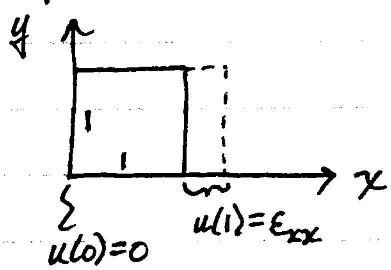
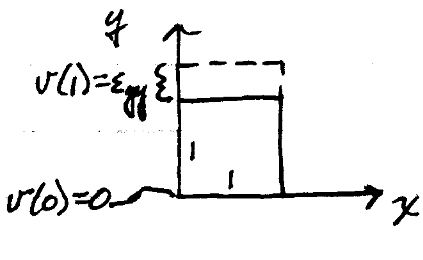
So where did $\varepsilon_{x'x'} = \frac{\gamma}{2}$ and $\varepsilon_{y'y'} = \frac{-\gamma}{2}$ come from for the 45° rotation?

To analyze how a rotation of coordinates affects the strain components we must analyze the kinematics of deformation. In my opinion, the easiest way to do this is to consider the simplest displacements that give us a general 2-D strain state at a point.

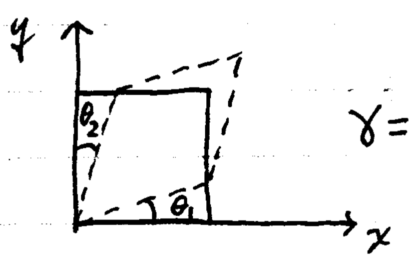


$u = \varepsilon_{xx} x$

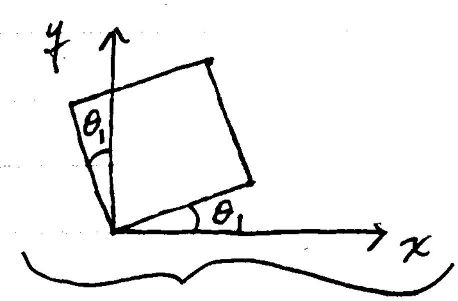


$v = \varepsilon_{yy} y$

Axial strains in x and y



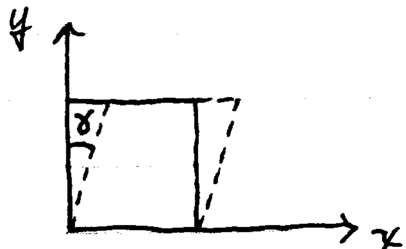
$\gamma = \theta_1 + \theta_2$



Pure rotation $\rightarrow \theta_2 = -\theta_1$

$\rightarrow \gamma = 0$

Simple shear strain $\rightarrow$



$u = \gamma_{xy} y$

There are other combinations of u & v that would also give γ.

So our displacement field that gives a general strain state is,

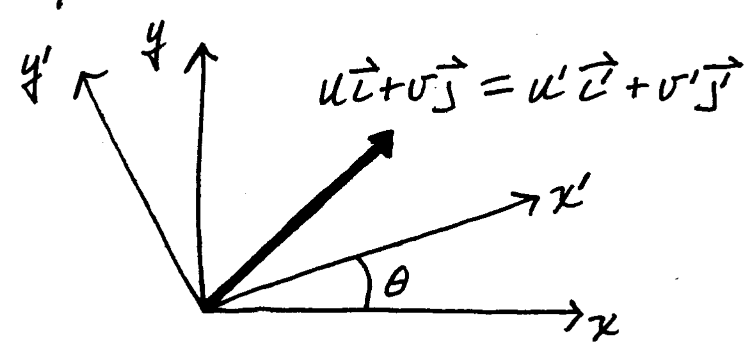
$$u = \varepsilon_{xx}\, x + \gamma_{xy}\, y$$

$$v = \varepsilon_{yy}\, y$$

Notice that : $\varepsilon_{xx} = \dfrac{\partial u}{\partial x}$, $\varepsilon_{yy} = \dfrac{\partial v}{\partial y}$

$$\theta_1 + \theta_2 = \underbrace{\frac{\partial v}{\partial x}}_{\theta_1} + \underbrace{\frac{\partial u}{\partial y}}_{\theta_2} = \gamma_{xy}$$

Now let's find u' and v' in a rotated (x', y') coordinate system.



$$\left.\begin{aligned} u' &= u\cos\theta + v\sin\theta \\ v' &= -u\sin\theta + v\cos\theta \end{aligned}\right\}$$ This is just a simple vector transformation of components.

These will have x & y in them, so we need x & y in terms of x' & y'.

$$\left.\begin{aligned} x &= x'\cos\theta - y'\sin\theta \\ y &= x'\sin\theta + y'\cos\theta \end{aligned}\right\}$$ This is another simple vector transformation.

$$u' = \underbrace{(\varepsilon_{xx} x + \gamma_{xy} y)}_{u} \cos\theta + \underbrace{\varepsilon_{yy} y}_{v} \sin\theta$$

$$= \varepsilon_{xx} \cos\theta \underbrace{(x'\cos\theta - y'\sin\theta)}_{x} + \gamma_{xy} \cos\theta \underbrace{(x'\sin\theta + y'\cos\theta)}_{y}$$

$$+ \varepsilon_{yy} \sin\theta \underbrace{(x'\sin\theta + y'\cos\theta)}_{y}$$

$$u' = (\varepsilon_{xx} \cos^2\theta + \varepsilon_{yy} \sin^2\theta + \gamma_{xy} \sin\theta\cos\theta)\, x'$$

$$+ (-\varepsilon_{xx} \sin\theta\cos\theta + \varepsilon_{yy} \sin\theta\cos\theta + \gamma_{xy} \cos^2\theta)\, y'$$

Showing fewer steps:

$$v' = -\sin\theta \left[\varepsilon_{xx} (x'\cos\theta - y'\sin\theta) + \gamma_{xy} (x'\sin\theta + y'\cos\theta) \right]$$

$$+ \cos\theta \left[\varepsilon_{yy} (x'\sin\theta + y'\cos\theta) \right]$$

$$v' = (-\varepsilon_{xx} \sin\theta\cos\theta + \varepsilon_{yy} \sin\theta\cos\theta - \gamma_{xy} \sin^2\theta)\, x'$$

$$+ (\varepsilon_{xx} \sin^2\theta + \varepsilon_{yy} \cos^2\theta - \gamma_{xy} \sin\theta\cos\theta)\, y'$$

So what are $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$, and $\gamma_{x'y'}$?

Essentially, these follow the same definitions that we used for ε_{xx}, ε_{yy}, and γ_{xy}.

$$\varepsilon_{x'x'} = \frac{\Delta L}{L} \text{ in the } x'\text{-direction} = \frac{\partial u'}{\partial x'}$$

$$\boxed{\begin{aligned} \varepsilon_{x'x'} &= \varepsilon_{xx}\cos^2\theta + \varepsilon_{yy}\sin^2\theta + \gamma_{xy}\sin\theta\cos\theta \\ \varepsilon_{x'x'} &= \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta + \frac{\gamma_{xy}}{2}\sin 2\theta \end{aligned}}$$

$$\varepsilon_{y'y'} = \frac{\Delta L}{L} \text{ in the } y'\text{-direction} = \frac{\partial v'}{\partial y'}$$

$$\boxed{\begin{aligned} \varepsilon_{y'y'} &= \varepsilon_{xx}\sin^2\theta + \varepsilon_{yy}\cos^2\theta - \gamma_{xy}\sin\theta\cos\theta \\ \varepsilon_{y'y'} &= \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} - \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos 2\theta - \frac{\gamma_{xy}}{2}\sin 2\theta \end{aligned}}$$

$$\gamma_{x'y'} = \theta_1' + \theta_2' = \frac{\partial v'}{\partial x'} + \frac{\partial u'}{\partial y'}$$

$$\boxed{\begin{aligned} \gamma_{x'y'} &= -(\varepsilon_{xx}-\varepsilon_{yy})\,2\sin\theta\cos\theta + \gamma_{xy}(\cos^2\theta - \sin^2\theta) \\ \gamma_{x'y'}/2 &= -\frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\sin 2\theta + \frac{\gamma_{xy}}{2}\cos 2\theta \end{aligned}}$$

So, if we define $\varepsilon_{x'y'} = \frac{\gamma_{x'y'}}{2}$ and $\varepsilon_{xy} = \frac{\gamma_{xy}}{2}$ then $\varepsilon_{x'x'}$, $\varepsilon_{y'y'}$, $\varepsilon_{x'y'}$ transform in exactly the same way as $\sigma_{x'x'}$, $\sigma_{y'y'}$, and $\sigma_{x'y'}$. In fact this is how a <u>second rank</u> tensor transforms.

For this reason ε_{xy} is called the <u>tensor shear strain</u>.

There is one other kinematic quantity of interest to look at, the rotation.

Looking back at our elements on page 154,
$\theta_1 + \theta_2$ = shear strain (actually <u>engineering shear strain</u>)

$\theta_1 - \theta_2 = 2\,\omega_z$ = twice the rotation about the z-axis.

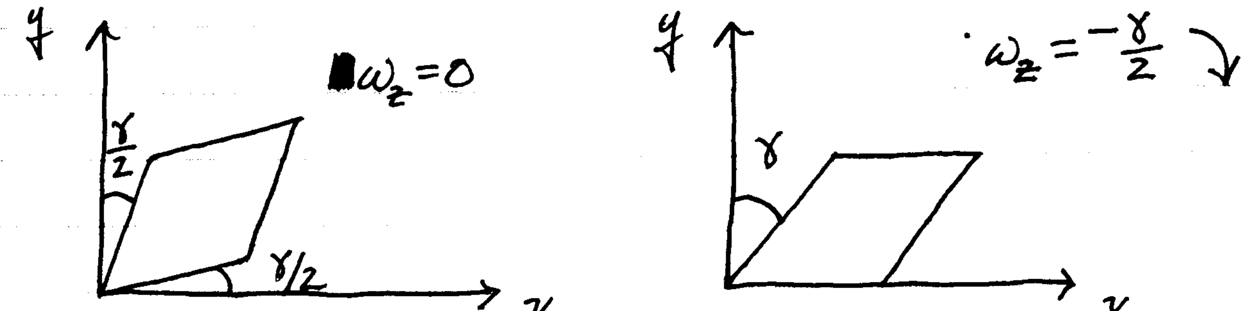
Notice that ω_z should <u>not</u> depend on θ (the orientation of the (x', y') system).

$$2\,\omega_z = \theta_1 - \theta_2 = \frac{\partial v}{\partial x} - \frac{\partial u}{\partial y} = -\gamma_{xy}$$

or $$2\omega_z = \theta_1' - \theta_2' = \frac{\partial v'}{\partial x'} - \frac{\partial u'}{\partial y'} = -\gamma_{xy}\sin^2\theta - \gamma_{xy}\cos^2\theta = -\gamma_{xy}$$

We must be careful with the interpretation of this example.

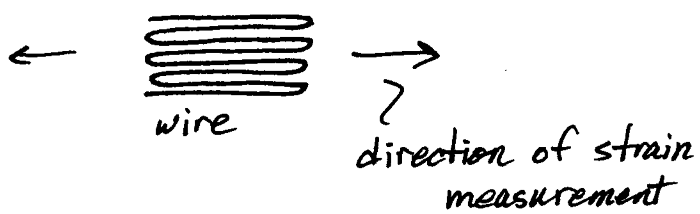
In general $2\omega_z \neq -\gamma_{xy}$. This was just the case in this specific example.



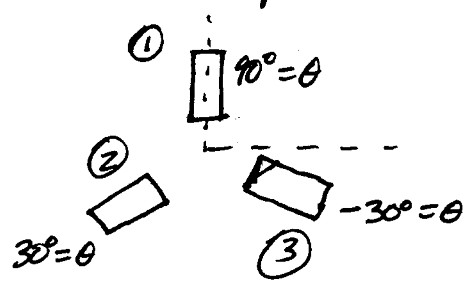
Same deformation, different rotations.

Application to Strain Gages

In order to measure the strain at a point on the surface of a specimen we need 3 measurements. Strain gages are comprised of metal wires that change their resistivity in a predictable way when stretched.



So one strain gage can measure an axial strain in one direction. A rosette is used to measure strains in multiple directions.



For the rosette shown assume that gage 1 reads $200\,\mu\varepsilon$, 2 reads $-60\,\mu\varepsilon$, and 3 reads $100\,\mu\varepsilon$. If $E = 210$ GPa and $\nu = 0.3$, what is the stress state at this point?

Gage 1 : $\varepsilon_{①} = \varepsilon_{yy} = 200 \times 10^{-6}$

Gage 2 : $\varepsilon_{②} = \dfrac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \dfrac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos(+60°) + \dfrac{\gamma_{xy}}{2}\sin(60°)$

$$\varepsilon_{②} = 0.75\,\varepsilon_{xx} + 0.25\,(200\times10^{-6}) + \frac{\sqrt{3}}{4}\gamma_{xy} = -60\times10^{-6}$$

Gage 3: $$\varepsilon_{③} = \frac{\varepsilon_{xx}+\varepsilon_{yy}}{2} + \frac{\varepsilon_{xx}-\varepsilon_{yy}}{2}\cos(-60°) + \frac{\gamma_{xy}}{2}\sin(-60°)$$

$$\varepsilon_{③} = 0.75\,\varepsilon_{xx} + 0.25\,(200\times10^{-6}) - \frac{\sqrt{3}}{4}\gamma_{xy} = 100\times10^{-6}$$

$$\varepsilon_{②} - \varepsilon_{③} = \frac{\sqrt{3}}{2}\gamma_{xy} = -160\times10^{-6}$$

$$\gamma_{xy} = -184.75\,\mu\varepsilon$$

$\rightarrow \varepsilon_{xx} =$ [redacted] $-40\,\mu\varepsilon$

$$\tau_{xy} = G\gamma_{xy} = \frac{E}{2(1+\nu)}\gamma_{xy} = \boxed{-14.9\text{ MPa} = \tau_{xy}}$$

$$\varepsilon_{xx} = -40\times10^{-6} = \frac{\sigma_{xx}}{E} - \frac{\nu\sigma_{yy}}{E}$$

$$-8.4\text{ MPa} = \sigma_{xx} - 0.3\sigma_{yy}$$

$$\varepsilon_{yy} = 200\times10^{-6} = -\frac{\nu\sigma_{xx}}{E} + \frac{\sigma_{yy}}{E}$$

$$42\text{ MPa} = -0.3\sigma_{xx} + \sigma_{yy}$$

$$42 = -0.3(0.3\sigma_{yy} - 8.4) + \sigma_{yy}$$

$\boxed{\sigma_{yy} = 43.4\text{ MPa} \rightarrow \sigma_{xx} = 4.6\text{ [redacted] MPa}}$

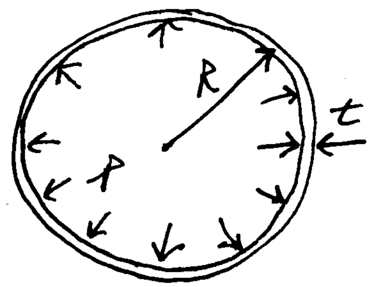
** Hooke's Law inverted (plane stress) $\longrightarrow$

$$\left.\begin{aligned} \sigma_{xx} &= \frac{E}{1-\nu^2}\varepsilon_{xx} + \frac{\nu E}{1-\nu^2}\varepsilon_{yy} \\ \sigma_{yy} &= \frac{\nu E}{1-\nu^2}\varepsilon_{xx} + \frac{E}{1-\nu^2}\varepsilon_{yy} \\ \tau_{xy} &= G\gamma_{xy} \end{aligned}\right\}$$ Useful with ε-gages

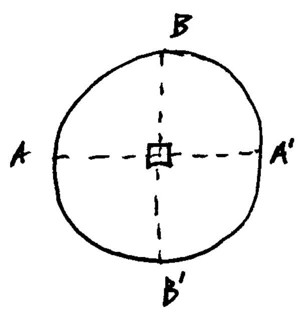
Pressure Vessels

We will study cases where the wall thickness of the pressure vessel is small compared to its radius and length.

Spherical Vessel:

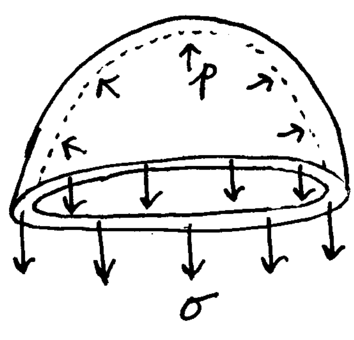


What are the stresses in the vessel?



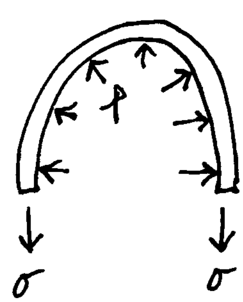
2 cuts

FBD for A-A':



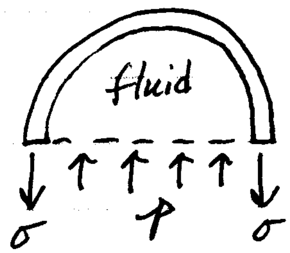
This FBD is if I cut the fluid out of the vessel.

A 2-D view may be clearer:



Now we have to do an equilibrium analysis of this, including the pressure acting at all different directions. Instead of doing this we can consider a simpler FBD.

In 2-D again:



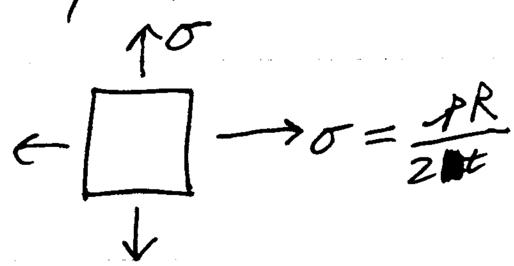
For this FBD I have not cut the fluid out of the vessel. The p acting on the dashed line is the force per unit area that the fluid below places on the fluid above. This FBD is simple to analyze.

$$\Sigma F_y = p\underbrace{(\pi R^2)}_{\text{area for } p} - \sigma\underbrace{(2\pi R t)}_{\text{area for } \sigma\text{, assuming } t \ll R} = 0$$

$$\rightarrow \boxed{\sigma = \frac{pR}{2t} \quad \text{for a thin-walled spherical pressure vessel}}$$

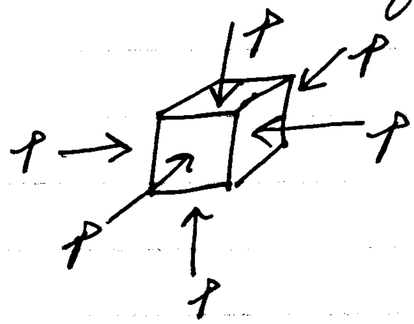
Note that cut B-B' looks exactly like A-A' but in the x-direction.

So the stress state is:

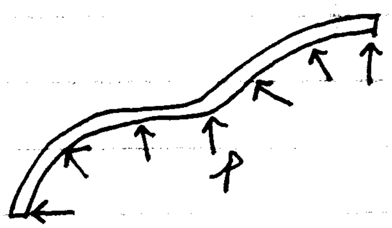


This "trick" for analyzing the net force due to a uniform pressure can be generalized to any arbitrary shape.

At static equilibrium fluids are not able to sustain shear stresses and so the stress state in a fluid must be an equi-triaxial pressure.

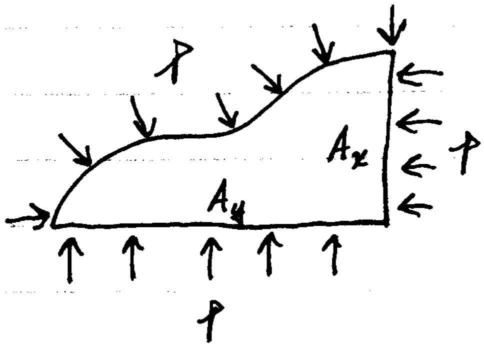


This is called a spherical or hydrostatic stress state. No matter how the stress element is rotated the stress components remain unchanged.



What are the net forces due to p in the x- and y-directions?

Consider an FBD of the fluid.



$$\Sigma F_x = F_x^p - pA_x = 0$$

$$\Sigma F_y = F_y^p + pA_y = 0$$

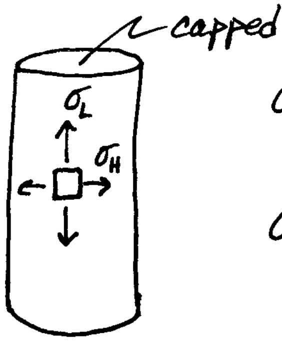
F_x^p and F_y^p are the resultants of the force due to the pressure on the curved surface in the x- and y-directions.

$$\left.\begin{aligned} F_x^p &= pA_x \\ F_y^p &= -pA_y \end{aligned}\right\}$$

What this says is that the magnitude of the resultant force is just the pressure times the projected area in the specific direction. The sign depends on the orientation of the cut and can be determined by inspection.

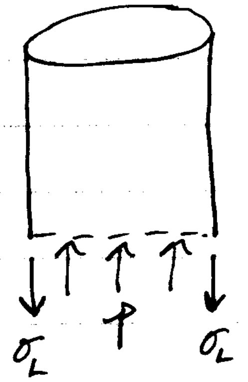
Also, note that F_x^p and F_y^p are the forces that the wall places on the air. So the forces that the air places on the wall are opposite to these.

Cylindrical Pressure Vessels



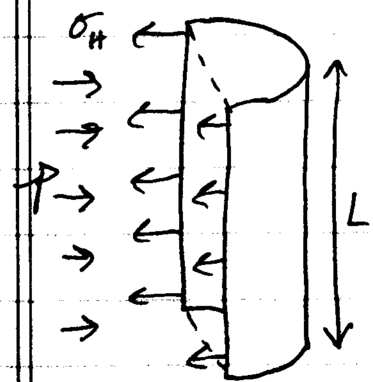
σ_L = longitudinal stress

σ_H = hoop stress



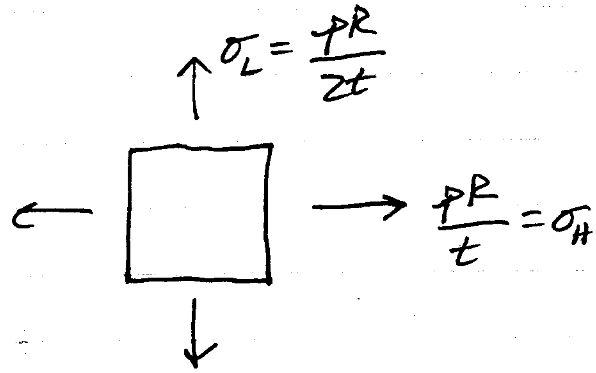
$$\Sigma F_y = p(\pi R^2) - \sigma_L(2\pi R t) = 0$$

$$\sigma_L = \frac{pR}{2t}$$

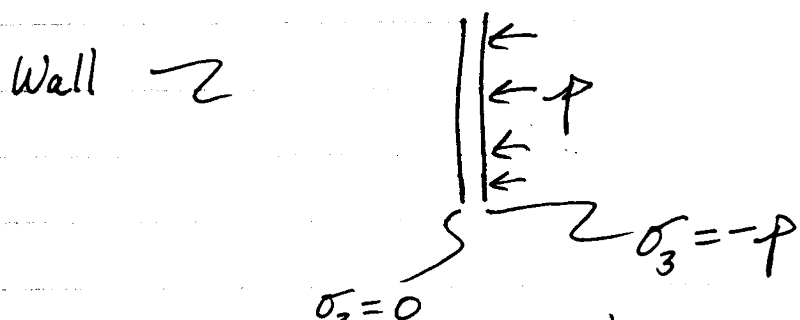


$$\Sigma F_x = p(2RL) - \sigma_H(2tL) = 0$$

$$\sigma_H = \frac{pR}{t}$$



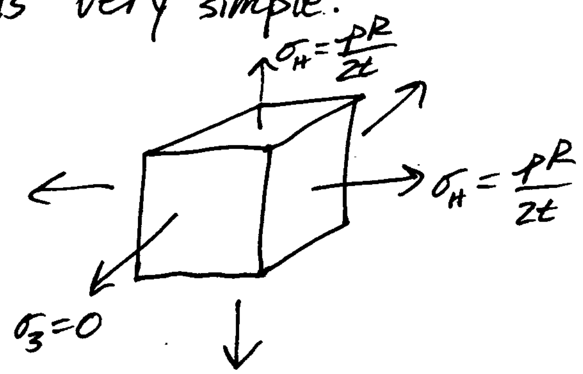
Out-of-plane stress



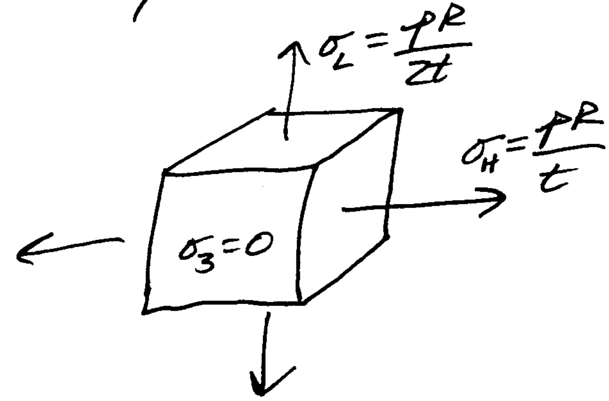
Inside $-p < \sigma_3 < 0$

for $t \ll R$, $p \ll \frac{pR}{t}$ so we make the approximation that $\sigma_3 \approx 0$.

If we want to consider shear failure in the wall we have to consider out-of-plane rotations of the stress element. Ultimately the result is very simple.



Spherical



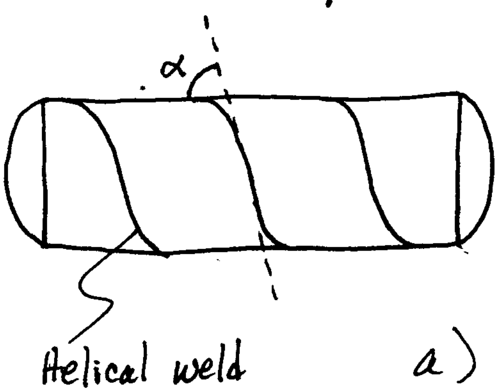
Cylindrical

$$\tau_{max} = \frac{\text{max principal stress} - \text{min principal stress}}{2}$$

$\tau_{max} = \frac{pR}{4t}$ for a spherical pressure vessel

$\tau_{max} = \frac{pR}{2t}$ for a cylindrical pressure vessel

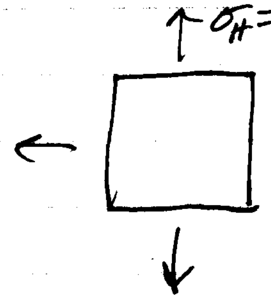
<u>Example</u>



$R = 0.6\,m$
$t = 18\,mm$
$p = 2.8\,MPa$
$\alpha = 55^\circ$
$E = 200\,GPa$
$\nu = 0.3$

a) σ_L & σ_H ?
b) Max in-plane & out-of-plane shear stresses?
c) ε_L & ε_H ?
d) Stresses on weld?

a)



$\sigma_H = \frac{pR}{t}$

$\sigma_L = \frac{pR}{2t}$ $\qquad \sigma_3 = 0$

b) in-plane: $\tau_{max} = \frac{\sigma_H - \sigma_L}{2} = \frac{pR}{4t}$

out-of-plane: $\tau_{max} = \frac{\sigma_H - \sigma_3}{2} = \frac{pR}{2t}$

c) $\varepsilon_L = \frac{\sigma_L}{E} - \frac{\nu \sigma_H}{E} = \frac{1-2\nu}{E} \frac{pR}{2t}$

$\varepsilon_H = \frac{\sigma_H}{E} - \frac{\nu \sigma_L}{E} = \frac{2-\nu}{E} \frac{pR}{2t}$

d) $\theta = 90^\circ - 55^\circ = 35^\circ$

$$\sigma_{x'x'} = \underbrace{\frac{\sigma_L + \sigma_H}{2}}_{C} + \underbrace{\frac{\sigma_L - \sigma_H}{2} \cos 70^\circ}_{-\Delta\sigma}$$

$$\sigma_{y'y'} = \frac{\sigma_L + \sigma_H}{2} - \frac{\sigma_L - \sigma_H}{2} \cos 70^\circ$$

$$\tau_{x'y'} = -\frac{\sigma_L - \sigma_H}{2} \sin 70^\circ$$