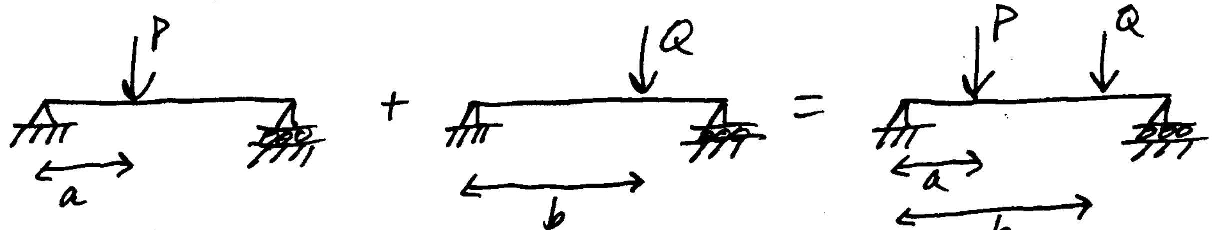
Beam Deflections using Superposition

The idea behind superposition is quite simple. Since $q = -\frac{1}{EI}\frac{d^4v}{dx^4}$ is a linear equation then if v_1 is a solution to $q_1 = \frac{-1}{EI}v_1''''$ and v_2 is a solution to $q_2 = \frac{-1}{EI}v_2''''$ then v_1+v_2 is a solution to $q_1+q_2 = \frac{-1}{EI}(v_1+v_2)''''$. Note that this is useful when each of these problem solutions fits the same boundary conditions.

For example:



v_P + v_Q = $v_P + v_Q$

Let's generate the solution for an arbitrarily located point load on a simply supported beam again.

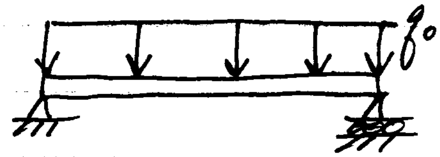
$$q(x) = P\langle x-a\rangle^{-1}$$
$$V(x) = -P\langle x-a\rangle^{0} + \frac{P(L-a)}{L}$$
$$M(x) = -P\langle x-a\rangle^{1} + \frac{P(L-a)}{L}x$$
$$\theta(x) = \frac{-P}{2EI}\langle x-a\rangle^{2} + \frac{P(L-a)}{2EIL}x^2 + c_1$$
$$v(x) = \frac{-P}{6EI}\langle x-a\rangle^{3} + \frac{P(L-a)}{6EIL}x^3 + c_1 x$$

$$v(x=L) = 0 = -\frac{P}{6EI}\underbrace{(L-a)^3}_{L^3-3L^2a+3aL^2-a^3} + \frac{P(L-a)}{6EI}L^2 + c_1 L$$

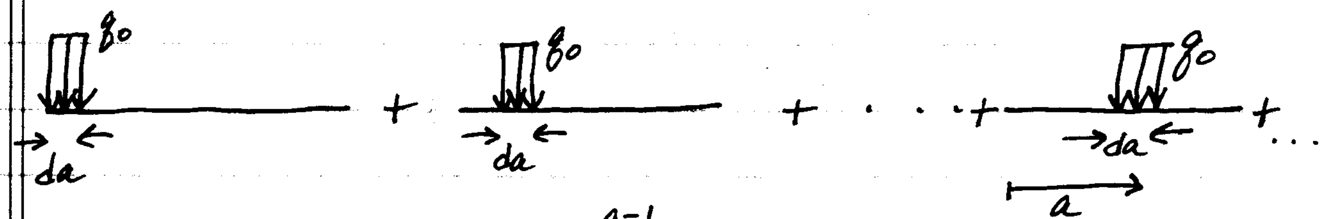
$$c_1 L = \frac{P}{6EI}\left(-2L^2a + 3La^2 - a^3\right)$$

$$v(x) = \frac{P}{6EI}\left[-\langle x-a\rangle^3 + \frac{L-a}{L}x^3 + \frac{a}{L}\left(-2L^2+3La-a^2\right)x\right]$$

Can we use this solution along with superposition to find the deflection for a uniform $q(x)=q_0$?



This distribution looks like:



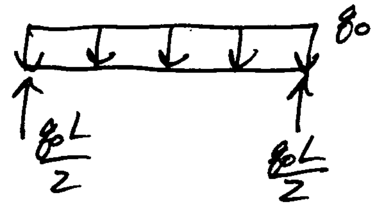
$$\text{Superposition} \rightarrow v(x) = \int_{a=0}^{a=L} \frac{q_0\,da}{6EI}\left[-\langle x-a\rangle^3 + \frac{L-a}{L}x^3 + \frac{a}{L}\left(-2L^2+3La-a^2\right)x\right]$$

$$v(x) = \int_{a=0}^{a=x} \frac{-q_0}{6EI}(x-a)^3\,da + \int_{a=0}^{a=L}\frac{q_0}{6EI}\left[\left(1-\frac{a}{L}\right)x^3 + \frac{1}{L}\left(-2L^2a+3La^2-a^3\right)x\right]da$$

$$= \left.\frac{+q_0}{24EI}(x-a)^4\right]_{a=0}^{a=x} + \frac{q_0}{6EI}\left\{Lx^3 - \frac{L}{2}x^3 - L^3x + L^3x - \frac{L^3}{4}x\right\}$$

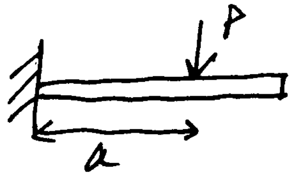
$$v(x) = \frac{-q_0}{24EI}x^4 + \frac{q_0}{12EI}x^3 - \frac{q_0L^3}{24EI}x$$

Is this correct?



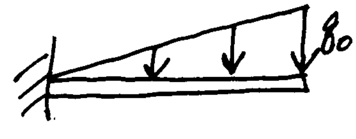
$$q(x) = q_0$$
$$V(x) = -q_0 x + \frac{q_0 L}{2}$$
$$M(x) = -\frac{q_0}{2}x^2 + \frac{q_0 L}{2}x$$
$$\theta(x) = -\frac{q_0}{6EI}x^3 + \frac{q_0 L}{4EI}x^2 + c_1$$
$$v(x) = -\frac{q_0}{24EI}x^4 + \frac{q_0 L}{12EI}x^3 + c_1 x$$
$$v(x=L) = 0 = -\frac{q_0}{24EI}L^4 + \frac{q_0}{12EI}L^4 + c_1 L$$
$$\rightarrow c_1 = -\frac{q_0 L^3}{24EI} \quad \checkmark$$

Used this way, the point force solution is called a Green's function. This function can be integrated to determine the solution for any distribution of $q(x)$ by using $P \rightarrow q(a)da$ as long as the beam is supported by simple supports on its ends. The Green's function for a cantilever support would be the solution to:



Following the steps on 120 yields

$$v(x) = -\frac{P}{6EI}\langle x-a\rangle^3 + \frac{P\,\text{[redacted]}}{6EI\,\text{[redacted]}}x^3 - \frac{Pa}{2EI}x^2$$

What is $v(x)$ for  ?

$$q(x) = q_0 \frac{x}{L} \quad \Rightarrow \quad P \to q_0 \frac{a}{L}\, da$$

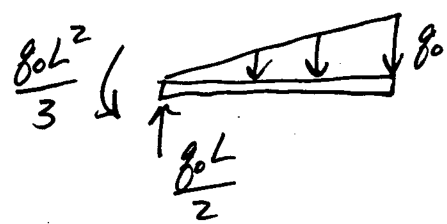
$$v(x) = \frac{1}{EI} \int_{a=0}^{a=L} -\frac{q_0 a}{6L}\langle x-a\rangle^3 + \frac{q_0 a}{6L} x^3 - \frac{q_0 a^2}{2L} x^2\, da$$

$$= \frac{q_0}{EIL}\left\{ \int_{a=0}^{a=x} \underbrace{-\frac{a}{6}(x-a)^3}_{-\frac{1}{6}(x^3a - 3x^2a^2 + 3xa^3 - a^4)} da + \int_{a=0}^{a=L} \frac{a}{6}x^3 - \frac{a^2}{2}x^2\, da \right\}$$

$$= \frac{q_0}{EIL}\left\{ \underbrace{-\frac{1}{12}x^3a^2 + \frac{1}{6}x^2a^3 - \frac{1}{8}xa^4 + \frac{1}{30}a^5 \Big]_{a=0}^{a=x}}_{-\frac{x^5}{120}} + \frac{L^2}{12}x^3 - \frac{L^3}{6}x^2 \right\}$$

$$v(x) = \frac{q_0}{EIL}\left(-\frac{x^5}{120} + \frac{L^2x^3}{12} - \frac{L^3x^2}{6} \right)$$

Is this correct?



$$q(x) = q_0 \frac{x}{L}$$

$$V(x) = -\frac{q_0}{2L}x^2 + \frac{q_0L}{2}$$

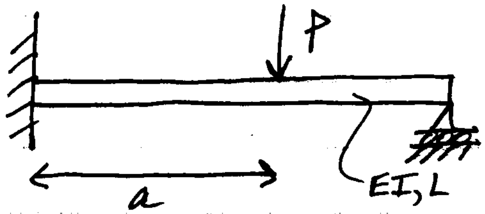
$$M(x) = -\frac{q_0}{6L}x^3 + \frac{q_0L}{2}x - \frac{q_0L^2}{3}$$

$$\theta(x) = -\frac{q_0}{24EIL}x^4 + \frac{q_0L}{4EI}x^2 - \frac{q_0L^2}{3EI}x$$

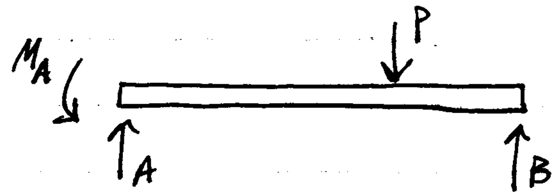
$$v(x) = -\frac{q_0}{120EIL}x^5 + \frac{q_0L}{12EI}x^3 - \frac{q_0L^2}{6EI}x^2 \quad \checkmark$$

This is just one use of superposition. Another important application is to statically indeterminate problems.

Consider the problem:



What are the reactions, and what is the deflection of the beam?

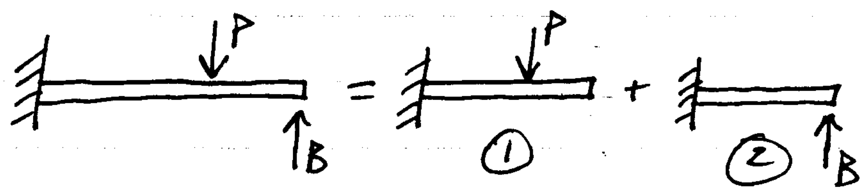


$$\Sigma F_y = A + B - P = 0$$

$$\Sigma M_z^A = M_A + BL - Pa = 0$$

Not enough equations for unknowns $\rightarrow$ statically indeterminate.

Superposition $\rightarrow$



as we just found on page 122,

$$v_1(x) = \frac{-P}{6EI}\langle x-a\rangle^3 + \frac{P}{6EI}x^3 - \frac{Pa}{2EI}x^2$$

The same formula can be used for $v_2(x)$.

$$v_2(x) = \frac{-B}{6EI}x^3 + \frac{BL}{2EI}x^2$$

So, $v(x) = v_1(x) + v_2(x) = \frac{-P}{6EI}\langle x-a\rangle^3 + \frac{P-B}{6EI}x^3 + \frac{BL-Pa}{2EI}x^2$

Now, we know P, but B is a reaction. How can we determine B?

We have the constraint that $v(x=L)=0$.

$$v(L) = -\frac{P}{6EI}(L-a)^3 + \frac{P-B}{6EI}L^3 + \frac{BL-Pa}{2EI}L^2 = 0$$

$$B = \frac{3P}{L^3}\left[\frac{1}{6}(L-a)^3 - \frac{1}{6}L^3 + \frac{a}{2}L^2\right]$$

$$\boxed{B = 3P\left[\frac{1}{6}\left(1-\frac{a}{L}\right)^3 - \frac{1}{6} + \frac{1}{2}\frac{a}{L}\right]}$$

Checks: $a=0 \rightarrow B=0 \qquad B=3P\left[\frac{1}{6}-\frac{1}{6}\right]$ ✓

$a=L \rightarrow B=P \qquad B=3P\left[-\frac{1}{6}+\frac{1}{2}\right]=P$ ✓

Now that we have B we know $v(x)$. Also, our equilibrium equations tell us A and M_A.

$$A = P-B$$
$$M_A = Pa - BL$$

We could also get A and M_A from $v(x)$.

$$A = V(x=0) = EI\,v'''\big|_{x=0}$$

$$M_A = -M(x=0) = -EI\,v''\big|_{x=0}$$

Another approach: $q(x) = P\langle x-a\rangle^{-1}$

$$\rightarrow V(x) = -P\langle x-a\rangle^{0} + A$$

$V(0)=A$

$$M(x) = -P\langle x-a\rangle^{1} + Ax - M_A$$

$M(0) = -M_A$

$$\theta(x) = \frac{-P}{2EI}\langle x-a\rangle^{2} + \frac{A}{2EI}x^2 - \frac{M_A}{EI}x + 0$$

$\theta(x=0)=0$

$$v(x) = \frac{-P}{6EI}\langle x-a\rangle^{3} + \frac{A}{6EI}x^3 - \frac{M_A}{2EI}x^2 + 0$$

$v(0)=0$

Boundary conditions at $x=L$:

$$v(L) = 0 = \frac{-P}{6EI}(L-a)^3 + \frac{A}{6EI}L^3 - \frac{M_A}{2EI}L^2$$

$$M(L) = EI\,v''(L) = 0 = -P(L-a) + AL - M_A$$

Solve for M_A & A $\rightarrow$ $M_A = AL - P(L-a)$

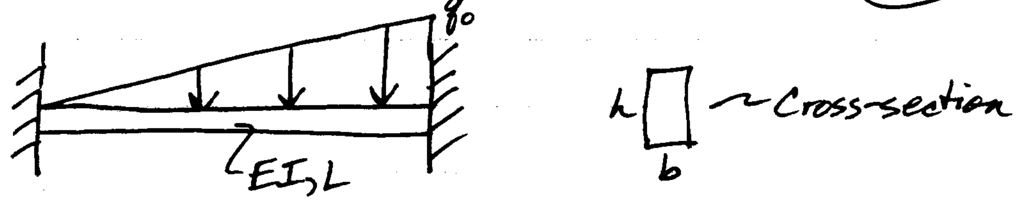
then $0 = -\frac{1}{3}P(L-a)^3 + \frac{1}{3}L^3A - L^3A + PL^2(L-a)$

$$A = -\frac{1}{2}P\left(1-\frac{a}{L}\right)^3 + \frac{3}{2}P\left(1-\frac{a}{L}\right)$$

From equilibrium $B = P - A = \frac{1}{2}P\left(1-\frac{a}{L}\right)^3 - \frac{1}{2}P + \frac{3}{2}P\frac{a}{L}$

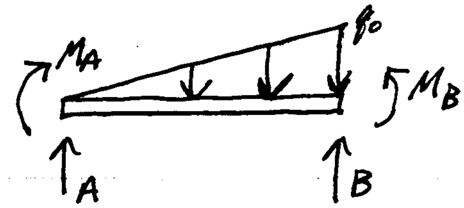
✓ this result is the same as our previous approach.

Example:



Determine τ_{max}, σ_{max}, v_{max}.

Integration approach:



$$\sum F_y = A + B - \frac{q_0 L}{2} = 0$$

$$\sum M_z^A = -M_A + M_B + BL - \frac{q_0 L}{2}\frac{2}{3}L = 0$$

2 equations for 4 unknowns

$$q(x) = q_0 \frac{x}{L}$$

$$V(x) = \frac{-q_0}{2L}x^2 + A \quad \text{from } V(0) = A$$

$$M(x) = \frac{-q_0}{6L}x^3 + Ax + M_A \quad \text{from } M(0) = M_A$$

$$\theta(x) = \frac{-q_0}{24EIL}x^4 + \frac{A}{2EI}x^2 + \frac{M_A}{EI}x \quad \text{using } \theta(0) = 0$$

$$v(x) = \frac{-q_0}{120EIL}x^5 + \frac{A}{6EI}x^3 + \frac{M_A}{2EI}x^2 \quad \text{using } v(0) = 0$$

Now we need boundary conditions at $x = L$.

$$v(L) = 0 = \frac{-q_0 L^5}{120EIL} + \frac{AL^3}{6EI} + \frac{M_A L^2}{2EI}$$

$$\theta(L) = 0 = \frac{-q_0 L^4}{24EIL} + \frac{AL^2}{2EI} + \frac{M_A L}{EI}$$

Solve for A & M_A.

$$M_A = \frac{q_0 L^2}{24} - \frac{AL}{2}$$

$$\rightarrow 0 = \frac{-q_0 L^2}{60} + \frac{AL}{3} + \frac{q_0 L^2}{24} - \frac{AL}{2}$$

$$\frac{A}{6} = \frac{q_0 L}{40} \rightarrow A = \frac{3 q_0 L}{20}$$

$$\rightarrow M_A = \frac{-4}{120} q_0 L^2$$

$$\text{Equilibrium} \rightarrow B = \frac{7}{20} q_0 L$$

$$M_B = \frac{-6}{120} q_0 L^2$$

Do the signs make sense? (Discuss)

Check B & M_B.

$$V(L) = -B \rightarrow \frac{-7}{20} q_0 L \overset{?}{=} -\frac{q_0}{2L} L^2 + \frac{3 q_0 L}{20} \overset{\checkmark}{=} \frac{-7}{20} q_0 L$$

$$M(L) = M_B \rightarrow \frac{-6}{120} q_0 L^2 \overset{?}{=} -\frac{q_0 L^2}{6} + \frac{3 q_0 L^2}{20} - \frac{4}{120} q_0 L^2 \overset{\checkmark}{=} \frac{-6}{120} q_0 L^2$$

$$v(x) = \frac{-q_0}{120EIL} x^5 + \frac{q_0 L}{40EI} x^3 - \frac{4 q_0 L^2}{240EI} x^2$$

v_{max} occurs where $\theta = 0 \rightarrow \frac{-q_0}{24EIL} x_c^4 + \frac{3 q_0 L}{40EI} x_c^2 - \frac{4 q_0 L^2}{120EI} x_c = 0$

$x_c = 0$ is a solution & we should get $x_c = L$ as well.

Factor $x_c = 0$ out $\rightarrow \frac{-1L^2}{24L^3} x_c^3 + \frac{3L^2}{40L} x_c - \frac{4L^2}{120} = 0$

Factor $\frac{x_c}{L} = 1$ out

$$\left(\frac{x_c}{L}-1\right)\left[-\frac{1}{24}\left(\frac{x_c}{L}\right)^2-\frac{1}{24}\left(\frac{x_c}{L}\right)+\frac{1}{30}\right]=0$$

$$\frac{x_c}{L} = \frac{\frac{1}{24} \pm \sqrt{\left(\frac{1}{24}\right)^2+\frac{1}{180}}}{-2/24}$$

$$\frac{x_c}{L} = -\frac{1}{2} + \left(\sqrt{1+\frac{576}{180}}\Big/2\right) \approx 0.525$$

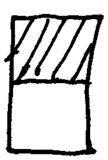
$$\rightarrow v_{max} = 0.0013085 \frac{q_0 L^4}{EI}$$

V_{max} occurs at one ~~of~~ the supports since $q \neq 0$ in the range $0 < x < L$.

$B > A \rightarrow \quad V_{max} = \frac{7}{20} q_0 L$ (Note that here it is OK to deal with magnitudes.)

$$\tau_{max} = \frac{V_{max}}{I}\left(\frac{Q}{b}\right)_{max}$$

For a rectangle Q_{max} occurs at $y=0$.



$$Q(y=0) = \int_0^{h/2} y' \, b \, dy' = \frac{b}{2} y'^2 \Big]_0^{h/2} = \frac{bh^2}{8}$$

$$\rightarrow \tau_{max} = \frac{7}{20} q_0 L \, \frac{12}{bh^3} \, \frac{bh^2}{8b} = \frac{21}{40} \frac{q_0 L}{bh}$$

M_{max} occurs where $V=0$.

$$V = -\frac{q_0}{2L}x_o^2 + \frac{3q_0L}{20} = 0$$

$$x_o = \sqrt{\frac{3}{10}}\,L$$

$$M\left(\sqrt{\tfrac{3}{10}}\,L\right) = q_0L^2\left[-\frac{1}{6}\left(\sqrt{\tfrac{3}{10}}\right)^3 + \frac{3}{20}\sqrt{\tfrac{3}{10}} - \frac{1}{30}\right]$$

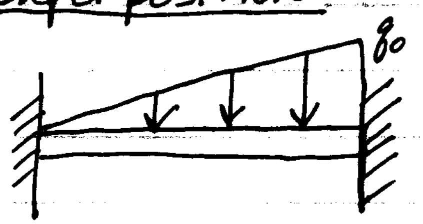
$$= 0.0214\, q_0L^2$$

$$M_A = -\frac{1}{30}q_0L^2 = -0.033\, q_0L^2$$

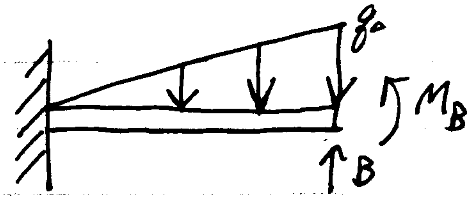
$$M_B = -\frac{1}{20}q_0L^2 = \boxed{-0.05\, q_0L^2}$$ max magnitude

$$\sigma_{max} = \frac{M_B\, h/2}{bh^3/12} = \frac{3}{10}\frac{q_0L^2}{bh^3}$$ tens / comp

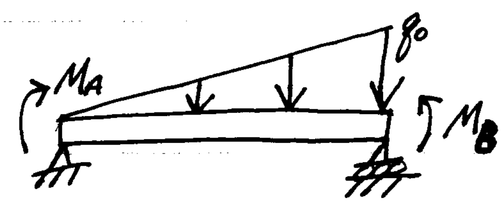
<u>Superposition</u>



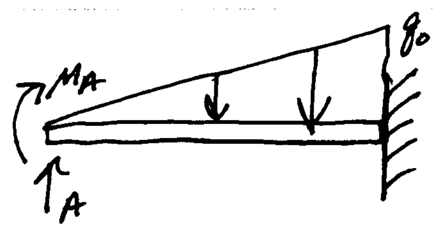
$=$



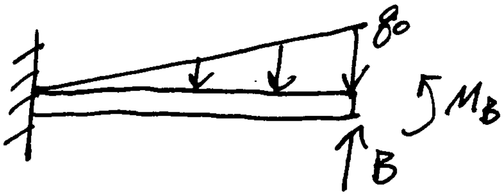
or $=$

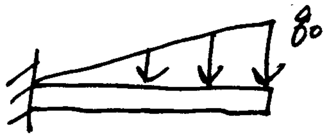


or $=$

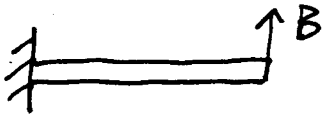


Any of these can be used for a superposition scheme. Let's look at the first.

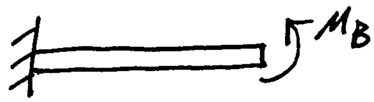




$$= \quad \Rightarrow \quad v_q = \frac{-q_0 x^2}{120EIL}\left(20L^3-10L^2x+x^3\right)$$



$$+ \quad \Rightarrow \quad v_B = \frac{Bx^2}{6EI}(3L-x)$$



$$+ \quad \Rightarrow \quad v_M = \frac{M_B x^2}{2EI}$$

$$v(x) = v_q + v_B + v_M = \frac{-q_0 x^2}{120EIL}\left(20L^3-10L^2x+x^3\right) + \frac{Bx^2}{6EI}(3L-x) + \frac{M_B x^2}{2EI}$$

Boundary conditions:

$$v(L) = 0 = \frac{-q_0 L^{\cancel{x}2}}{120\cancel{EI}}\underbrace{(20-10+1)}_{11} + \frac{BL^{\cancel{3}}}{6\cancel{EI}}\underbrace{(3-1)}_{2} + \frac{M_B \cancel{L^2}}{2\cancel{EI}}$$

$$\theta(L) = v'(L) = 0 = -\frac{q_0 L^{\cancel{3}2}}{120\cancel{EI}}\underbrace{(40-30+5)}_{15} + \frac{BL^{\cancel{2}}}{6\cancel{EI}}\underbrace{(6-3)}_{3} + \frac{M_B \cancel{L}}{\cancel{EI}}$$

$$M_B = \frac{15}{120}q_0L^2 - \frac{BL}{2}$$

$$0 = -\frac{11}{120}q_0L^2 + \frac{BL}{3} - \frac{BL}{4} + \frac{15}{240}q_0L^2$$

$\rightarrow B = \frac{7}{20}q_0L$ ✓ same as previous approach

Use $q = -EIv''''$, $V = EIv'''$, $M = EIv''$ to find max/min.