{2}a$

$$\bar{x} = \frac{\bar{x}_1 L_1 + \bar{x}_2 L_2 + \bar{x}_3 L_3 + \bar{x}_4 L_4}{L_1 + L_2 + L_3 + L_4}$$

$$= \frac{0 + 0 + \frac{a}{2}\cancel{a} + a \cdot \cancel{a}}{\left(\frac{\pi}{2} + 3\right)\cancel{a}} = \underline{\underline{\frac{3}{\pi + 6}\, a}}$$

$$\bar{y} = \frac{\bar{y}_1 L_1 + \bar{y}_2 L_2 + \bar{y}_3 L_3 + \bar{y}_4 L_4}{L_1 + L_2 + L_3 + L_4}$$

$$= \frac{-\frac{2}{\pi} a \, \frac{\pi}{2}\cancel{a} + 0 + 0 + 0}{\left(\frac{\pi}{2} + 3\right)\cancel{a}} = \underline{\underline{\frac{-2}{\pi + 6}\, a}}$$

$$\bar{z} = \frac{\bar{z}_1 L_1 + \bar{z}_2 L_2 + \bar{z}_3 L_3 + \bar{z}_4 L_4}{L_1 + L_2 + L_3 + L_4}$$

$$= \frac{\left(1 - \frac{2}{\pi}\right) a \, \frac{\pi}{2}\cancel{a} + \frac{a}{2}\cancel{a} + 0 + \frac{a}{2}\cancel{a}}{\left(\frac{\pi}{2} + 3\right)\cancel{a}}$$

$$= \underline{\underline{\frac{\pi}{\pi + 6}\, a}}$$