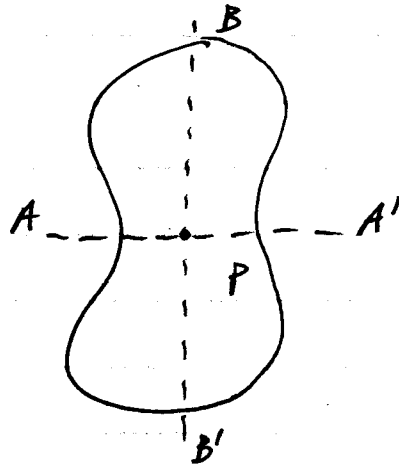


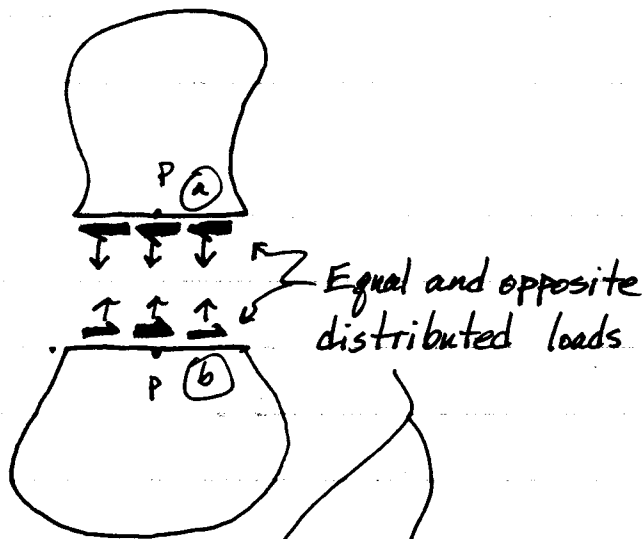
Stress at a Point (Plane stress)

Consider a point in an object.

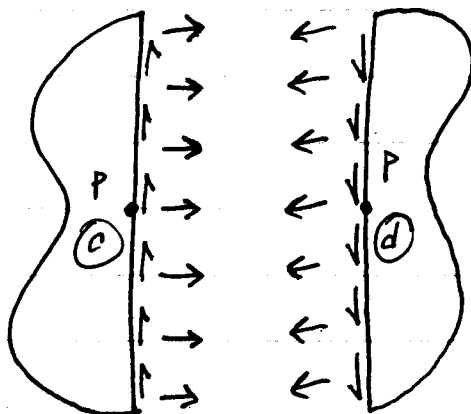


AA' and BB' are orthogonal planes through P where we are going to cut.

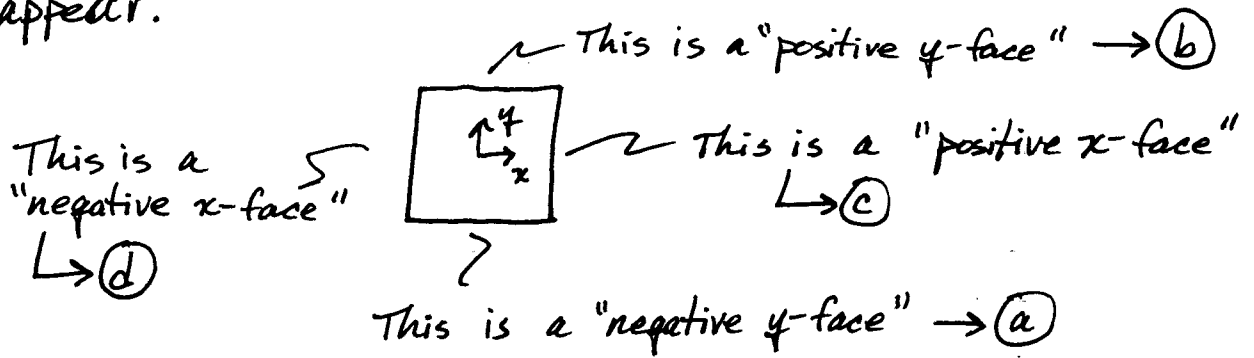
AA'



BB'

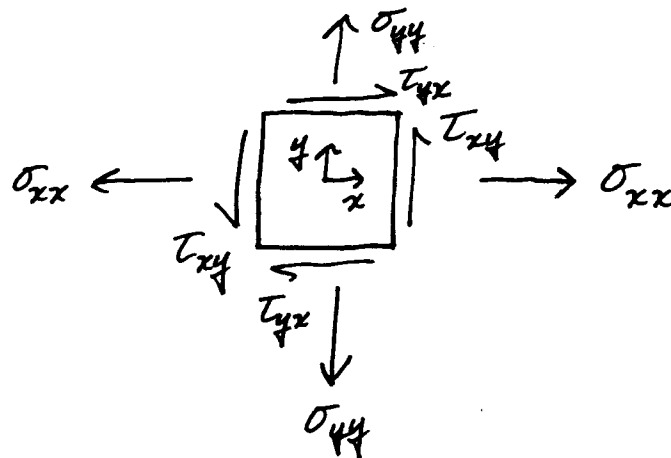


Stress is more complicated than a vector. It is not possible to simply represent it with a directed arrow. So what can we do? The idea is to draw a little square (cube in 3D) around P. This square is not a differential element, but rather is used to represent the surfaces where equal but opposite forces appear.

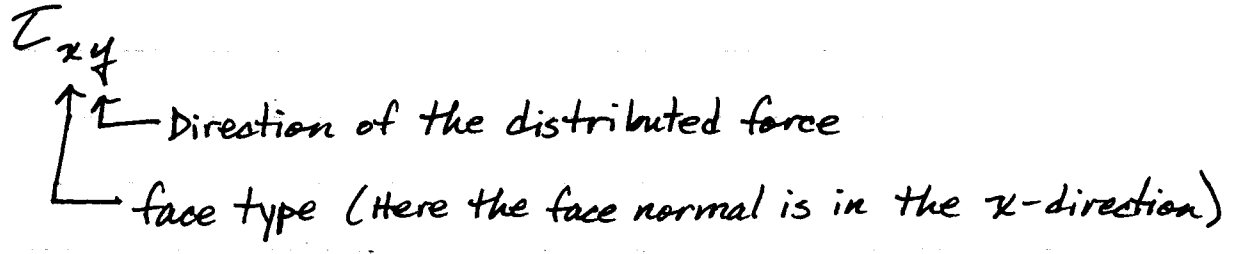


Positive/negative faces are denoted as such due to the direction that an outward pointing normal vector would be oriented toward.

Now we need to represent the forces that are present on each of these "face orientations"

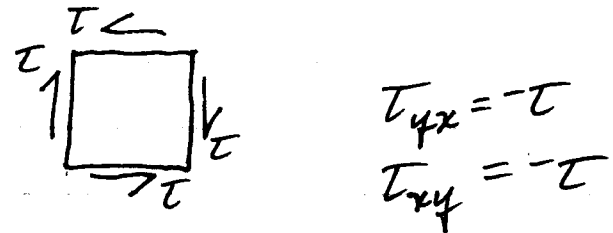
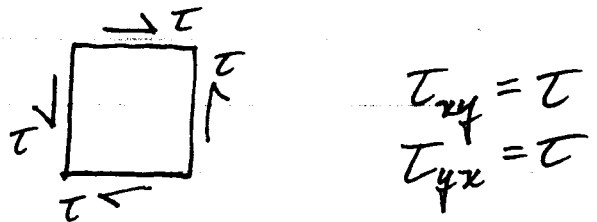
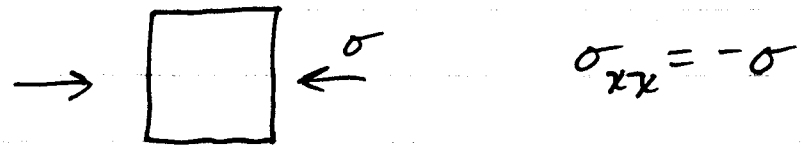
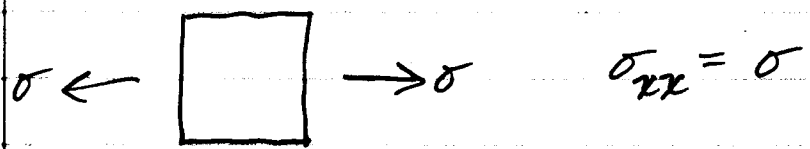


Here I have used the following conventions.



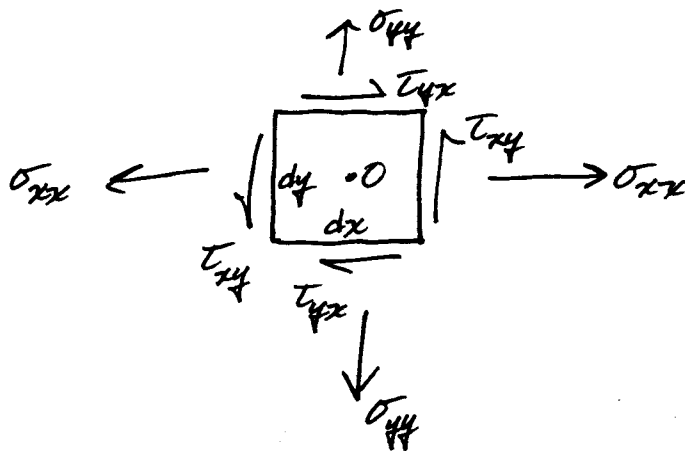
Signs - If the distributed force is in the same direction as the face normal then the stress component is positive.

If the distributed force is in the opposite direction to the face normal then the stress is negative.



Newton's 3rd gives us that the distributed loads must be in opposite directions on opposing faces, but what about τ_{xy} and τ_{yx} ?

To answer this question rigorously we have to look at a differential element and expand the stresses into Taylor series. However, we can get away with looking at a uniform stress state on our element.



$$\sum M_z^o = (\tau_{xy} dy dz) \frac{dx}{2} + (\tau_{yx} dx dz) \frac{dy}{2}$$

$$- (\tau_{yx} dx dz) \frac{dy}{2} - (\tau_{xy} dy dz) \frac{dx}{2} = 0$$

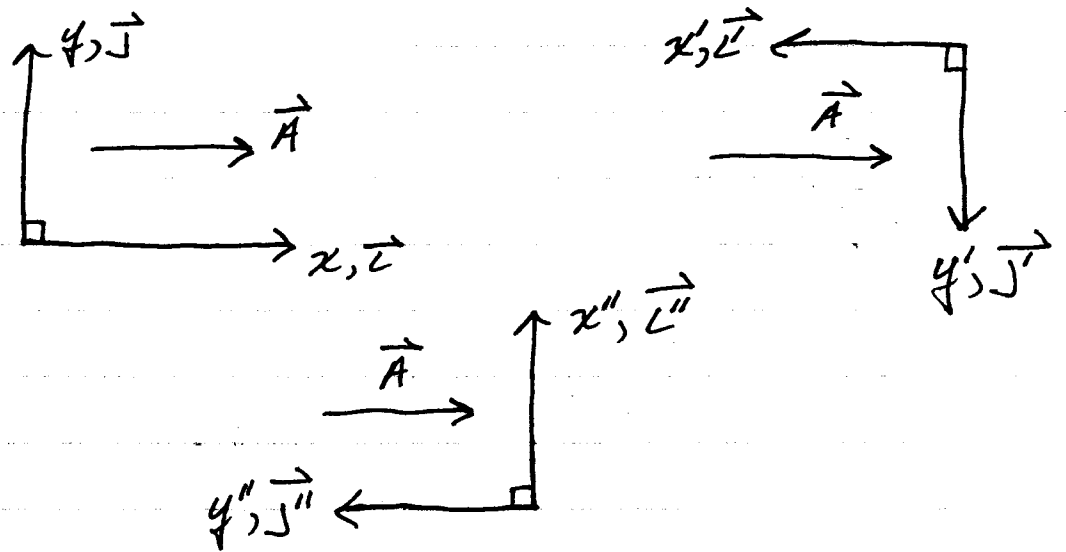
force
moment arm

$$\rightarrow \boxed{\tau_{xy} = \tau_{yx}}$$

If we did expand our stresses into Taylor series, then any modifications to this equation are of higher order in dx and dy .

Next we want to discuss how components of stress "change" when we rotate the orientation of our coordinate system.

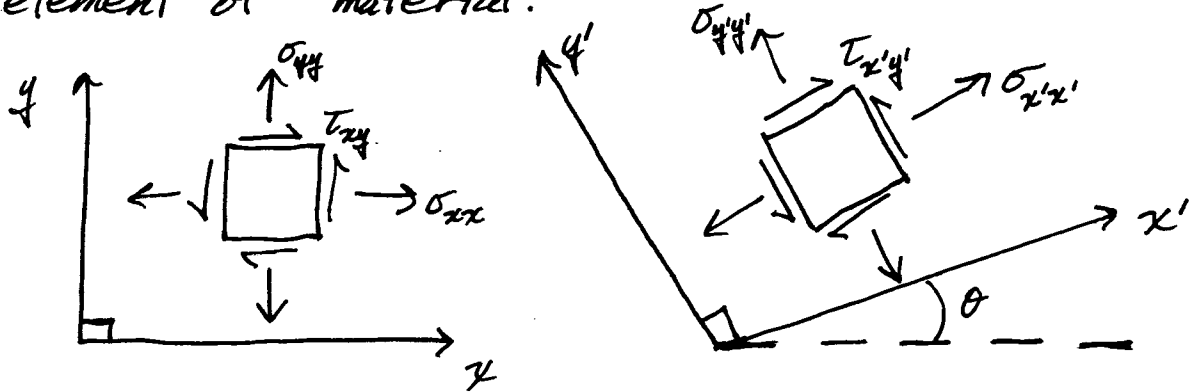
The idea here is that the stress at a point is what it is, but the description of its components depends upon the coordinate system used. This is something that you are already comfortable with for vectors.



$$\vec{A} = \underbrace{A}_{A_x} \vec{l} = -\underbrace{A}_{A_{x'}} \vec{l}' = -\underbrace{A}_{A_{y''}} \vec{j}''$$

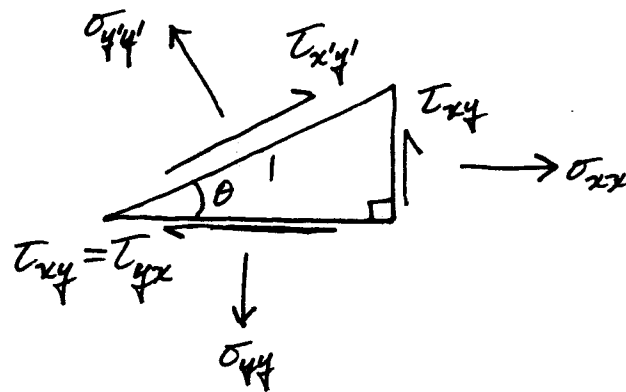
\vec{A} is the same vector in all cases, but the components of \vec{A} depend on the coordinate system.

To analyze the transformation of stress components under a rotation of coordinate system we can consider a homogeneous stress state in a small element of material.



Given σ_{xx} , σ_{yy} , τ_{xy} , what are $\sigma_{x'x'}$, $\sigma_{y'y'}$, $\tau_{x'y'}$?

Let's cut our element to the left at an angle θ .



Let's arbitrarily assign the length of the hypotenuse to be 1, then the length of the base is $\cos\theta$ and the height is $\sin\theta$.

This piece that we have cut out must be in equilibrium.

$$\begin{aligned} \Sigma F_{x'} = & \underbrace{\sigma_{xx} \cos \theta \sin \theta}_{\text{force comp. area}} + \underbrace{\tau_{xy} \sin \theta \sin \theta}_{\text{force comp. area}} \\ & - \underbrace{\sigma_{yy} \sin \theta \cos \theta}_{\text{force comp. area}} - \underbrace{\tau_{xy} \cos \theta \cos \theta}_{\text{force comp. area}} \\ & + \underbrace{\tau_{x'y'}}_{\text{force}} \cdot \underbrace{1}_{\text{area}} = 0 \end{aligned}$$

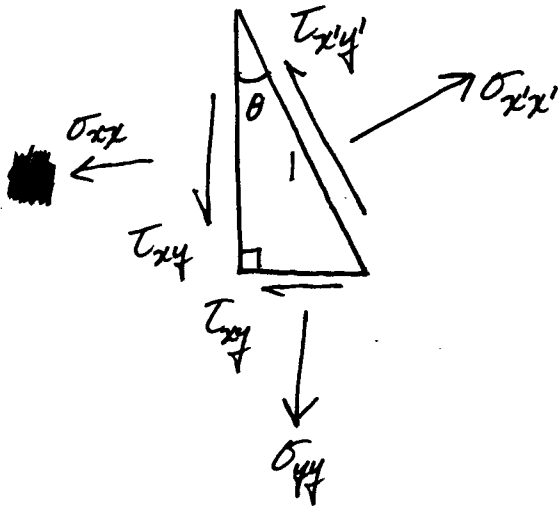
$$\begin{aligned} \rightarrow \tau_{x'y'} &= -(\sigma_{xx} - \sigma_{yy}) \sin \theta \cos \theta + \tau_{xy} (\cos^2 \theta - \sin^2 \theta) \\ \tau_{x'y'} &= -\frac{\sigma_{xx} - \sigma_{yy}}{2} \sin 2\theta + \tau_{xy} \cos 2\theta \end{aligned}$$

$$\begin{aligned} \Sigma F_{y'} = & -\sigma_{xx} \sin \theta \sin \theta + \tau_{xy} \cos \theta \sin \theta \\ & - \sigma_{yy} \cos \theta \cos \theta + \tau_{xy} \sin \theta \cos \theta + \underbrace{\sigma_{y'y'}}_{\text{force}} \cdot \underbrace{1}_{\text{area}} = 0 \end{aligned}$$

$$\begin{aligned} \sigma_{y'y'} &= \sigma_{yy} \cos^2 \theta + \sigma_{xx} \sin^2 \theta - 2\tau_{xy} \sin \theta \cos \theta \\ \sigma_{y'y'} &= \frac{\sigma_{yy} + \sigma_{xx}}{2} - \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta - \tau_{xy} \sin 2\theta \end{aligned}$$

What about $\sigma_{x'x'}$?

We have to do another cut.



$$\sum F_{x'} = \sigma_{x'x'} \cdot 1 - \sigma_{xx} \cos\theta \cos\theta - \tau_{xy} \sin\theta \cos\theta - \sigma_{yy} \sin\theta \sin\theta - \tau_{xy} \cos\theta \sin\theta = 0$$

$$\sigma_{x'x'} = \sigma_{xx} \cos^2\theta + \sigma_{yy} \sin^2\theta + 2\tau_{xy} \sin\theta \cos\theta$$

$$\sigma_{x'x'} = \frac{\sigma_{xx} + \sigma_{yy}}{2} + \frac{\sigma_{xx} - \sigma_{yy}}{2} \cos 2\theta + \tau_{xy} \sin 2\theta$$

$$\sum F_{y'} = \tau_{x'y'} \cdot 1 + \sigma_{xx} \sin\theta \cos\theta - \tau_{xy} \cos\theta \cos\theta - \sigma_{yy} \cos\theta \sin\theta + \tau_{xy} \sin\theta \sin\theta = 0$$

$$\tau_{x'y'} = \tau_{xy} (\cos^2\theta - \sin^2\theta) - (\sigma_{xx} - \sigma_{yy}) \sin\theta \cos\theta \checkmark$$

* Note that a series expansion of the stresses would only alter these equations by higher order terms in the lengths of the sides of the element.