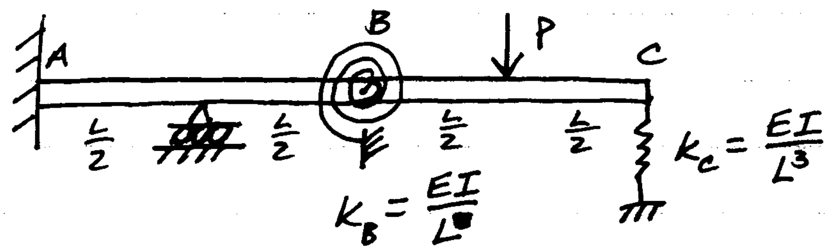
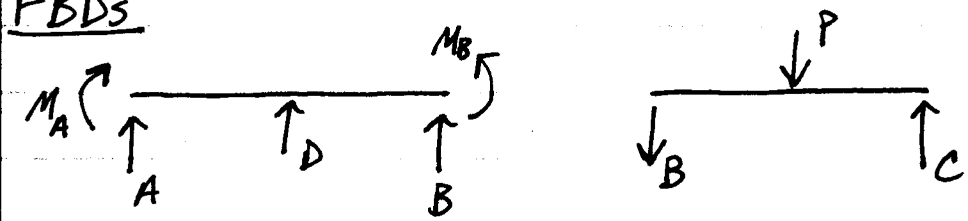


The torsional spring is attached to beam AB.
The beams are joined by a frictionless hinge at B.

<u>FBDs</u>



M_B = moment that spring places on beam AB
B = internal force between beams AB and BC
C = force of spring on beam BC.

Start with BC since it is statically determinate.

$$\Sigma M_z^B = CL - P\frac{L}{2} = 0 \rightarrow C = \frac{P}{2}$$

$$\Sigma F_y \rightarrow B = -\frac{P}{2}$$ (drawn in wrong direction)

<u>AB</u>

$$\Sigma F_y = A + D + B = 0 \rightarrow A + D = \frac{P}{2}$$

$$\Sigma M_z^D = -A\frac{L}{2} + B\frac{L}{2} + M_B - M_A = 0$$

Too many unknowns $\rightarrow$ we need deflections.

For AB we have

$$q(x) = -D\langle x - \tfrac{L}{2}\rangle^{-1}$$

$$V(x) = D\langle x - \tfrac{L}{2}\rangle^{0} + A$$

$$M(x) = D\langle x - \tfrac{L}{2}\rangle^{1} + Ax + M_A$$

$$\theta(x) = \frac{D}{2EI}\langle x-\tfrac{L}{2}\rangle^2 + \frac{A}{2EI}x^2 + \frac{M_A}{EI}x + 0 \quad (\theta(0)=0)$$

$$v(x) = \frac{D}{6EI}\langle x-\tfrac{L}{2}\rangle^3 + \frac{A}{6EI}x^3 + \frac{M_A}{2EI}x^2 \quad (v(0)=0)$$

Now, we have a torsional spring attached to AB at $x=L$. As M_B is drawn on the previous page, the spring will produce this direction of moment if the angle θ is like  , but this is a <u>negative</u> slope

So, $\quad M_B = -k_B\,\theta_B = -\frac{EI}{L}\,\theta(x=L)$

$$\rightarrow M_B = -\left(\frac{DL}{8} + \frac{AL}{2} + M_A\right) \quad (a)$$

We also know $v(\tfrac{L}{2})=0$

$$v(\tfrac{L}{2}) = \frac{AL^3}{48EI} + \frac{M_A L^2}{8EI} = 0 \rightarrow M_A = \frac{-AL}{6} \quad (b)$$

$$\Sigma F_y = D+A+B=0 \rightarrow D = \frac{P}{2} - A$$

$$\Sigma M_z^D \rightarrow M_B = M_A + \frac{AL}{2} + \frac{PL}{4} \overset{(b)}{=} \frac{AL}{3} + \frac{PL}{4}$$

$$(a) \rightarrow \underbrace{\frac{AL}{3} + \frac{PL}{4}}_{M_B} = \underbrace{-\frac{PL}{16} + \frac{AL}{8}}_{-DL/8} - \frac{AL}{2} + \underbrace{\frac{AL}{6}}_{-M_A}$$

$$\frac{13}{24}AL = \frac{-5}{16}PL \rightarrow A = \frac{-15}{26}P$$

$$\rightarrow M_A = \frac{5}{52} PL$$

$$D = \frac{14}{13} P$$

$$M_B = +\frac{3}{52} PL$$

Beam BC: $q(x) = P\langle x - \frac{3L}{2}\rangle^{-1}$

$$V(x) = -P\langle x - \frac{3L}{2}\rangle^{0} + \underbrace{\frac{P}{2}}_{-B}$$

$$M(x) = -P\langle x - \frac{3L}{2}\rangle + \frac{P}{2}x - \frac{PL}{2} \quad (M(L)=0)$$

$$\theta(x) = -\frac{P}{2EI}\langle x - \frac{3L}{2}\rangle^2 + \frac{P}{4EI}x^2 - \frac{PL}{2}x + c_1$$

$$v(x) = -\frac{P}{6EI}\langle x - \frac{3L}{2}\rangle^3 + \frac{P}{12EI}x^3 - \frac{PL}{4}x^2 + c_1 x + c_2$$

How do we find c_1 and c_2?

$v(x=L) = v_B$ = deflection at $x=L$ for beam AB

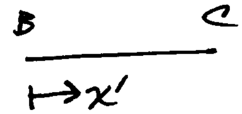
$$\frac{PL^3}{12\,EI} - \frac{PL^3}{4EI} + c_1 L + c_2 = \frac{14}{13}\frac{PL^3}{48EI} - \frac{15}{26}\frac{PL^3}{6EI} + \frac{5}{52}\frac{PL^3}{2EI}$$

$$c_1 L + c_2 = \frac{PL^3}{EI}\,\frac{11}{78}$$

$$v(x=2L) = -\frac{C}{k_c} = -\frac{PL^3}{2EI} = -\frac{PL^3}{48EI} + \frac{8PL^3}{12EI} - \frac{PL^3}{EI} + 2c_1 L + c_2$$

$$2c_1 L + c_2 = -\frac{7}{48}\frac{PL^3}{EI}$$

$$\rightarrow c_1 = -\frac{179}{624}\frac{PL^2}{EI}, \quad c_2 = \frac{89}{208}\frac{PL^3}{EI}$$

Using a shifted axis for beam BC 

$q(x') = P\langle x' - \frac{L}{2}\rangle^{-1}$

$V(x') = -P\langle x' - \frac{L}{2}\rangle^{0} + \frac{P}{2}$

$M(x') = -P\langle x' - \frac{L}{2}\rangle^{1} + \frac{P}{2}x'$

$\theta(x') = -\frac{P}{2EI}\langle x' - \frac{L}{2}\rangle^{2} + \frac{P}{4EI}x'^2 + c_1'$

$v(x') = \frac{-P}{6EI}\langle x' - \frac{L}{2}\rangle^{3} + \frac{P}{12EI}x^3 + c_1' x + c_2'$

$v(x'=0) = v_{AB}(x=L)$

$\rightarrow c_2' = \frac{14}{13}\frac{PL^3}{48EI} - \frac{15}{26}\frac{PL^3}{6EI} + \frac{5}{52}\frac{PL^3}{2EI} = -\frac{1}{39}\frac{PL^3}{EI}$

$v(x'=L) = -\frac{C}{k_c}$

$\hookrightarrow \frac{-P}{6EI}\frac{L^3}{8} + \frac{PL^3}{12EI} + c_1' L - \frac{1}{39}\frac{PL^3}{EI} = \frac{-PL^3}{2EI}$

$\rightarrow c_1' = -\frac{335}{624}\frac{PL^2}{EI}$

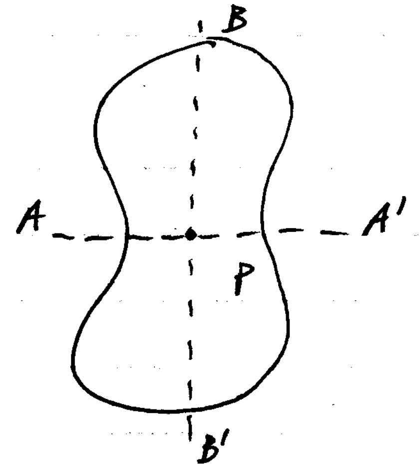
Check: $\frac{P}{12EI}x'^3 - \frac{335}{624}\frac{PL^2}{EI}x' - \frac{1}{39}\frac{PL^3}{EI} \overset{?}{=} \frac{P}{12EI}x^3 - \frac{PL}{4EI}x^2 - \frac{179}{624}\frac{PL^2}{EI}x + \frac{89}{208}\frac{PL^3}{EI}$

$x' = x - L$

$\frac{1}{12}(x-1)^3 - \frac{335}{624}(x-1) - \frac{1}{39} = \frac{x^3}{12} - \frac{x^2}{4} - \frac{179}{624}x + \frac{89}{208}$ ✓

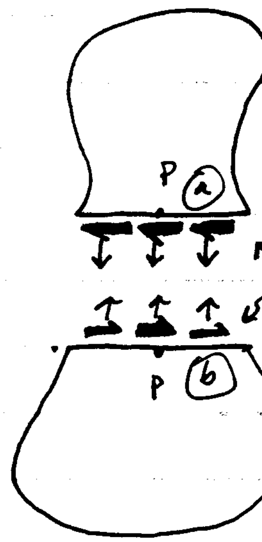
Stress at a Point (Plane stress)

Consider a point in an object.



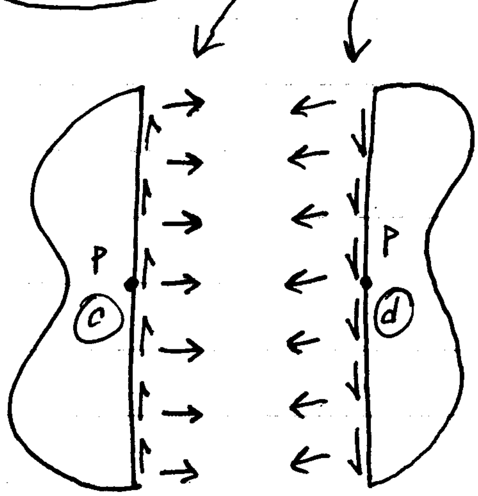
AA' and BB' are orthogonal planes through P where we are going to cut.

AA'

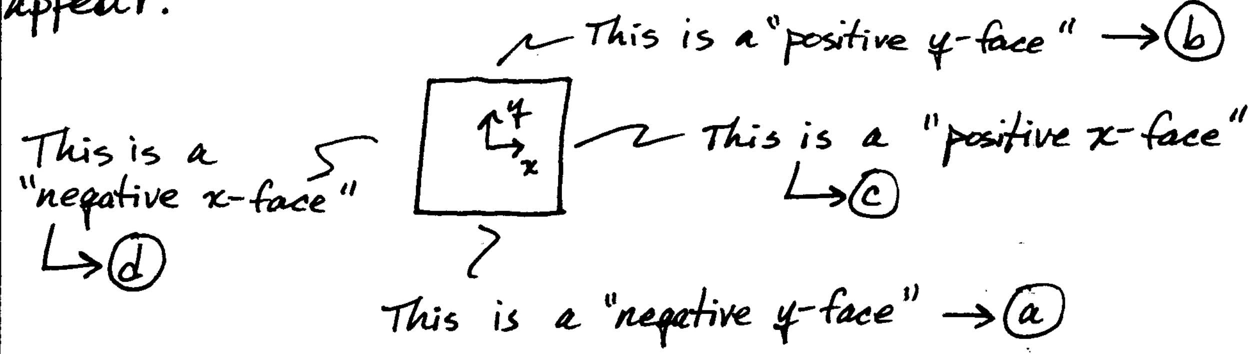


Equal and opposite distributed loads

BB'

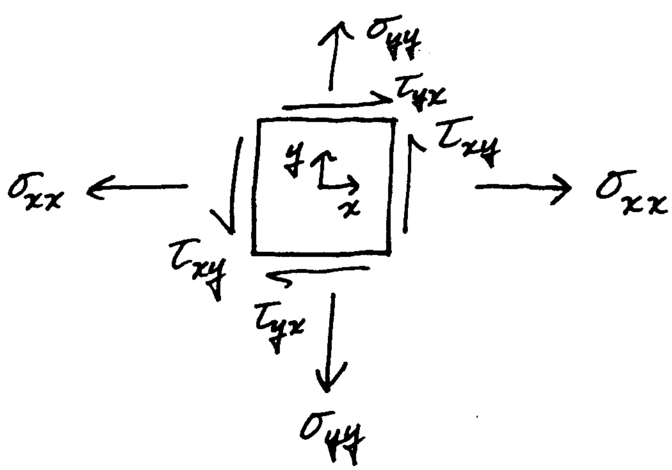


Stress is more complicated than a vector. It is not possible to simply represent it with a directed arrow. So what can we do? The idea is to draw a little square (cube in 3D) around P. This square is <u>not</u> a differential element, but rather is used to represent the surfaces where equal but opposite forces appear.



Positive/negative faces are denoted as such due to the direction that an outward pointing normal vector would be oriented toward.

Now we need to represent the forces that are present on each of these "face orientations"



Here I have used the following conventions.

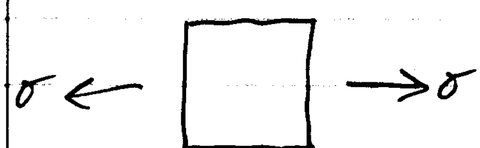
τ_{xy}

y — Direction of the distributed force

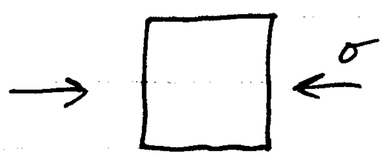
x — face type (Here the face normal is in the x-direction)

Signs — If the distributed force is in the same direction as the face normal then the stress component is positive.

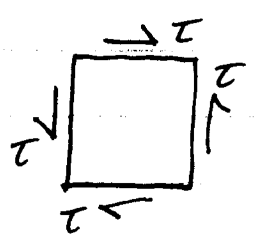
If the distributed force is in the opposite direction to the face normal then the stress is negative.



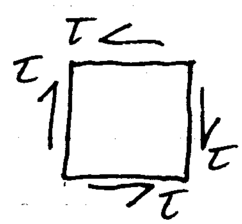
$\sigma_{xx} = \sigma$



$\sigma_{xx} = -\sigma$



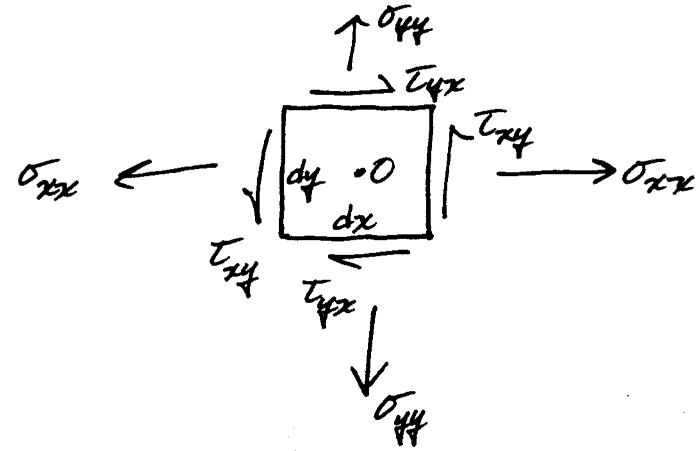
$\tau_{xy} = \tau$
$\tau_{yx} = \tau$



$\tau_{yx} = -\tau$
$\tau_{xy} = -\tau$

Newton's 3rd gives us that the distributed loads must be in opposite directions on opposing faces, but what about τ_{xy} and τ_{yx}?

To answer this question rigorously we have to look at a differential element and expand the stresses into Taylor series. However, we can get away with looking at a uniform stress state on our element.



$$\Sigma M_z^O = (\tau_{xy}\,dy\,dz)\frac{dx}{2} + (\tau_{xy}\,dy\,dz)\frac{dx}{2}$$

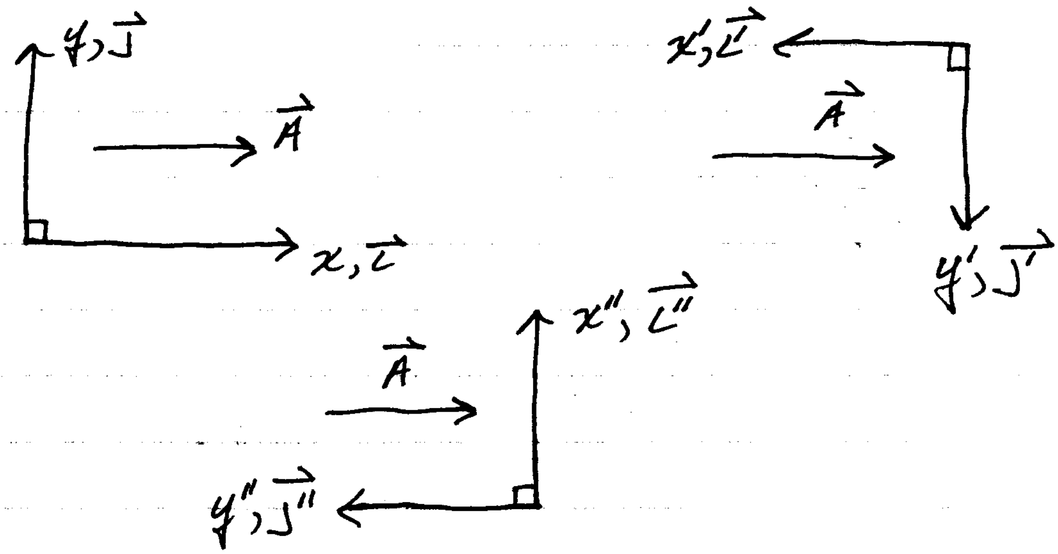
$$- \underbrace{(\tau_{yx}\,dx\,dz)}_{\text{force}}\underbrace{\frac{dy}{2}}_{\text{moment arm}} - (\tau_{yx}\,dx\,dz)\frac{dy}{2} = 0$$

$$\rightarrow \quad \boxed{\tau_{xy} = \tau_{yx}}$$

If we did expand our stresses into Taylor series, then any modifications to this equation are of higher order in dx and dy.

Next we want to discuss how components of stress "change" when we rotate the orientation of our coordinate system.

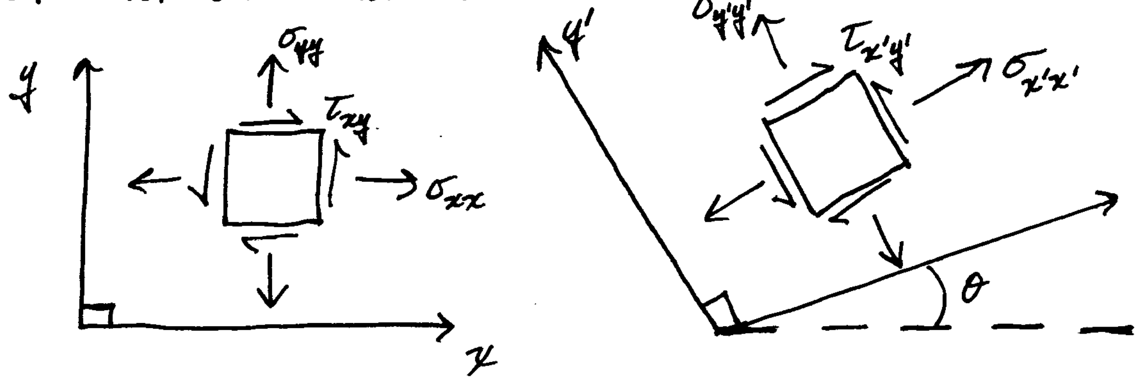
The idea here is that the stress at a point is what it is, but the description of its components depends upon the coordinate system used. This is something that you are already comfortable with for vectors.



$$\vec{A} = \underbrace{A}_{A_x}\vec{i} = \underbrace{-A}_{A_{x'}}\vec{i}' = \underbrace{-A}_{A_{y''}}\vec{j}''$$

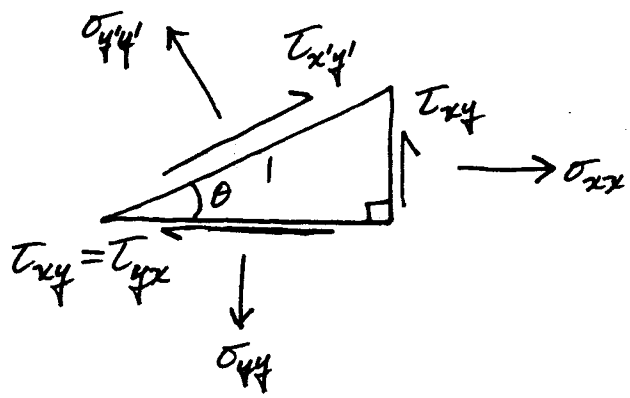
$\vec{A}$ is the same vector in all cases, but the components of $\vec{A}$ depend on the coordinate system.

To analyze the transformation of stress components under a rotation of coordinate system we can consider a homogeneous stress state in a small element of material.



Given σ_{xx}, σ_{yy}, τ_{xy}, what are $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\tau_{x'y'}$?

Let's cut our element to the left at an angle θ.



Let's arbitrarily assign the length of the hypotenuse to be 1, then the length of the base is $\cos\theta$ and the height is $\sin\theta$.

This piece that we have cut out must be in equilibrium.

$$\Sigma F_{x'} = \underbrace{\sigma_{xx}\cos\theta}_{\text{force comp.}}\underbrace{\sin\theta}_{\text{area}} + \underbrace{\tau_{xy}\sin\theta}_{\text{force comp.}}\underbrace{\sin\theta}_{\text{area}}$$

$$-\underbrace{\sigma_{yy}\sin\theta}_{\text{force comp.}}\underbrace{\cos\theta}_{\text{area}} - \underbrace{\tau_{xy}\cos\theta}_{\text{force comp.}}\underbrace{\cos\theta}_{\text{area}}$$

$$+ \underbrace{\tau_{x'y'}}_{\text{force}}\cdot\underbrace{1}_{\text{area}} = 0$$

$$\longrightarrow \boxed{\begin{aligned} \tau_{x'y'} &= -(\sigma_{xx}-\sigma_{yy})\sin\theta\cos\theta + \tau_{xy}(\cos^2\theta-\sin^2\theta) \\ \tau_{x'y'} &= -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta \end{aligned}}$$

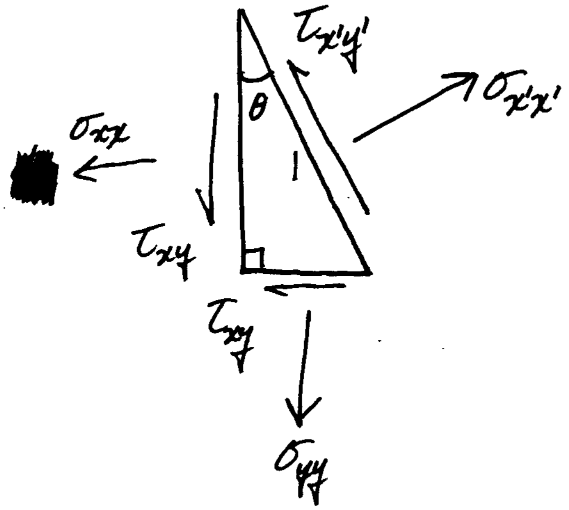
$$\Sigma F_{y'} = -\sigma_{xx}\sin\theta\sin\theta + \tau_{xy}\cos\theta\sin\theta$$

$$-\sigma_{yy}\cos\theta\cos\theta + \tau_{xy}\sin\theta\cos\theta + \sigma_{y'y'}\cdot 1 = 0$$

$$\boxed{\begin{aligned} \sigma_{y'y'} &= \sigma_{yy}\cos^2\theta + \sigma_{xx}\sin^2\theta - 2\tau_{xy}\sin\theta\cos\theta \\ \sigma_{y'y'} &= \frac{\sigma_{yy}+\sigma_{xx}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta \end{aligned}}$$

What about $\sigma_{x'x'}$?

We have to do another cut.



$$\Sigma F_{x'} = \sigma_{x'x'} \cdot 1 - \sigma_{xx}\cos\theta\cos\theta - \tau_{xy}\sin\theta\cos\theta$$

$$- \sigma_{yy}\sin\theta\sin\theta - \tau_{xy}\cos\theta\sin\theta = 0$$

$$\boxed{\begin{aligned} \sigma_{x'x'} &= \sigma_{xx}\cos^2\theta + \sigma_{yy}\sin^2\theta + 2\tau_{xy}\sin\theta\cos\theta \\ \sigma_{x'x'} &= \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta \end{aligned}}$$

$$\Sigma F_{y'} = \tau_{x'y'} \cdot 1 + \sigma_{xx}\sin\theta\cos\theta - \tau_{xy}\cos\theta\cos\theta$$

$$- \sigma_{yy}\cos\theta\sin\theta + \tau_{xy}\sin\theta\sin\theta = 0$$

$$\tau_{x'y'} = \tau_{xy}(\cos^2\theta - \sin^2\theta) - (\sigma_{xx} - \sigma_{yy})\sin\theta\cos\theta \checkmark$$

*Note that a series expansion of the stresses would only alter these equations by higher order terms in the lengths of the sides of the element.