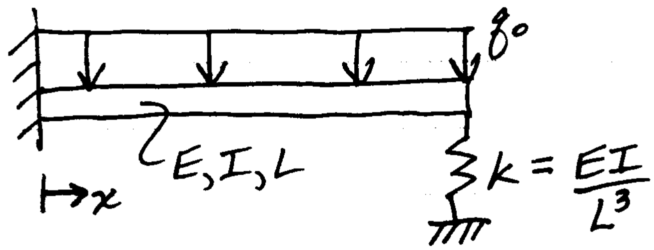
Statically Indeterminate Beams



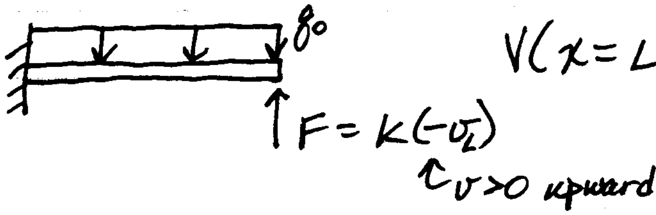
Integration : $q(x) = q_0$

$$\rightarrow V(x) = -q_0 x + c_1$$
$$\rightarrow M(x) = -\frac{1}{2} q_0 x^2 + c_1 x + c_2$$
$$\rightarrow EI\theta(x) = -\frac{1}{6} q_0 x^3 + \frac{1}{2} c_1 x^2 + c_2 x + c_3$$
$$\rightarrow EIv(x) = -\frac{1}{24} q_0 x^4 + \frac{1}{6} c_1 x^3 + \frac{1}{2} c_2 x^2 + c_3 x + c_4$$

BCs: $v(x=0) = 0 \rightarrow c_4 = 0$ *

$\theta(x=0) = 0 \rightarrow c_3 = 0$ **

$M(x=L) = 0 \rightarrow -\frac{1}{2} q_0 L^2 + c_1 L + c_2 = 0$ (a)

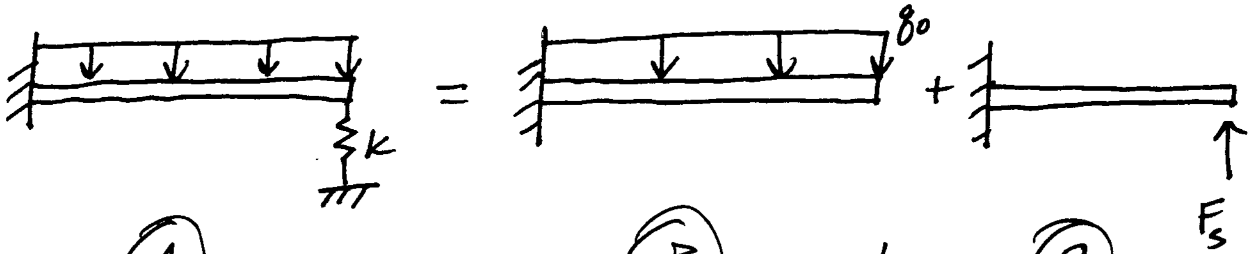


$V(x=L) = -F = k v_L$ (b)

$$\text{(b)} \rightarrow -q_0 L + c_1 = \frac{1}{L^3}\left[-\frac{1}{24} q_0 L^4 + \frac{1}{6} c_1 L^3 + \frac{1}{2} c_2 L^2\right]$$
$$= -\frac{1}{24} q_0 L + \frac{1}{6} c_1 + \frac{1}{2L} c_2$$
$$-\frac{23}{24} q_0 L + \frac{5}{6} c_1 = \frac{1}{2L} c_2$$
$$\text{(a)} \rightarrow -\frac{12}{24} q_0 L + \frac{6}{6} c_1 = -\frac{2}{2L} c_2$$

$$-\frac{29}{24} q_0 L + \frac{8}{6} c_1 = 0$$
$$\rightarrow c_1 = \frac{29}{32} q_0 L \quad ***$$
$$\rightarrow c_2 = -\frac{13}{32} q_0 L^2 \quad ****$$

Superposition



A = B + C

Problems B and C can be found in tables.

B : $v_B(x) = -\dfrac{q_0 x^2}{24EI}(6L^2 - 4Lx + x^2)$

C : $v_C(x) = \dfrac{F_s x^2}{6EI}(3L - x)$

So, $v_A(x) = v_B(x) + v_C(x)$, but what about F_s ?

Again, $F_s = k(-v_L) = -k\, v_A(x=L)$

$$F_s = \underbrace{-\frac{EI}{L^3}}_{-k}\left[\frac{-q_0L^2}{24EI}\underbrace{(6L^2-4L^2+L^2)}_{3L^2} + \frac{F_sL^2}{6EI}\underbrace{(3L-L)}_{2L}\right]$$

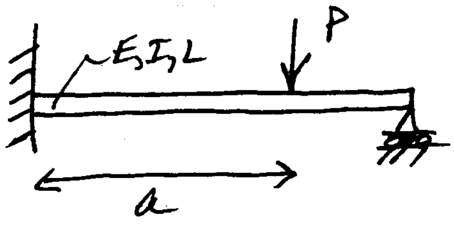
$$F_s = \frac{q_0L}{8} + \frac{-F_s}{3} \rightarrow \boxed{F_s = \frac{3}{32}q_0L}$$

Check with previous solution: $v(x=L) = \dfrac{q_0L^4}{EI}\left[\dfrac{-1}{24} + \dfrac{29}{192} - \dfrac{13}{64}\right]$

$$= \frac{q_0L^4}{EI}\left(\frac{-3}{32}\right)$$

$$F_s = -\frac{EI}{L^3}v(x=L) = \frac{3}{32}q_0L \checkmark$$

Example:

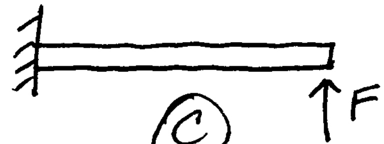


A

A =

B

+



C

B and C are in tables. How do we find F? The force F must have a magnitude such that $v_A(x=L) = 0$.

Tables $\rightarrow$ $$v_B(x=L) = -\frac{Pa^2}{6EI}(3L-a)$$

$$v_C(x=L) = +\frac{FL^3}{3EI}$$

$$v_A(x=L) = 0 \rightarrow -\frac{Pa^2}{6\cancel{EI}}(3L-a) + \frac{FL^3}{3\cancel{EI}} = 0$$

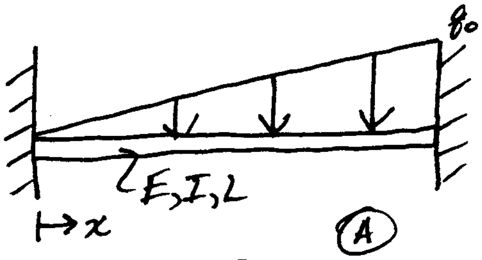
$$\rightarrow \boxed{F = \frac{P}{2}\frac{a^2}{L^2}\left(3-\frac{a}{L}\right)}$$

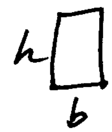
Then $v_A(x) = v_B(x) + v_C(x)$

where: $$v_C(x) = \frac{P}{2}\frac{a^2}{L^2}\left(3-\frac{a}{L}\right)\frac{x^2(3L-x)}{6EI}$$

$$v_B(x) = \begin{cases} -\frac{Px^2}{6EI}(3a-x) & \text{for } 0 \le x \le a \\ -\frac{Pa^2}{6EI}(3x-a) & \text{for } a \le x \le L \end{cases}$$

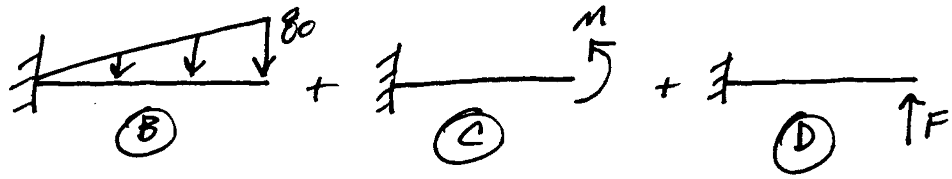
Example:





τ_{max}, σ_{max}, v_{max} ?

Superposition:



How do we find M and F? They must satisfy the constraints at the wall, i.e. $v_A(L)=0$ and $\theta_A(L)=0$.

$$v_A(L)=0 \rightarrow -\frac{q_0 L^2}{120EIL}\underbrace{(20L^3-10L^3+L^3)}_{11L^3}+\frac{ML^2}{2EI}+\frac{FL^3}{3EI}=0 \quad (1)$$

$$\theta_A(L)=0 \rightarrow -\frac{q_0 L}{24EIL}\underbrace{(8L^3-6L^3+L^3)}_{3L^3}+\frac{ML}{EI}+\frac{FL^2}{2EI}=0 \quad (2)$$

$$2\times\frac{(1)}{L}-(2) \rightarrow \frac{-7}{120}q_0L^3+\frac{FL^2}{6}=0 \rightarrow \boxed{F=\frac{7}{20}q_0L}$$

$$\text{Then } (2) \rightarrow -\frac{q_0L^3}{8}+ML+\frac{7}{40}q_0L^3=0 \rightarrow \boxed{M=\frac{-1}{20}q_0L^2}$$

Now we can write $v_A(x)$.

$$v_A(x)=\frac{-q_0x^2}{120EIL}(20L^3-10L^2x+x^3)$$
$$-\frac{q_0L^2x^2}{40EI}+\frac{7q_0Lx^2}{120EI}(3L-x)$$

$$\theta(x) = \frac{dv}{dx} = \frac{-q_0}{120EIL}\left(40L^3x - 30L^2x^2 + 5x^4\right) - \frac{q_0L^2x}{20EI} + \frac{7q_0L}{120EI}\left(6Lx - 3x^2\right)$$

$$M(x) = EI\frac{d\theta}{dx} = \frac{-q_0}{120L}\left(40L^3 - 60L^2x + 20x^3\right) - \frac{q_0L^2\cancel{EI}}{20\cancel{EI}} + \frac{7q_0L\cancel{EI}}{120\cancel{EI}}\left(6L - 6x\right)$$

$$V(x) = \frac{dM}{dx} = \frac{-q_0}{120L}\left(-60L^2 + 60x^2\right) - \frac{7q_0L}{20}$$

$$q(x) = -\frac{dV}{dx} = q_0 x \checkmark$$

$$\tau_{max} = \frac{|V|_{max}}{I}\left(\frac{Q}{b}\right)_{max}$$

V_{max} occurs where $q(x)=0$ or boundaries

$V(0) = \frac{3q_0L}{20}$

$V(L) = -\frac{7q_0L}{20}$ ← max absolute value

* $\tau_{max} = \frac{3}{2}\,\frac{7q_0L/20}{bh}$

$\sigma_{max/min} = \pm\left|\frac{My}{I}\right|_{max}$ for symmetric sections

M_{max} occurs where $V=0 \rightarrow \frac{-q_0}{2L}(x^2 - L^2) = \frac{7q_0L}{20}$ or boundaries

$$\frac{-q_0x^2}{2L} = \frac{-3q_0L}{20}$$

$$x = \pm\sqrt{\frac{3}{10}}\,L$$

$M(0) = -\frac{1}{30}q_0L^2 = -0.033q_0L^2$

$M(L) = -\frac{1}{20}q_0L^2 = \underline{-0.05q_0L^2}$

$M(\sqrt{\tfrac{3}{10}}L) = 0.0214\,q_0L^2$

$$|M|_{max} = \frac{q_0L^2}{20} \rightarrow \sigma_{max/min} = \frac{\pm 6q_0L^2/20}{bh^2}$$

$v_{max/min}$ occurs where $\theta = 0$.

$$\rightarrow \frac{-\cancel{w_0} x}{120\cancel{EI}L}(40L^3 - 30L^2x + 5x^3) - \frac{\cancel{w_0}L^2x}{20\cancel{EI}} + \frac{7\cancel{w_0}Lx}{120\cancel{EI}}(6L-3x) = 0$$

$$-\frac{x}{\cancel{120L}}\left[4L^3 - 9L^2x + 5x^3\right] = 0$$

$$-x\,(L-x)(4L^2 - 5Lx - 5x^2) = 0$$

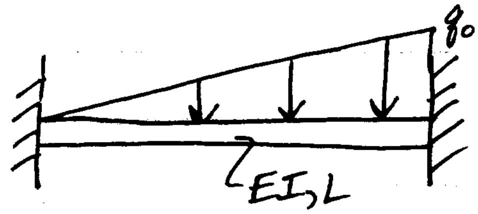
$$x = 0,\ x = L,\quad x = \frac{5L \pm \sqrt{25L^2 + 80L^2}}{-10} = -1.5247L,\ \underline{\underline{0.5247L}}$$

$v(0) = 0$

$v(L) = 0$

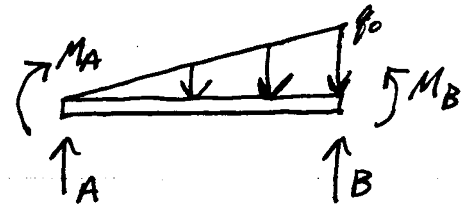
$$v(0.5247L) = -0.0013\,\frac{w_0 L^2}{EI} = v_{min}$$

Example:



Determine τ_{max}, σ_{max}, v_{max}.

Integration approach:



$$\sum F_y = A + B - \frac{q_0 L}{2} = 0$$

$$\sum M_z^A = -M_A + M_B + BL - \frac{q_0 L}{2}\frac{2}{3}L = 0$$

2 equations for 4 unknowns

$$q(x) = q_0 \frac{x}{L}$$

$$V(x) = \frac{-q_0}{2L}x^2 + \underline{A} \text{ from } V(0) = A$$

$$M(x) = \frac{-q_0}{6L}x^3 + Ax + \underline{M_A} \text{ from } M(0) = M_A$$

$$\theta(x) = \frac{-q_0}{24EIL}x^4 + \frac{A}{2EI}x^2 + \frac{M_A}{EI}x \quad \text{using } \theta(0) = 0$$

$$v(x) = \frac{-q_0}{120EIL}x^5 + \frac{A}{6EI}x^3 + \frac{M_A}{2EI}x^2 \quad \text{using } v(0) = 0$$

Now we need boundary conditions at $x = L$.

$$v(L) = 0 = \frac{-q_0 L^5}{120EIL} + \frac{AL^3}{6EI} + \frac{M_A L^2}{2EI}$$

$$\theta(L) = 0 = \frac{-q_0 L^4}{24EIL} + \frac{AL^2}{2EI} + \frac{M_A L}{EI}$$

Solve for A & M_A.

$$M_A = \frac{q_0 L^2}{24} - \frac{AL}{2}$$

$$\rightarrow 0 = \frac{-q_0 L^2}{60} + \frac{AL}{3} + \frac{q_0 L^2}{24} - \frac{AL}{2}$$

$$\frac{A}{6} = \frac{q_0 L}{40} \rightarrow A = \frac{3 q_0 L}{20}$$

$$\rightarrow M_A = \frac{-4}{120} q_0 L^2$$

$$\text{Equilibrium} \rightarrow B = \frac{7}{20} q_0 L$$

$$M_B = \frac{-6}{120} q_0 L^2$$

Do the signs make sense? (Discuss)

Check B & M_B.

$$V(L) = -B \rightarrow \frac{-7}{20} q_0 L \stackrel{?}{=} -\frac{q_0}{2L} L^2 + \frac{3 q_0 L}{20} \checkmark = \frac{-7}{20} q_0 L$$

$$M(L) = M_B \rightarrow \frac{-6}{120} q_0 L^2 \stackrel{?}{=} -\frac{q_0 L^2}{6} + \frac{3 q_0 L^2}{20} - \frac{4}{120} q_0 L^2 \checkmark = \frac{-6}{120} q_0 L^2$$

$$v(x) = \frac{-q_0}{120 EIL} x^5 + \frac{q_0 L}{40 EI} x^3 - \frac{4 q_0 L^2}{240 EI} x^2$$

v_{max} occurs where $\theta = 0 \rightarrow \frac{-q_0}{24EIL} x_c^4 + \frac{3 q_0 L}{40 EI} x_c^2 - \frac{4 q_0 L^2}{120 EI} x_c = 0$

$x_c = 0$ is a solution & we should get $x_c = L$ as well.

Factor $x_c = 0$ out $\rightarrow \frac{-1 L^2}{24 L^3} x_c^3 + \frac{3L^2}{40L} x_c - \frac{4L^2}{120} = 0$

Factor $\frac{x_c}{L} = 1$ out

$$\left(\frac{x_c}{L}-1\right)\left[-\frac{1}{24}\left(\frac{x_c}{L}\right)^2-\frac{1}{24}\left(\frac{x_c}{L}\right)+\frac{1}{30}\right]=0$$

$$\frac{x_c}{L} = \frac{\frac{1}{24}\pm\sqrt{\left(\frac{1}{24}\right)^2+\frac{1}{180}}}{-2/24}$$

$$\frac{x_c}{L} = -\frac{1}{2}+\left(\sqrt{1+\frac{576}{180}}\Big/2\right)\approx 0.525$$

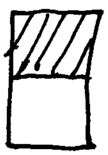
$$\rightarrow\ v_{max} = 0.0013085\,\frac{q_0L^4}{EI}$$

V_{max} occurs at one of the supports since $q \neq 0$ in the range $0<x<L$.

$$B>A \rightarrow \quad V_{max}=\frac{7}{20}q_0L$$ (Note that here it is OK to deal with magnitudes.)

$$\tau_{max} = \frac{V_{max}}{I}\left(\frac{Q}{b}\right)_{max}$$

For a rectangle Q_{max} occurs at $y=0$.



$$Q(y=0)=\int_0^{h/2} y'\,b\,dy' = \frac{b}{2}y'^2\Big]_0^{h/2} = \frac{bh^2}{8}$$

$$\rightarrow\ \tau_{max} = \frac{7}{20}q_0L\,\frac{12}{bh^3}\,\frac{bh^2}{8b} = \frac{21}{40}\,\frac{q_0L}{bh}$$

M_{max} occurs where $V=0$.

$$V = -\frac{q_0}{2L} x_\sigma^2 + \frac{3 q_0 L}{20} = 0$$

$$x_\sigma = \sqrt{\frac{3}{10}}\, L$$

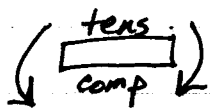
$$M\left(\sqrt{\tfrac{3}{10}}\, L\right) = q_0 L^2 \left[-\frac{1}{6}\left(\sqrt{\tfrac{3}{10}}\right)^3 + \frac{3}{20}\sqrt{\tfrac{3}{10}} - \frac{1}{30} \right]$$

$$= 0.0214\, q_0 L^2$$

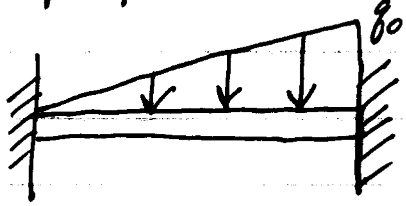
$$M_A = -\frac{1}{30} q_0 L^2 = -0.033\, q_0 L^2$$

$$M_B = -\frac{1}{20} q_0 L^2 = \boxed{-0.05\, q_0 L^2} \quad \text{max magnitude}$$

$$\sigma_{max} = \frac{M_B\, h/2}{b h^3/12} = \frac{3}{10} \frac{q_0 L^2}{b h^3}$$



Superposition



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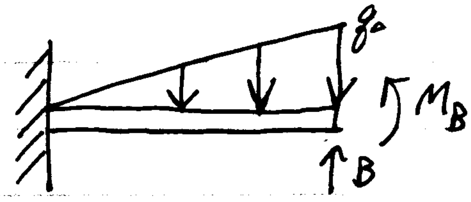
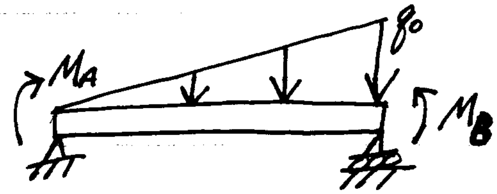


or =



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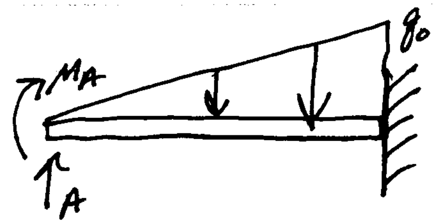
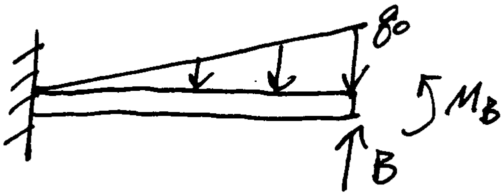
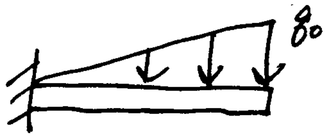
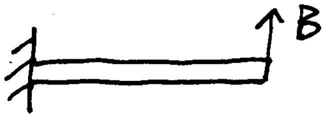
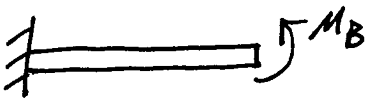


Any of these can be used for a superposition scheme. Let's look at the first.



$=$  $\Rightarrow v_q = \dfrac{-q_0 x^2}{120EIL}(20L^3 - 10L^2 x + x^3)$

$+$  $\Rightarrow v_B = \dfrac{Bx^2}{6EI}(3L - x)$

$+$  $\Rightarrow v_M = \dfrac{M_B x^2}{2EI}$

$$v(x) = v_q + v_B + v_M = \frac{-q_0 x^2}{120EIL}(20L^3 - 10L^2 x + x^3) + \frac{Bx^2}{6EI}(3L - x) + \frac{M_B x^2}{2EI}$$

Boundary conditions:

$$v(L) = 0 = \frac{-q_0 L^{\cancel{4}2}}{120E\cancel{I}}\underbrace{(20-10+1)}_{11} + \frac{BL^{\cancel{3}}}{6E\cancel{I}}\underbrace{(3-1)}_{2} + \frac{M_B \cancel{L^2}}{2\cancel{EI}}$$

$$\theta(L) = v'(L) = 0 = -\frac{q_0 L^{\cancel{3}2}}{120\cancel{EI}}\underbrace{(40-30+5)}_{15} + \frac{BL^{\cancel{2}}}{6\cancel{EI}}\underbrace{(6-3)}_{3} + \frac{M_B \cancel{L}}{\cancel{EI}}$$

$$M_B = \frac{15}{120} q_0 L^2 - \frac{BL}{2}$$

$$0 = -\frac{11}{120} q_0 L^2 + \frac{BL}{3} - \frac{BL}{4} + \frac{15}{240} q_0 L^2$$

$\rightarrow B = \dfrac{7}{20} q_0 L$ ✓ same as previous approach

Use $q = -EIv''''$, $V = EIv'''$, $M = EIv''$ to find max/min.