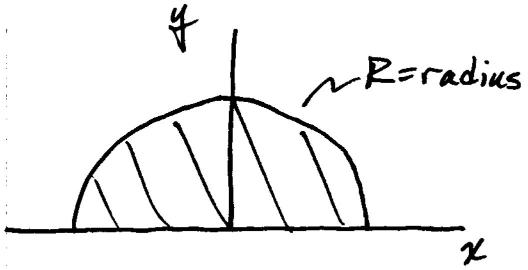
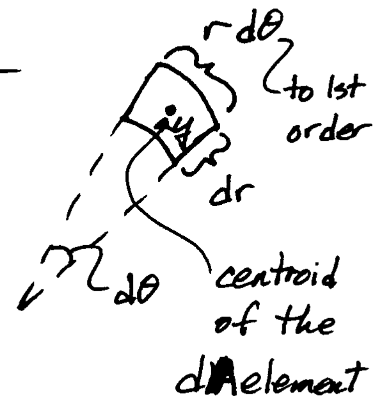
An alternative (and less straight-forward) approach.



$\bar{x} = 0$ due to symmetry

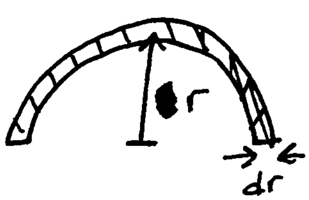
$$\bar{y} = \frac{\int_A y \, dA}{A}$$



if we use $dA = r\,dr\,d\theta$
then y is simply $y = r\sin\theta$

$$\rightarrow \bar{y} = \frac{\int_0^{\pi}\int_0^{R} r^2 \sin\theta \, dr\, d\theta}{\pi R^2/2} = \frac{4}{3\pi}R$$

What if we use the following dA element?



$dA = \pi r\,dr$

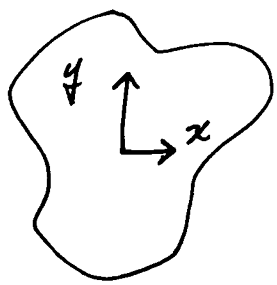
But what is y for this element? y has many values, so we must choose y_c for the differential element itself, $y_c = \frac{2r}{\pi}$

$$\rightarrow \bar{y} = \frac{\int_0^{R} \frac{2r}{\pi}\pi r\,dr}{\pi R^2/2} = \frac{4}{3\pi}R$$

Moments of Inertia

When you study strength of materials and dynamics the concepts of moment of inertia of an area and moment of inertia of a mass will be useful.

Definitions:



$$I_x \text{ or } I_{xx} = \int_A y^2 \, dA$$

$$I_y \text{ or } I_{yy} = \int_A x^2 \, dA$$

$$I_{xy} = I_{yx} = \int_A xy \, dA$$ (also called a product of inertia)

$$I_z \text{ or } I_P = \int_A r^2 \, dA$$ (also called polar moment of inertia)

$$= \int_A x^2 + y^2 \, dA$$

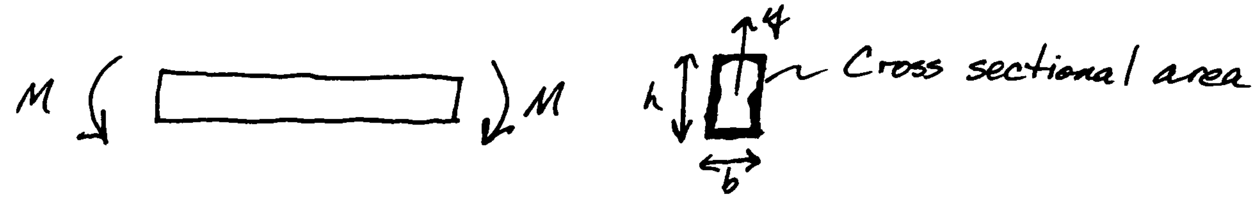
$$= \int_A x^2 \, dA + \int_A y^2 \, dA$$

$$= I_x + I_y$$

* These are moments of inertia of the area A.

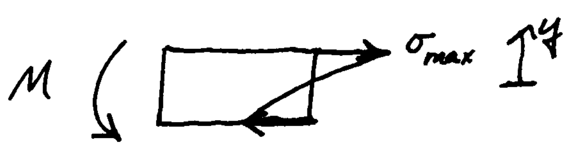
Why are these interesting?

Consider a beam being bent by a moment M.



How is the moment (or the forces due to the moment) distributed over the area?

The top of the beam is in tension and the bottom is in compression. You will learn in strength of materials that in many interesting situation the distribution of stress due to the moment is linear with respect to the y-coordinate.



$$\sigma = \sigma_{max} \frac{2y}{h}$$

$\sigma \equiv$ stress , stress is a force per unit area

Moment equilibrium $\rightarrow$ $M - \int_A \sigma \, \underbrace{y}_{\text{moment arm}} \, dA = 0$

$\sigma dA = dF$

$$\rightarrow \quad M = \int_A \sigma y \, dA$$

$$M = \frac{2\sigma_{max}}{h} \underbrace{\int_A y^2 \, dA}_{I}$$

$$= \frac{2\sigma_{max}}{h} \int_{-\frac{b}{2}}^{\frac{b}{2}} \int_{-\frac{h}{2}}^{\frac{h}{2}} y^2 \underbrace{dy\,dx}$$

* Note that this choice of coordinates implies z is along the axis of the beam

$$= \frac{2\sigma_{max}}{h} \left[b \cdot \frac{1}{3} y^3 \Big]_{-\frac{h}{2}}^{h/2} \right]$$

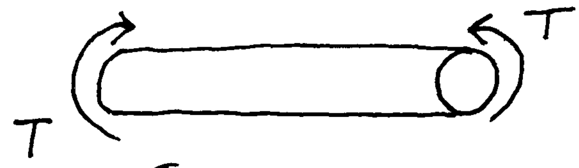
$$= \frac{2\sigma_{max}}{h} \underbrace{\left(\frac{1}{12} b h^3 \right)}_{I \text{ for a rectangular beam}}$$

$$M = \frac{2I}{h} \sigma_{max}$$

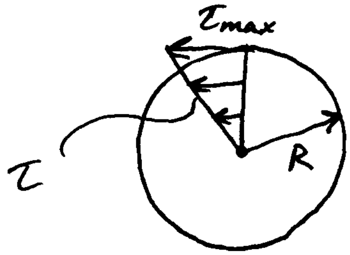
$$\rightarrow \quad \boxed{\sigma = \frac{My}{I}}$$

* We have derived this for a rectangular cross-section, but the result is valid for any arbitrary cross-section.

Similar features occur when we twist a circular shaft.



torque T 



τ = shear stress
Shear stresses have the force acting tangentially to the area they are acting on.

$$\tau = \tau_{max} \frac{r}{R}$$

$$T = \int_A \underbrace{r}_{\text{moment arm}} \underbrace{\tau \, dA}_{dF}$$

$$= \frac{\tau_{max}}{R} \underbrace{\int_A r^2 \, dA}_{I_p}$$

$$= \frac{\tau_{max}}{R} \int_0^{2\pi} \int_0^{R} r^2 \, r \, dr \, d\theta$$

$$= \frac{\tau_{max}}{R} \left(2\pi \left. \frac{1}{4} r^4 \right]_0^R \right)$$

$$= \frac{\tau_{max}}{R} \underbrace{\frac{\pi R^4}{2}}_{\text{I_p for a circular cross-section}}$$

$$\rightarrow \boxed{\tau = \frac{Tr}{I_p}}$$

Again, we derived this for a circular section, but it is generally valid.

So, that is enough motivation for now.

Moments of inertia of areas are tabulated for many geometries, usually about axes that pass through the <u>centroid</u> of the area.

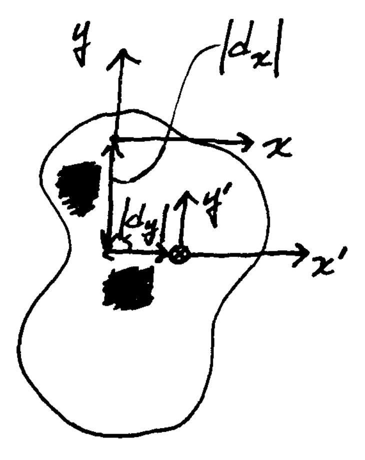
When computing the moment of inertia of a <u>composite</u> area we must add (or subtract) the moments of inertia of the component pieces about a <u>common</u> axis. To do this we must use a theorem called <u>the parallel axis theorem</u>.

<u>Parallel Axis Theorem</u>

$$I_x = \int_A y^2 \, dA$$

$$I_y = \int_A x^2 \, dA$$

$$I_z = \int_A r^2 \, dA$$



⊗ = location of the centroid

$$\bar{I}_x = \int_A (y')^2\, dA$$

$$\bar{I}_y = \int_A (x')^2\, dA$$

$$\bar{I}_z = \int_A (r')^2\, dA$$

***

$$\left.\begin{aligned} A\cdot\overline{y'} &= \int_A y'\, dA = 0 \\ A\cdot\overline{x'} &= \int_A x'\, dA = 0 \end{aligned}\right\}$$ The x', y' system has origin at the centroid.

$$y = y' + \underbrace{d_x}$$

distance between x and x' axes

$$x = x' + \underbrace{d_y}$$

distance between y and y' axes

$$r^2 = x^2 + y^2 = (x' + d_y)^2 + (y' + d_x)^2$$

$$= \underbrace{(x')^2 + (y')^2}_{(r')^2} + \underbrace{d_x^2 + d_y^2}_{d^2} + 2x'd_y + 2y'd_x$$

$$I_x = \int_A y^2\, dA = \int_A (y' + d_x)^2\, dA$$

$$= \int_A (y')^2\, dA + \int_A 2y'd_x\, dA + \int_A d_x^2\, dA$$

But d_x is just a constant distance.

$$\rightarrow I_x = \underbrace{\int_A (y')^2 dA}_{\bar{I}_x} + d_x 2 \underbrace{\int_A y' dA}_{0} + d_x^2 \underbrace{\int_A dA}_{A}$$

$$= \bar{I}_x + A d_x^2$$

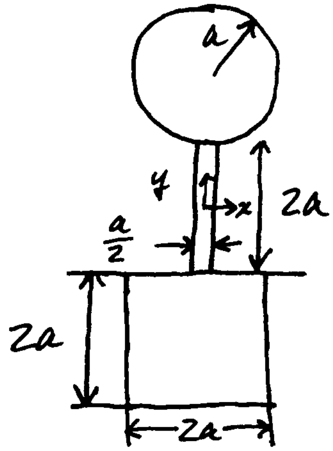
By similar procedures: $I_y = \bar{I}_y + A d_y^2$

$$I_z = \bar{I}_z + A d^2$$

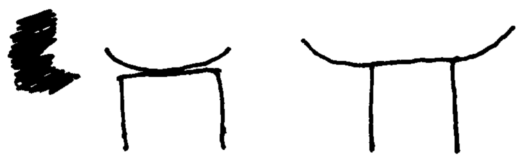
* The barred quantities are <u>always</u> about an axis <u>passing through the centroid</u>.

Note that the moment of inertia about an axis passing through the centroid is always <u>lower</u> than the moment of inertia about another parallel axis passing through some other point.

What are the moments of inertia of this composite area about [redacted] axes passing through its centroid?



Neglect difference due to the connection at the circle.



1) Pick a location for a set of axes. We will take the center of the slender bar.

2) Compute the location of the centroids.

$\bar{x} = 0$ due to symmetry

$\bar{y}_c = 2a \qquad A_c = \pi a^2$

$\bar{y}_R = 0 \qquad A_R = a^2$

$\bar{y}_s = -2a \qquad A_s = 4a^2$

$$\bar{y} = \frac{\bar{y}_c A_c + \bar{y}_R A_R + \bar{y}_s A_s}{A_c + A_R + A_s} = \frac{2a}{(5+\pi)a^2}(\pi - 4)a^2$$

$$\bar{y} = -\frac{8-2\pi}{\pi+5}a$$

3) Look up or compute $\bar{I}_x, \bar{I}_y, \bar{I}_z$ for each component

	Circle	Rectangle	Square
$\bar{I}_x^{shape}$	$\frac{\pi}{4}a^4$	$\frac{1}{3}a^4$	$\frac{4}{3}a^4$
$\bar{I}_y^{shape}$	$\frac{\pi}{4}a^4$	$\frac{1}{48}a^4$	$\frac{4}{3}a^4$
$\bar{I}_z^{shape}$	$\frac{\pi}{2}a^4$	$\frac{17}{48}a^4$	$\frac{8}{3}a^4$

4) Compute I_x, I_y, I_z for each component (or you could go directly to $I_{\bar{x}}$ ...)

	Circle	Rectangle	Square
I_x	$\frac{\pi}{4}a^4 + 4a^2 \cdot \pi a^2 = \frac{17\pi}{4}a^4$	$\frac{1}{3}a^4 + 0 = \frac{1}{3}a^4$	$\frac{4}{3}a^4 + 4a^2 \cdot 4a^2 = \frac{52}{3}a^4$
I_y	$\frac{\pi}{4}a^4$	$\frac{1}{48}a^4$	$\frac{4}{3}a^4$
I_z	$\frac{9\pi}{2}a^4$	$\frac{17}{48}a^4$	$\frac{56}{3}a^4$

Totals: $I_x = \left(\frac{53}{3} + \frac{17\pi}{4}\right)a^4 = 31.02\,a^4$

$$I_y = \left(\frac{65}{48} + \frac{\pi}{4}\right)a^4 = 2.14\,a^4$$

$$I_z = \left(\frac{913}{48} + \frac{9\pi}{2}\right)a^4 = 33.16\,a^4$$

5) Compute $\bar{I}_x, \bar{I}_y, \bar{I}_z$

$$I_x = \bar{I}_x + A\,d_x^2$$

$$31.02\,a^4 = \bar{I}_x + (5+\pi)a^2\left(\frac{8-2\pi}{5+\pi}\right)^2 a^2$$

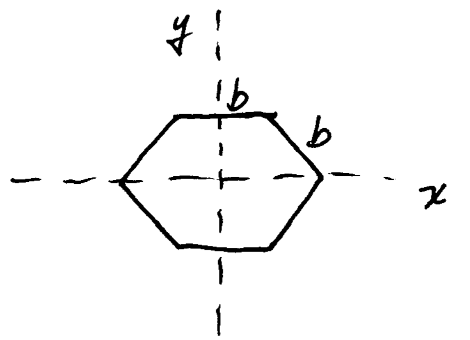
$$= \bar{I}_x + 0.362\,a^4$$

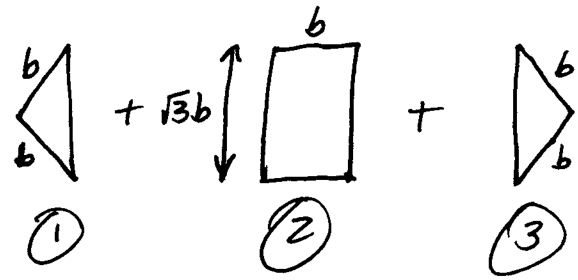
$$\boxed{\bar{I}_x = 30.66\,a^4}$$

$$\bar{I}_y = \bar{I}_y + A\cancel{d_y^2}^{\,0} \Rightarrow \boxed{\bar{I}_y = 2.14\,a^4}$$

$$\boxed{\bar{I}_z = \bar{I}_x + \bar{I}_y = 32.80\,a^4}$$

Moment of Inertia of a Hexagon





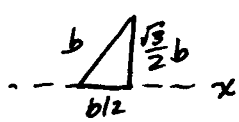
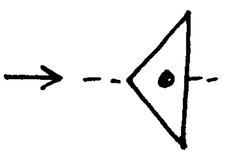


Table $\rightarrow$ $I_x = \dfrac{b/2 \cdot \frac{3\sqrt{3}}{8} b^3}{12} = \dfrac{\sqrt{3}}{64} b^4$



$\rightarrow$ $I_x = \bar{I}_x^{①} = \dfrac{\sqrt{3}}{32} b^4$ (x-axis passes through the centroid)

Table $\rightarrow$ $\bar{I}_y^{①} = \dfrac{\sqrt{3}b \cdot b^3/8}{36} = \dfrac{\sqrt{3}}{288} b^4$

$\bar{I}_x^{③} = \bar{I}_x^{①}$, $\bar{I}_y^{③} = \bar{I}_y^{①}$

Table $\rightarrow$ $\bar{I}_x^{②} = \frac{1}{12} b\, 3\sqrt{3} b^3 = \frac{\sqrt{3}}{4} b^4$, $\bar{I}_y^{②} = \frac{1}{12}\sqrt{3}b\, b^3 = \frac{\sqrt{3}}{12} b^4$

$\bar{y}_{①} = \bar{y}_{②} = \bar{y}_{③} = 0 \rightarrow \bar{I}_x = \bar{I}_x^{①} + \bar{I}_x^{②} + \bar{I}_x^{③}$

$$\rightarrow \bar{I}_x = \left(\frac{\sqrt{3}}{32} + \frac{\sqrt{3}}{4} + \frac{\sqrt{3}}{32}\right)b^4 = \frac{5\sqrt{3}}{16}b^4$$

$\bar{x}_{①} = -\frac{b}{2} - \frac{1}{3}b/2 = -\frac{2}{3}b \quad , \bar{x}_{②} = 0 \, , \quad \bar{x}_{③} = \frac{2}{3}b$

$$\rightarrow \bar{I}_y = \bar{I}_y^{①} + A_{①}\bar{x}_{①}^2 + \bar{I}_y^{②} + \bar{I}_y^{③} + A_{③}\bar{x}_{③}^2$$

$$= \left(2\frac{\sqrt{3}}{288} + 2\frac{1}{2}\frac{\sqrt{3}}{2}\frac{4}{9} + \frac{\sqrt{3}}{12}\right)b^4$$

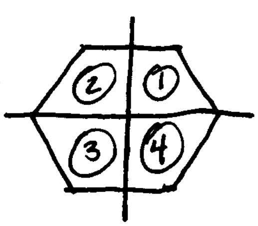
$$= \frac{5\sqrt{3}}{16}b^4$$

$$\rightarrow \boxed{\bar{I}_x = \bar{I}_y = \frac{5\sqrt{3}}{16}b^4 \text{ for a regular hexagon}}$$

What about $\bar{I}_{xy}$?

$$\bar{I}_{xy} = \int_A xy \, dA$$

We can show this equals zero in the following way.

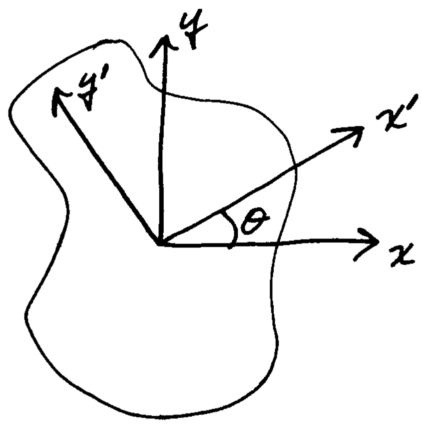


For every (x,y) in ①, there is a $(-x,y)$ in ② $\rightarrow \int_{A_1} xy \, dA = -\int_{A_2} xy \, dA$

Similar arguments for ③ & ④ apply.

$\rightarrow \bar{I}_{xy} = 0$ for a regular hexagon in this orientation.

What about other orientations of the x & y axes?



$$x' = x\cos\theta + y\sin\theta$$

$$y' = -x\sin\theta + y\cos\theta$$

$$\begin{aligned}
I_{x'} &= \int_A (y')^2 \, dA \\
&= \int_A (-x\sin\theta + y\cos\theta)^2 \, dA \\
&= \int_A x^2\sin^2\theta - 2xy\sin\theta\cos\theta + y^2\cos^2\theta \, dA \\
&= \cos^2\theta \int_A y^2 \, dA + \sin^2\theta \int_A x^2 \, dA - 2\sin\theta\cos\theta \int_A xy \, dA
\end{aligned}$$

$$\boxed{\begin{aligned}
&I_{x'} = I_x\cos^2\theta + I_y\sin^2\theta - 2I_{xy}\sin\theta\cos\theta \\
&\text{or} \quad I_{x'} = \frac{I_x + I_y}{2} + \frac{I_x - I_y}{2}\cos 2\theta - I_{xy}\sin 2\theta
\end{aligned}}$$

Similar procedure gives:

$$\boxed{\begin{aligned}
I_{y'} &= I_y\cos^2\theta + I_x\sin^2\theta + 2I_{xy}\sin\theta\cos\theta \\
&= \frac{I_x + I_y}{2} + \frac{I_y - I_x}{2}\cos 2\theta + I_{xy}\sin 2\theta
\end{aligned}}$$

$$I_{x'y'} = \int_A x'y' \, dA$$

$$= \int_A (x\cos\theta + y\sin\theta)(-x\sin\theta + y\cos\theta) \, dA$$

$$= \int_A -x^2\sin\theta\cos\theta + y^2\sin\theta\cos\theta + xy\cos^2\theta - xy\sin^2\theta \, dA$$

$$\boxed{\begin{aligned} I_{x'y'} &= (I_x - I_y)\sin\theta\cos\theta + I_{xy}(\cos^2\theta - \sin^2\theta) \\ &= \frac{I_x - I_y}{2}\sin 2\theta + I_{xy}\cos 2\theta \end{aligned}}$$

Note: $I_{x'} + I_{y'} = I_x + I_y \rightarrow I_z = I_{z'}$

Also: The angle θ that maximizes/minimizes $I_{x'}$ and $I_{y'}$ is the same angle where $I_{x'y'} = 0$. The x'-y' axes for this special angle are called the principal axes, or principal directions.

This critical or "principal" angle is given by

$$\tan 2\theta_p = \frac{2I_{xy}}{I_y - I_x}$$

then

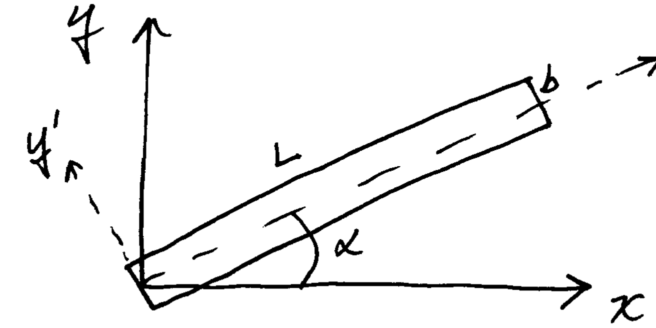
$$I_{max} = \frac{I_x + I_y}{2} + \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

$$I_{min} = \frac{I_x + I_y}{2} - \frac{1}{2}\sqrt{(I_x - I_y)^2 + 4I_{xy}^2}$$

For the regular hexagon $I_x = I_y = \frac{5\sqrt{3}}{16} b^4$ and $I_{xy} = 0$.

This is a special shape, such that $I_{x'} = I_{y'} = \frac{5\sqrt{3}}{16} b^4$ and $I_{x'y'} = 0$ for any rotation of the axes.

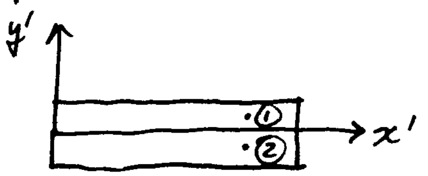
This is not always the case.



Determine I_x, I_y, I_{xy}.

$$\bar{I}_{x'} = I_{x'} = \frac{1}{12} L b^3$$

$$\bar{I}_{y'} = \frac{1}{12} b L^3 \quad , \quad I_{y'} = \frac{1}{12} b L^3 + bL\left(\frac{L}{2}\right)^2 = \frac{1}{3} b L^3$$



For every (x', y') there is a $(x', -y') \rightarrow I_{x'y'} = 0$.

$$I_x = \frac{I_{x'} + I_{y'}}{2} + \frac{I_{x'} - I_{y'}}{2} \cos 2\theta - I_{x'y'} \sin 2\theta$$

$$I_y = \frac{I_{x'} + I_{y'}}{2} - \frac{I_{x'} - I_{y'}}{2} \cos 2\theta + I_{x'y'} \sin 2\theta$$

$$I_{xy} = \frac{I_{x'} - I_{y'}}{2} \sin 2\theta + I_{x'y'} \cos 2\theta$$

** $\theta = -\alpha$ in this case

$$\rightarrow I_x = \frac{Lb^3}{24} + \frac{bL^3}{6} + \frac{Lb^3 - 4bL^3}{24}\cos 2\alpha$$

$$I_y = \frac{Lb^3}{24} + \frac{bL^3}{6} - \frac{Lb^3 - 4bL^3}{24}\cos 2\alpha$$

$$I_{xy} = -\frac{Lb^3 - 4bL^3}{24}\sin 2\alpha$$

Limiting case where $b \ll L$

$$\rightarrow I_x = \frac{bL^3}{6}(1 - \cos 2\alpha) = \frac{bL^3}{3}\sin^2\alpha$$

$$I_y = \frac{bL^3}{6}(1 + \cos 2\alpha) = \frac{bL^3}{3}\cos^2\alpha$$

$$I_{xy} = \frac{bL^3}{6}\sin 2\alpha = \frac{bL^3}{3}\sin\alpha\cos\alpha$$

By integration with $b \ll L$

$$I_x = \int_0^L \underbrace{(x'\sin\alpha)^2}_{y^2}\,\underbrace{b\,dx'}_{dA} = \frac{bL^3}{3}\sin^2\alpha$$

$$I_y = \int_0^L (x'\cos\alpha)^2\, b\,dx' = \frac{bL^3}{3}\cos^2\alpha$$

$$I_{xy} = \int_0^L (x'\sin\alpha)(x'\cos\alpha)\, b\,dx' = \frac{bL^3}{3}\sin\alpha\cos\alpha$$