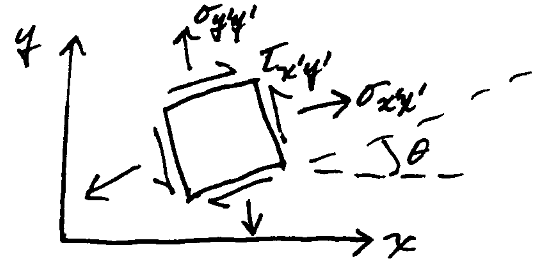
## Maximum Axial and Shear Stresses

Recall




$$\sigma_{x'x'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\sigma_{y'y'} = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta - \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

At what angle $\theta$ are the axial stresses maximized or minimized?

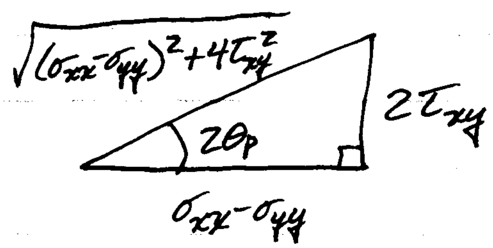
$$\frac{\partial \sigma_{x'x'}}{\partial \theta} = \underbrace{-(\sigma_{xx}-\sigma_{yy})\sin 2\theta + 2\tau_{xy}\cos 2\theta}_{\text{Notice that this is } 2\tau_{x'y'}.} = 0$$

So max/min $\sigma_{x'x'}$ occurs where $\tau_{x'y'}=0$

$$\boxed{\begin{aligned} \tan 2\theta_p &= \frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}} \\ \theta_p &= \frac{1}{2}\arctan\left(\frac{2\tau_{xy}}{\sigma_{xx}-\sigma_{yy}}\right) + \frac{n\pi}{2}, \quad n\in I \end{aligned}}$$

$$\frac{\partial \sigma_{y'y'}}{\partial \theta} = -\frac{\partial \sigma_{x'x'}}{\partial \theta} \longrightarrow \text{same angles are found}$$

We next need to plug this angle back into our formulas for $\sigma_{x'x'}$ and $\sigma_{y'y'}$ (we know $\tau_{x'y'} = 0$). To do this we can construct a trigonometric trick for $\cos 2\theta_p$ and $\sin 2\theta_p$.




$$\cos 2\theta = \frac{\sigma_{xx}-\sigma_{yy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}$$

$$\sin 2\theta = \frac{2\tau_{xy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}$$

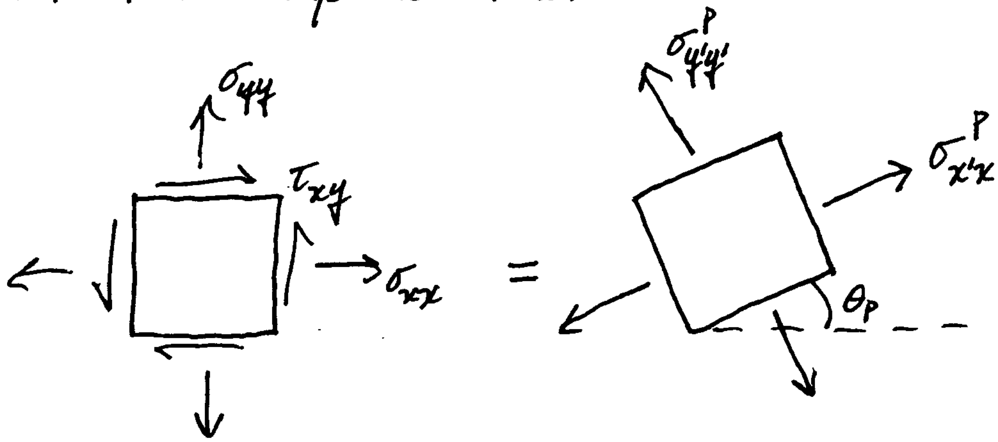
$$\sigma_{x'x'}^{p} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{(\sigma_{xx}-\sigma_{yy})^2}{2\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}} + \frac{4\tau_{xy}^2}{2\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}$$

$$\boxed{\sigma_{x'x'}^{p} = \frac{\sigma_{xx}+\sigma_{yy}}{2} + \frac{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}{2}}$$

By similar manipulations we can show that

$$\boxed{\sigma_{y'y'}^{p} = \frac{\sigma_{xx}+\sigma_{yy}}{2} - \frac{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}{2}}$$

What this says is that:




$\tau^{p}_{x'y'} = 0$

$\sigma^{p}_{x'x'}$ and $\sigma^{p}_{y'y'}$ are called the <u>principal stresses</u>.

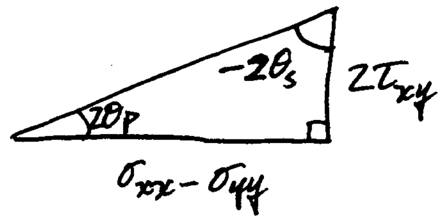
Note that if we rotate our coordinate system by $90°$ such that $\theta \to \theta_p + 90°$ then the stresses are the same numerical values, but we would have $\sigma_{x'x'}$ and $\sigma_{y'y'}$ interchanged.

Also note that $\sigma^{p}_{x'x'} + \sigma^{p}_{y'y'} = \sigma_{xx} + \sigma_{yy}$, which of course is true for any angle $\theta$.

What about the maximum shear stress?

$$\frac{\partial \tau_{x'y'}}{\partial \theta} = -(\sigma_{xx} - \sigma_{yy})\cos 2\theta - 2\tau_{xy}\sin 2\theta = 0$$

$$\boxed{\begin{gathered} \tan 2\theta_s = -\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}} \\[1em] \theta_s = \tfrac{1}{2}\arctan\left(-\frac{\sigma_{xx} - \sigma_{yy}}{2\tau_{xy}}\right) + \frac{n\pi}{2}, \; n \in I \end{gathered}}$$



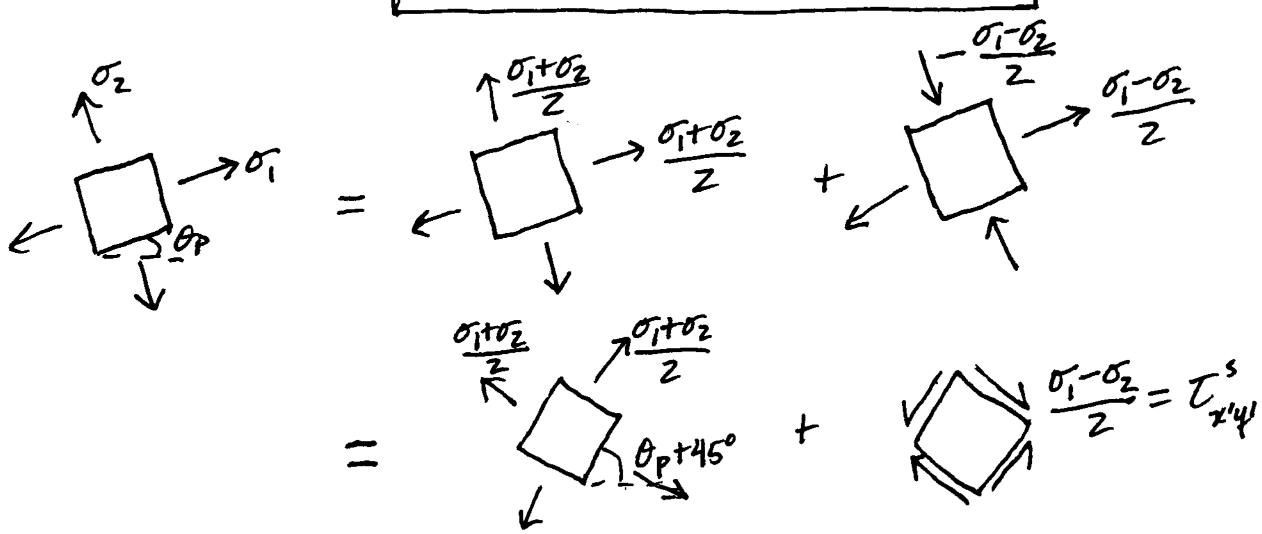

$$\rightarrow \quad 2\theta_p - 2\theta_s = \frac{\pi}{2} \quad \text{or} \quad \theta_s = \theta_p - \frac{\pi}{4}$$ (and we can add or subtract multiples of $\frac{\pi}{2}$)

Note however that the axial stresses are not equal to zero at $\theta_s$. We can find all of our stresses at this angle as:

$$\tau^s_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\,\frac{-(\sigma_{xx}-\sigma_{yy})}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}} + \tau_{xy}\frac{2\tau_{xy}}{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}$$

$$\boxed{\tau^s_{x'y'} = \frac{\sqrt{(\sigma_{xx}-\sigma_{yy})^2+4\tau_{xy}^2}}{2} = \frac{\sigma^p_{x'x'}-\sigma^p_{y'y'}}{2}}$$

Also, we find: $\boxed{\sigma^s_{x'x'} = \sigma^s_{y'y'} = \dfrac{\sigma_{xx}+\sigma_{yy}}{2}}$



# Mohr's Circle

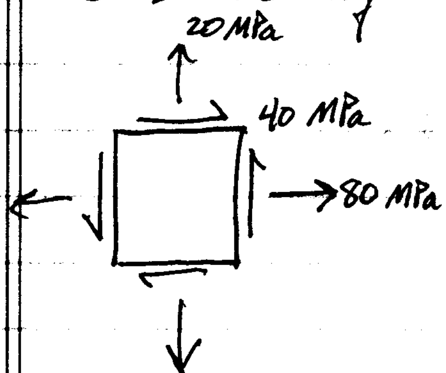
$$\sigma_{x'x'} - \frac{\sigma_{xx}+\sigma_{yy}}{2} = \frac{\sigma_{xx}-\sigma_{yy}}{2}\cos 2\theta + \tau_{xy}\sin 2\theta$$

$$\tau_{x'y'} = -\frac{\sigma_{xx}-\sigma_{yy}}{2}\sin 2\theta + \tau_{xy}\cos 2\theta$$

$$* \left(\sigma_{x'x'} - \frac{\sigma_{xx}+\sigma_{yy}}{2}\right)^2 + \tau_{x'y'}^2 = \underbrace{\left(\frac{\sigma_{xx}-\sigma_{yy}}{2}\right)^2 + \tau_{xy}^2}_{R^2}$$

* This is the equation for a circle centered at $(\sigma_{x'x'}, \tau_{x'y'}) = \left(\frac{\sigma_{xx}+\sigma_{yy}}{2}, 0\right)$ with radius $R$.

- Constructing Mohr's Circle by an example

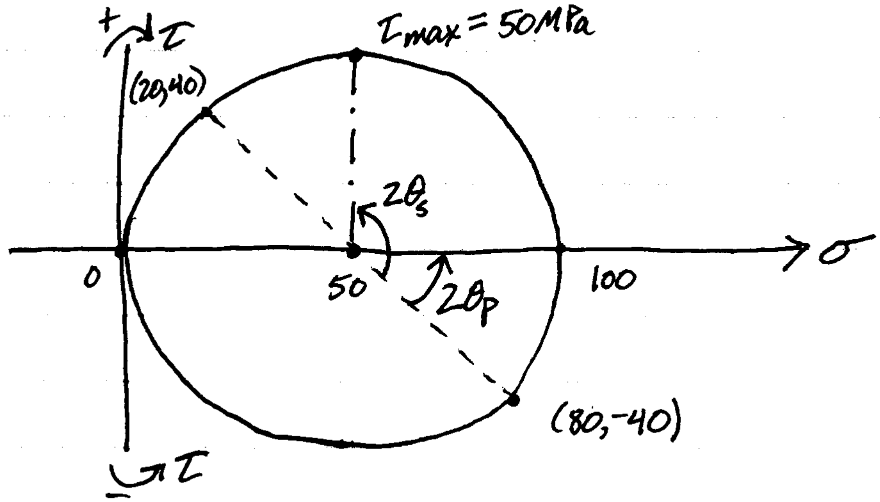



$$\text{Center} = \frac{80+20}{2} = 50\text{ MPa}$$

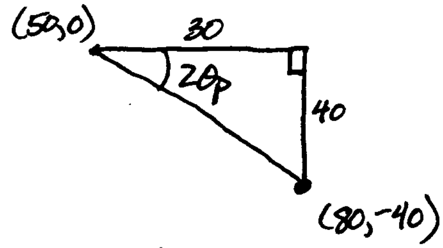
$$\text{Radius} = \sqrt{30^2+40^2} = 50\text{ MPa}$$

Point 1 = $(80, -40)$ from $x$-face

Point 2 = $(20, +40)$ from $y$-face

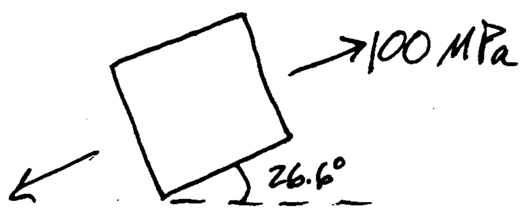


$\theta_p$ can be easily determined from trigonometry.




$\rightarrow \tan 2\theta_p = \frac{4}{3}$

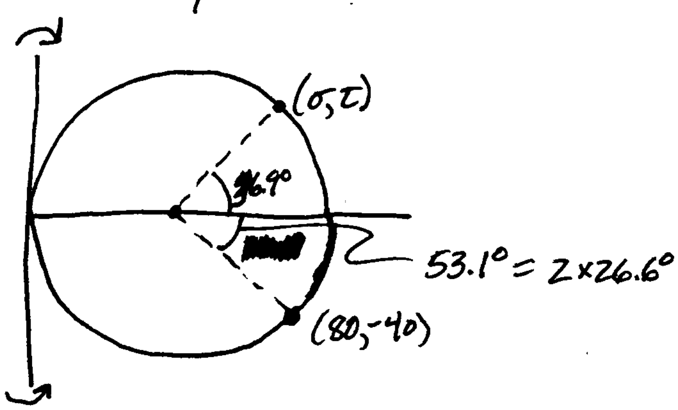
$\theta_p = 26.6°$



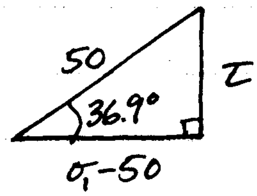

Note that on the face at 90° from this you look to the opposite side of the circle. In this case that stress is zero.

What this demonstrates is that this rather complicated looking stress state is actually a simple uniaxial stress applied at 26.6° from our original x-axis.

What is the stress element for a coordinate system rotated by 45° from the original system?



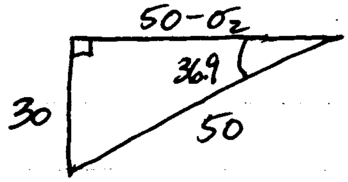

Total $2\theta = 90°$



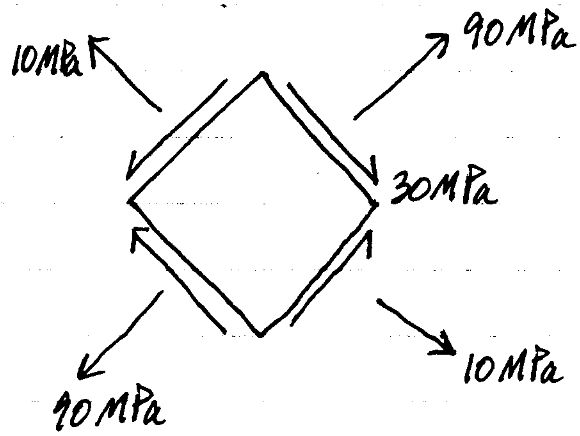

$$\tau = 50 \sin 36.9° = 30 \text{ MPa}$$

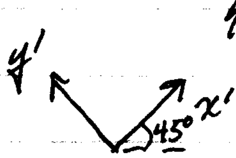
$$\sigma_1 - 50 = 50 \cos 36.9° = 40 \text{ MPa}$$

$$\rightarrow \sigma_1 = 90 \text{ MPa}$$




$$50 - \sigma_2 = 40 \text{ MPa} \rightarrow \sigma_2 = 10 \text{ MPa}$$



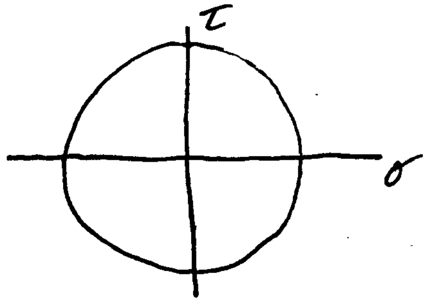

$$\sigma_{x'x'} = 90 \text{ MPa}$$

$$\sigma_{y'y'} = 10 \text{ MPa}$$

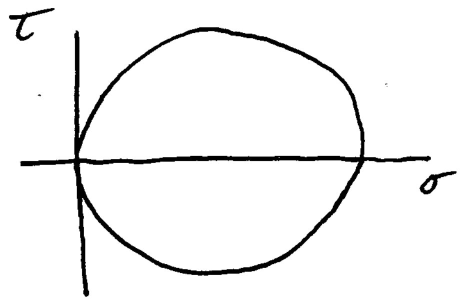
$$\tau_{x'y'} = -30 \text{ MPa}$$

Note that Mohr's circle has a unique +/- convention that is required to plot the circle that is different from the face/direction convention.

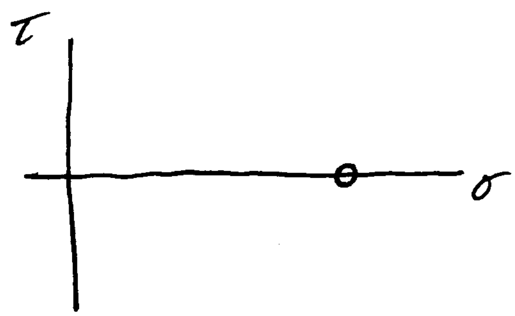
Mohr's circle can tell us a great deal about a stress state at a glance.




This is a pure shear stress state and the radius of the circle denotes the "magnitude" of the stress.

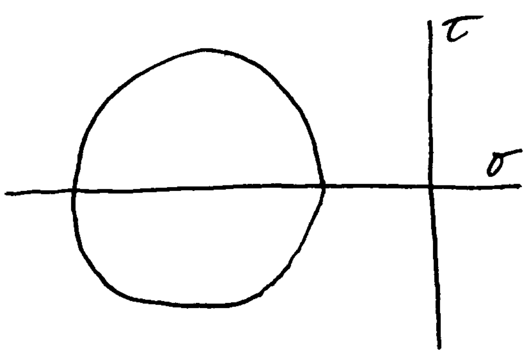



This is a uniaxial tension. Note that $\tau_{max} = \sigma_{max}/2$ for this state.




This is very close to an equi-biaxial stress state. (A single point would be equi-biaxial.)

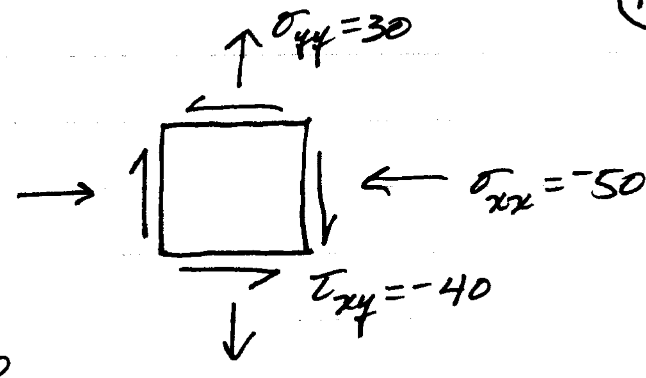
Note that the shear stress is close to zero for any orientation and the two axial stresses are nearly identical.




This stress state has no orientation with a tensile axial stress. Such a state is desired in concrete structures.

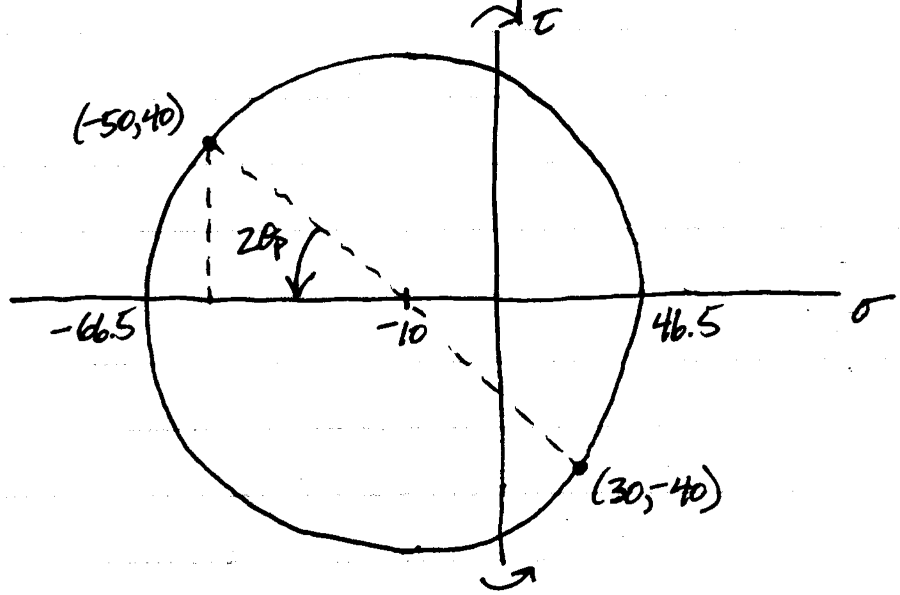
The radius of the circle and its location along the $\sigma$-axis are both measures of the stress "magnitude".

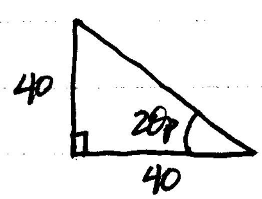
Another example:



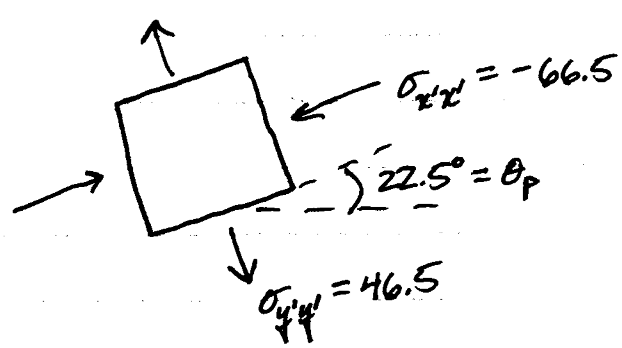

$$\text{Center} = \frac{\sigma_{xx} + \sigma_{yy}}{2} = -10$$

$$\text{Radius} = \sqrt{\left(\frac{\sigma_{xx} - \sigma_{yy}}{2}\right)^2 + \tau_{xy}^2} = 56.5$$




$\rightarrow 2\theta_p = 45° \rightarrow \theta_p = 22.5°$



## Hooke's Law for Plane Stress

$$\varepsilon_{xx} = \underbrace{\frac{1}{E}\sigma_{xx}}_{\text{axial strain}} - \underbrace{\frac{\nu}{E}\sigma_{yy}}_{\text{transverse strain}}$$

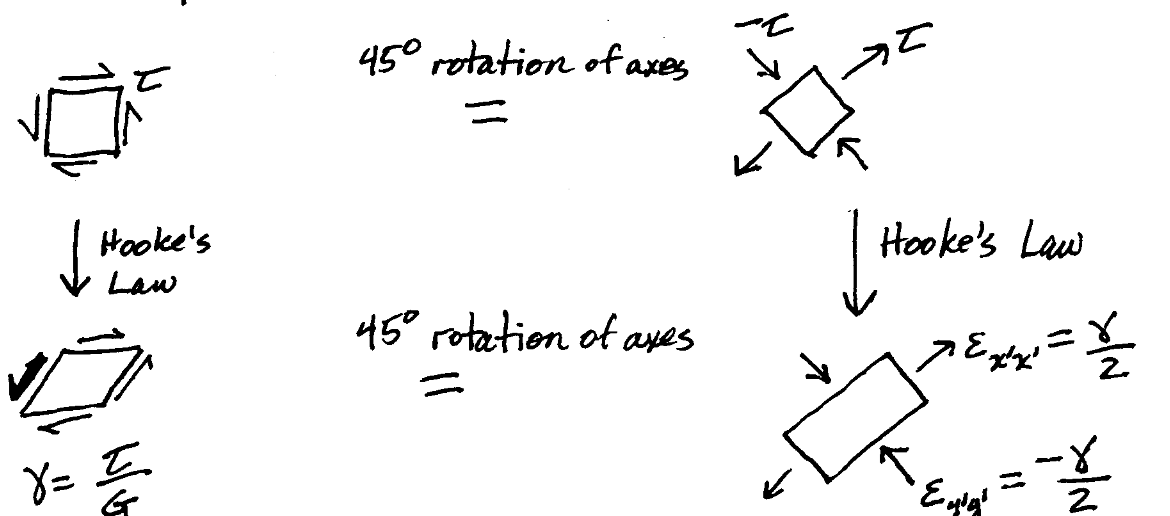
$$\varepsilon_{yy} = \frac{1}{E}\sigma_{yy} - \frac{\nu}{E}\sigma_{xx}$$

$$\gamma_{xy} = \frac{1}{G}\tau_{xy}$$

$E$ = Young's modulus
$\nu$ = Poisson's ratio
$G$ = shear modulus

We can show that for an isotropic material only two of these are independent.

Consider a pure shear stress state.




$$\varepsilon_{x'x'} \overset{\text{Hooke's Law}}{=} \frac{1}{E}\underbrace{\tau}_{\sigma_{x'x'}} - \frac{\nu}{E}\underbrace{(-\tau)}_{\sigma_{y'y'}} = \underbrace{\frac{1+\nu}{E}\tau \underset{\text{45° rotation}}{=} \frac{\gamma}{2} \overset{\text{Hooke's Law}}{=} \frac{1}{2G}\tau}$$

$$\therefore \boxed{G = \frac{E}{2(1+\nu)}}$$