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Convergence Rates for Direct Transcription of Optimal Control Problems Using Second Derivative Methods

Dayaram Sonawane, Manan Pathak, and Venkat R. Subramanian

***Abstract*— In this paper, Second Derivative Method (SDM) of numerical discretization is applied to optimal control problems. Convergence rates for the error between the discretized solution of SDM and the corresponding analytical solution of optimal control problems are analyzed. Illustrative examples are included to demonstrate the applicability and benefits of SDM. The comparison of the convergence rates of SDM with implicit Runge-Kutta methods (third order, 2-stage RadauIIA and fourth order, 3-stage LobattoIIIA) is also presented. Using SDM, for optimal control problems with non-stiff type of state equations, the fourth order convergence for states and second order convergence for controls is observed, while for certain stiff/oscillatory equations, it results in reduced order of convergence as observed in other approaches. Depending on the choice of optimization algorithms/platforms used, the proposed method is found to be comparable to other approaches and for certain cases, more efficient.**

I. INTRODUCTION

Optimal control problems defined by ordinary differential equations arise in wide range of engineering applications. Numerical methods used to discretize optimal control problems are classified as indirect methods and direct methods. In the indirect methods, the calculus of variations [1] and Pontryagin's minimum principle [2] are used to form the first order optimality conditions and then the resulting Hamiltonian boundary-value problem (HBVP) is solved to obtain an optimal solution. On the other hand, in the direct methods, the infinite dimensional optimal control problem is converted to the finite dimensional non-linear programming problem (NLP) by discretizing the governing state equations and objective functional [3]. Within direct methods, the three main methods are control vector parameterization, multiple shooting and simultaneous discritization [4,5].

In the simultaneous approach, numerical discritization of state equations is combined with optimization; the method of discritization can include multistep or single step Runge-Kutta methods [6]. To evaluate the efficiency of discretization methods used in numerical optimization, it is necessary to study the rate of convergence of the discretized solution to the continuous solution of the optimal control problems. As far as convergence rates are concerned, previously Budak [7] and Cullum [8] studied the convergence for the solution of a discrete approximation of unconstrained optimal control problems. In these papers, authors have analyzed the rate of convergence using standard one-step and multi-step integration schemes. Using explicit Runge-Kutta schemes, William Hager [9] proved the second order convergence of controls for unconstrained optimal control problems using both one step and multi-step discritization methods. Dontchev [10] presented Euler discritization of optimal control problems; it was the first attempt to prove the convergence of constrained optimization. Numerical results given in [6, 19, 23] demonstrates that for the range of Runge-Kutta discretization methods even for simple optimal control problems, the convergence rate of controls are often found to be lower than states and adjoint variables. Several other notable research contributions were made in the area of order of convergence of numerical discretization methods and their implications to optical control problems. The reader is urged to refer the related work of previous researchers [11-15] in this direction. It is important to note that multistep methods cannot provide more than third order of convergence for states irrespective of the method used [9]. This means that while backward difference formulas [16] are very good for the simulation of stiff ordinary differential equations (ODEs) and differential-algebraic equations (DAEs), they are not necessarily good for direct transcription of optimal control problems. As proved by Hager [9], multistep methods perform very poorly in particular, near the boundaries where the order of convergence can be reduced to one.

Single step methods provide good convergence properties. In fact, an *s-stage* Gauss method will provide an order of *2s* for optimal control problems for the state variables at the terminal nodes. However, Gauss RK is only A-stable not L-stable and might fail for stiff problems. RadauIIA methods with *s-stages* can provide *2s-1* accuracy for the state variables at the terminal nodes, they can handle stiff problems as well. LobattoIIIA methods are A-stable and provide *2s-2* order of convergence at the terminal nodes.

A-stability: As explained in [22], A *k-step* method is a A-stable, if all its solution tend to zero as $n \to \infty$, when the method is applied to any ODE of the form $dx/dt = qx$; where q is a complex constant with negative real part and its stability domain $S:\{z;|R(z) \le 1\}$ covers the entire left half of plane.

L-stability: A method is L-stable if it is A-stable and $\phi(z) \to 0$ as $z \to \infty$ where ϕ is the stability function of the method.

It is important to note that the theory of Superconvergence gives *2s*, *2s-1* and *2s-2* order of accuracy [17] at the terminal nodes for s-stage implicit RK methods defined by Gauss, RadauIIA and LobattoIIIA methods. This is a great advantage for these methods when implemented from simulation of stiff ODEs and DAEs point of view.

Dayaram Sonawane and Manan Pathak are with Department of Chemical Engineering University of Washington Seattle, WA 98195 USA (emails: sonawanedn@gmail.com, mananp@uw.edu)

Corresponding Author: Venkat R. Subramanian is with the Department of Chemical Engineering, University of Washington, Seattle, WA 98195 USA, he also has a joint position at the Pacific Northwest National Laboratory (PNNL), Richland, WA 99352 USA, phone: 206-543-2271; fax: 206-543-3778 (email-: vsubram@uw.edu)

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However, at the internal stage points (collocation points), the order of accuracy is only <=*s* and is similarly observed when applied to optimal control problems. The SDM method on the other hand avoids any internal collocation point, but needs the second derivative of the dependent variables.

To our knowledge, the application of SDM for optimal control problems has not been reported in the literature. This paper studies the convergence rates of SDM [18] and its applicability for general optimal control problems. Numerical results demonstrate that, it is possible to achieve third or fourth order convergence for states and second order convergence for controls without post-computing an estimate of controls as suggested in [6] with lesser number of discretization variables compared to implicit Runge-Kutta methods. Though the improvement of sparse solvers and optimizers help to address large number of optimization variables, SDM methods reduce the number of discretized variables, thereby reducing the RAM requirement and providing improved computational efficiency when standard non-sparse solvers are used for optimization.

II. DIRECT TRANSCRIPTION OF OPTIMAL CONTROL PROBLEM

A. Optimal Control Problem

Optimal control problems governed by differential state variables and control variables arise in wide range of applications. If there are no path constraints on the states or control variables, and if the initial and final times are fixed, a general continuous time optimal control problem in Mayer form is defined as

$$\min_{u(t)} \phi\left(y\left(t_f\right)\right) \quad (1a)$$

s.t.

$$\frac{dy}{dt}(t) = f\left(y\left(t\right),u\left(t\right)\right), y\left(t_0\right) = \tilde{y}_0 \quad (1b)$$

where, $y:\left[t_0,t_f\right] \mapsto \mathbb{R}^{n_y}$ are the state variables, $u:\left[t_0,t_f\right] \mapsto \mathbb{R}^{n_u}$ are the control variables to be determined, $\phi:\mathbb{R}^{n_y} \times \mathbb{R} \mapsto \mathbb{R}^{n_y}$ is a cost functional, $f:\mathbb{R}^{n_y} \times \mathbb{R}^{n_u} \mapsto \mathbb{R}^{n_y}$ is vector form of given function, assumed to be continuously differentiable and $\tilde{y}_0 \in \mathbb{R}^{n_y}$ is the given initial condition. Eq. (1a) represents the cost functional to be minimized and eq. (1b) represents the system dynamics and initial state conditions.

Let $(y^*(t),u^*(t))$ be the continuous-time solution of the optimal control problem (1a-1b) that satisfies the following set of necessary optimality conditions.

$$\frac{dy^*}{dt}(t) = f\left(y^*\left(t\right),u^*\left(t\right)\right), y^*\left(t_0\right) = \tilde{y}_0 \quad (2a)$$

$$\frac{d\lambda^*}{dt}(t) = -f_y\left(y^*\left(t\right),u^*\left(t\right)\right)\lambda^*\left(t\right), \lambda^*\left(t_f\right) = -\phi_y\left(y^*\left(t_f\right)\right) \quad (2b)$$

$$0 = f_u \quad y^* \quad t \quad ,u^* \quad t \quad \lambda^* \quad t \quad (2c)$$

where, eq. (2b) is called an adjoint equation and eq. (2c) is called a gradient equation.

B. Direct Transcription with Second Derivative Method

As discussed before, with direct transcriptional approach, optimal control problems are typically first *discretized* using Euler, Trapezoidal or a range of Runge-Kutta methods on a uniform grid of N points covering the time interval $t \in \left[t_0,t_f\right]$ and then the resulting NLP problem is *optimized*. Such direct transcription methods, especially the range of Runge-Kutta methods have received significant attention recently [19, 20, 23]. One reason for this is the fast convergence of the solutions of discretized optimal control problem to the solution of the underlying continuous time optimal control problem. Though IRK methods have proved to be more popular, the SDM gives an advantage over the number of discretized variables which will help to reduce RAM size and CPU time for non-sparse type of optimizers. For certain optimal control problems where system dynamics are described by stiff ODEs, IRK methods result in oscillations for the state variables at internal node points because of lower stage accuracy; in such cases SDM is competitive.

To describe SDM method, the fourth order single step formula is given as:

$$y_{n+1} = y_n + \frac{h}{2}\left(f_n + f_{n+1}\right) + \frac{h^2}{12}\left(g_n - g_{n+1}\right); n = 1...N-1$$

(3)

where, f_n and f_{n+1}, are the discrete approximations of states, g_n and g_{n+1}, represent the analytical differentiation of f at grid points t_n and t_{n+1} respectively. Eq. (3) provides a fourth order accurate integration scheme. The scheme is A-stable and not L-stable in a region ϕ of the $z-$plane with the stability function given by $\phi(z) = \frac{(1+z/2+z^2/12)}{(1-z/2+z^2/12)}$. The third order SDM formula [18] is written as

$$y_{n+1} = y_n + \frac{h}{3}\left(2f_{n+1} + f_n\right) - \frac{h^2}{6}\left(g_{n+1}\right); n = 1...N-1 \quad (4)$$

Eq. (4) is accurate to the third order and is both A-stable and L-stable.

For the direct transcription of optimal control problem, we discretized the optimal control problem (1a-1b) using SDM over a finite horizon $t \in \left[t_o,t_f\right]$ with a uniform grid of $N-1$ intervals and initial conditions. Using SDM, the state equation (1b) is approximated by eq. (3); it will also involve the derivative of u, the control variable. This cannot be left as an additional optimization variable as it will lead to oscillations. For the SDM method, the differentiation of control vector u is approximated by its finite difference approximation i.e. $\frac{du}{dt} = up_i = (u_{i+1} - u_i)/h; h = t_f/\left(N-1\right)$ this leads to the following SDM discretization of (1a-1b).

$$\min_{u_n,y_n,y_f} \phi\left(y_f\right) \quad (5a)$$

$$s.t.\quad y_{n+1}-y_n-\frac{h}{2}\left(f_n+f_{n+1}\right)+\frac{h^2}{12}\left(g_n-g_{n+1}\right)=0; n=1....N-1 \qquad (5b)$$

The optimality conditions for control problem (5a-5b) are obtained by forming the Lagrangian function, where we assumed $\frac{du}{dt}=up$ as a free variable.

$$L\left(y_n,u_n,y_f,\lambda_n,\nu_n,up_n\right)=\phi\left(y_f\right)+\sum_{n=1}^{N-1}\lambda_n^T\left\{\begin{array}{l}\left(y_{n+1}-y_n\right)-\frac{h}{2}\left(f\left(y_n,u_n\right)+f\left(y_{n+1},u_{n+1}\right)\right)\\-\frac{h^2}{12}\left(g\left(y_n,u_n,up_n\right)-g\left(y_{n+1},u_{n+1},up_{n+1}\right)\right)\end{array}\right\}+\sum_{n=1}^{N-1}\nu_n^T\left(y_n-y_0\right) \qquad (6)$$

The necessary conditions of optimality can be written as:

$$\nabla_{\lambda_n}L=\left(y_{n+1}-y_n\right)-\frac{h}{2}\left(f\left(y_n,u_n\right)+f\left(y_{n+1},u_{n+1}\right)\right)-\frac{h^2}{12}\left(g\left(y_n,u_n,up_n\right)-g\left(y_{n+1},u_{n+1},up_{n+1}\right)\right)=0; n=1...N-1; \qquad (7a)$$

$$\nabla_{u_n}L=\lambda_n\left\{-\frac{h}{2}f_u\left(y_n,u_n\right)-\frac{h^2}{12}g_u\left(y_n,u_n,up_n\right)\right\}+\lambda_{n-1}\left\{-\frac{h}{2}f_u\left(y_{n+1},u_{n+1}\right)+\frac{h^2}{12}g_u\left(y_{n+1},u_{n+1},up_{n+1}\right)\right\}=0; \qquad (7b)$$

$n=1...N-1;$

$$\nabla_{y_n}L=\lambda_n\left\{-1-\frac{h}{2}f_y\left(y_n,u_n\right)-\frac{h^2}{12}g_y\left(y_n,u_n,up_n\right)\right\}+\lambda_{n-1}\left\{1-\frac{h}{2}f_y\left(y_{n+1},u_{n+1}\right)+\frac{h^2}{12}g_y\left(y_{n+1},u_{n+1},up_{n+1}\right)\right\}=0; \qquad (7c)$$

$n=1...N-1;$

$$\nabla_{\nu_n}L=(y_n-y_0)=0; n=1....N-1 \qquad (7d)$$

$$\nabla_{up_n}L=\lambda_n\left\{-\frac{h^2}{12}g_{up}\left(y_n,u_n,up_n\right)\right\}+\lambda_{n-1}\left\{\frac{h^2}{12}g_{up}\left(y_{n+1},u_{n+1},up_{n+1}\right)\right\}=0; n=1..N-1 \qquad (7e)$$

$$\nabla_{y_f}L=\phi_y\left(y_f\right)+\lambda_{N-1}\left\{1-\frac{h}{2}f_y\left(y_f,u_f\right)+\frac{h^2}{12}g_y\left(y_f,u_f\right)\right\}=0 \qquad (7f)$$

$$\nabla_{u_f}L=\lambda_{N-1}\left\{-\frac{h}{2}f_u\left(y_f,u_f\right)+\frac{h^2}{12}g_u\left(y_f,u_f\right)\right\}=0 \qquad (7g)$$

After the earlier attempts in the seventies, SDM methods lost out to IRK based on Gauss, Lobatto, and Radau type families for simulation of stiff ODEs and DAEs because of the need to find second derivatives. This manuscript suggests for direct transcription of optimal control problems; SDM methods can be competitive. To make our discussion more concrete, we have discussed some optimal control examples and the order of convergence obtained using SDM in the next section, and it is compared with fourth order 3-stage LobattoIIIA and third order 2-stage RadauIIA as implicit Runge-Kutta methods [6, 19, 20]. In this text, we refer Second Derivative Method as SDM, fourth order, 3-stage LobattoIIIA method as LobattoIIIA, fourth order, 3-stage LobattoIIIA with midpoint approximation for control $Uint=(u_1+u_2)/2$ as LobattoIIIA*, third order, 2-stage RadauIIA as RadauIIA and third order, 2-stage RadauIIA with midpoint approximation for control $Uint=(2/3u_1+1/3u_2)$ as RadauIIA*.

III. NUMERICAL EXAMPLES AND RESULTS

This section describes the order of convergence obtained using SDM on some of the typical optimal control examples illustrated in literature. The order of convergence obtained using SDM is compared with LobattoIIIA, RadauIIA, LobattoIIIA* and RadauIIA* implicit Runge-Kutta discretization methods.

Problem 1- P1, The first problem considered is

$$\min_u -y(t_f)$$
$$\dot{y}(t)=-y(t)+y(t)u(t)-u(t)^2; t\in[0,5],$$
$$y(0)=1.$$

The analytical solution of state and control is given as

$$y^*(t)=\frac{4}{\left(1+3e^t\right)}; u^*(t)=\frac{y^*(t)}{2}$$

The value of optimal cost can be computed as -0.008963796. We solved this problem using direct transcription approach on uniform grids of N points with $N-1$ intervals. A discrete solution of P1 is obtained using 4th order SDM (eq. (3)), LobattoIIIA and RadauIIA discretization methods. The discrete solutions obtained using SDM with $N=11$ for state and control is compared with the analytical solution. Figure (1a-1b) shows the optimal solution y^* and u^* of problem P1.

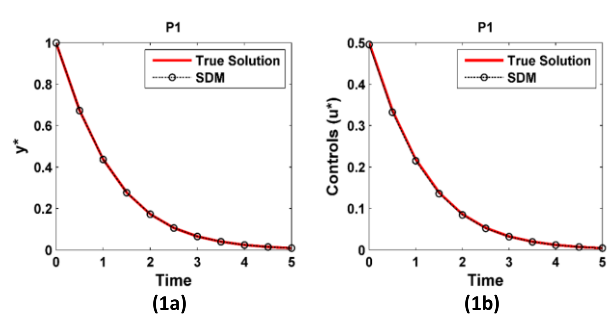

Figure 1. Graph of y^* and u^* for P1 with $N=11$

From figure 1, it is observed that, for problem P1 the discrete optimal solution obtained using SDM for state and controls is much closer to that of analytical solution, LobattoIIIA and RadauIIA discritization methods also reveals the same nature and no oscillations are observed for internal collocation points.

The error, in numerical analysis is generally measured in supremum (sup) norm which is computed as $-\log_2$ of the infinity norm error. As mentioned by the authors in [6, 22], even for simple linear control problems, the controls are often found to be less accurate than the states and objective functional. To observe and compare the order of convergence of discretization methods under study, the sup norm error for optimal discrete control is plotted versus $-\log_2(h)$ with h being the step length, for the values of $h = 1/5, 1/10, 1/20, 1/40..$ as shown in figure 2. The slope of the line in figure 2 is the convergence rate. The numerical results for sup norm error and maximum error for problem P1 are given in Table I & II.

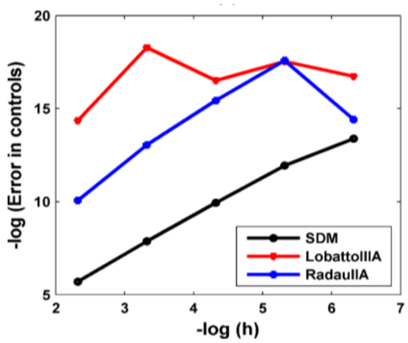

Figure 2. Maximum norm error in controls (u^*) for P1

TABLE I. $-\log_2$ OF INFINITY NORM ERROR IN P1

N	$\|y^* - y_N\|$			$\|u^* - u_N\|$		
	SDM	Lobatto IIIA	Radau IIA	SDM	Lobatto IIIA	Radau IIA
6	11.8130	13.3537	9.0657	5.700	14.351	10.066
11	15.9144	17.3600	11.9822	7.867	18.282	13.046
21	19.9357	21.3447	14.9193	9.940	16.509	15.434
41	23.9222	25.3343	17.8924	11.942	17.536	17.581
81	27.9154	29.3214	20.9021	13.386	16.733	14.411
Order	4	4	3	2	4	3

TABLE II. MAXIMUM ERROR IN OUTPUTS OF P1

N	$\|y^* - y_N\|$			$\|u^* - u_N\|$		
	SDM	Lobatto IIIA	Radau IIA	SDM	Lobatto IIIA	Radau IIA
6	2.77e-4	9.55e-5	1.86e-3	1.92e-2	4.78e-5	9.32e-4
11	1.61e-5	5.94e-6	2.47e-4	4.28e-3	3.13e-6	1.18e-4
21	9.97e-7	3.75e-7	3.22e-5	1.01e-3	1.05e-5	2.25e-5
41	6.29e-8	2.36e-8	4.10e-6	2.54e-4	5.26e-6	5.09e-6

From numerical results, we observed for SDM 4th order convergence for states and 2nd order convergence for controls for LobattoIIIA 4th order convergence in states as well as in controls while, RadauIIA gives 3rd order convergence in state and controls.

Problem 2- P2, This example is adapted from Anna Engelsone, et. al., [11].

$$\min_u \int_0^1 \left[\frac{5}{4} y(t)^2 + y(t)u(t) + u(t)^2 \right] dt$$

$$\dot{y}(t) = \frac{1}{2} y(t) + u(t); t \in [0,1],$$

$$y(0) = 1.$$

The analytical solution of state and control is given as

$$y^*(t) = \frac{\cosh(1-t)}{\cosh(1)}; u^*(t) = \frac{-(\tanh(1-t) + 0.5)\cosh(1-t)}{\cosh(1)}$$

The value of optimal cost is 0.7615941557. This example was chosen as most direct transcription approaches based on multistep methods exhibit order reduction and oscillations in the optimal solution. As explained in [9], IRK methods prove to be better compared to multistep methods (but still exhibit order reduction and oscillation for the control variable). We have attempted this problem using 4th order SDM (eq. (3)) and obtained the optimal solution by discretizing on uniform grids of N points with $N-1$ intervals. The discrete solution obtained with $N = 11$ for state and control is compared with the analytical solution. Figure (3a-3b) shows the optimal solution of P2. From figure 3, it is observed that, the discrete optimal solution obtained using SDM for state and controls closely match with the analytical solution. The optimal solution obtained using LobattoIIIA and RadauIIA discritization methods also appear very similar.

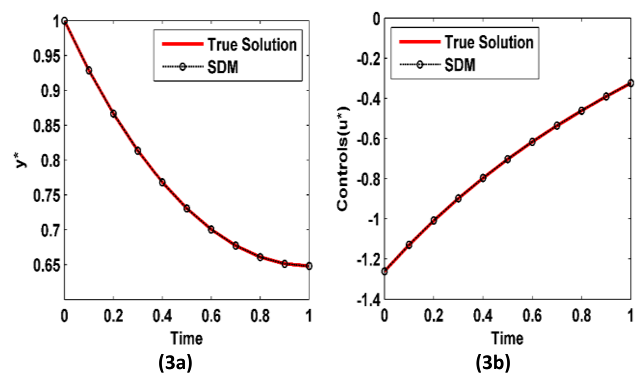

Figure 3. Graph of y^* and u^* for P2

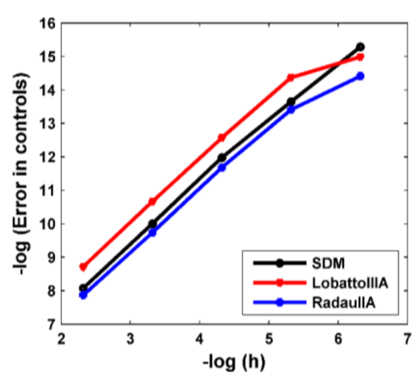

Figure 4. Maximum norm error in controls (u^*) for P2

The sup norm error for optimal discrete control is plotted versus $-\log_2(h)$ as shown in figure 4 and the numerical results for sup norm error and maximum error for P2 are tabulated in Table III & IV.

TABLE III. $-\log_2$ OF INFINITY NORM ERROR IN P2

N	$\|y^* - y_N\|$			$\|u^* - u_N\|$		
	SDM	Lobatto IIIA	Radau IIA	SDM	Lobatto IIIA	Radau IIA
6	14.9523	17.6910	14.1420	8.0753	8.7187	7.8667
11	18.6012	21.5501	17.1526	10.006	10.6698	9.7413
21	20.4297	22.2099	20.2914	11.979	12.5704	11.680
41	20.8025	22.4036	21.1629	13.651	14.3704	13.409
81	21.0389	21.0949	20.1710	15.283	14.9897	14.413
Order	4	4	3	2	2	2

TABLE IV. MAXIMUM ERROR IN OUTPUTS OF P2

N	$\|y^* - y_N\|$			$\|u^* - u_N\|$		
	SDM	Lobatto IIIA	Radau IIA	SDM	Lobatto IIIA	Radau IIA
6	3.15e-5	4.72e-6	5.53e-5	3.70e-3	2.37e-3	4.28e-3
11	2.51e-6	3.25e-7	6.86e-6	9.72e-4	6.13e-4	1.16e-3
21	7.08e-7	2.06e-7	7.79e-7	2.38e-4	1.63e-4	3.04e-4
41	5.46e-7	1.80e-7	4.25e-7	7.77e-5	4.70e-5	9.19e-5

From numerical results of problem P2, we observed 4th order convergence for states and 2nd order convergence for controls with SDM and LobattoIIIA methods, while, RadauIIA gives 3rd order convergence in state and 2nd order convergence in controls. But, even though the order of convergence of controls is the same for all three methods, SDM gives an advantage over the other two discretization methods, as it uses a lesser number of optimization variables and all the variables are of same accuracy unlike IRK methods in which internal collocation/stage variables are less accurate. This example confirms the effectiveness and benefit of the SDM approach of numerical discretization over IRK methods for optimal control problems.

Problem 3- P3, A control problem formulated from a stiff BVP is adapted from [21]. The second order problem is converted to a first order problem (by integration) with an arbitrary constant of integration, k.

$$\min_u \int_0^1 (u)dt$$

$$s.t.$$

$$\dot{y}_1(t) = u(t)\,y_1(t) + k; t \in [0,1]$$

$$0 \ll u \ll Pe(=100) \;\; \& \;\; y_1(0) = 1$$

The analytical solution for problem P3:

$$y_1^*(t) = \frac{e^{Pe}}{-1+e^{Pe}} - \frac{e^{(Pe*t)}}{-1+e^{Pe}}$$

$$u^*(t) = Pe(const.)$$

The value of optimal cost is Pe (Peclet number) over a time. The optimal solution of P3 for state and controls is obtained with $N = 21$ using 4th order SDM, LobattoIIIA* and RadauIIA methods. For discrete controls, we observed the same nature and order of convergence for all three discritization methods but for optimal state, we witnessed oscillations at mesh points for LobattoIIIA* and sustained oscillations at internal collocation points with RadauIIA methods, whereas RadauIIA* did not converge. The optimal solution for state y_1^* is plotted versus time as shown in figure 5. The sup norm error and order of convergence for optimal state is given in Table V.

TABLE V. $-\log_2$ OF INFINITY NORM ERROR IN OPTIMAL DISCRETE STATE OF P3

N	$\|y^* - y_N\|$		
	SDM	LobattoIIIA*	RadauIIA
11	1.7260	1.7262	0.0026
21	3.3549	3.3550	0.5551
41	5.9968	5.9968	3.0296
81	9.5554	9.5556	6.0621
161	13.6522	13.6520	9.4046
Order	4	4	3

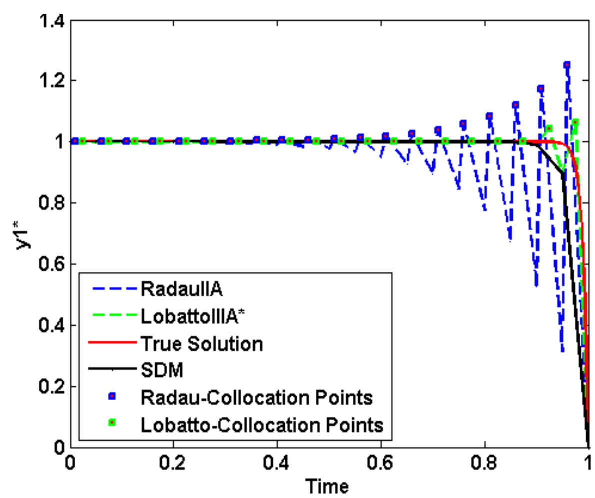

Figure 5. Graph of optimal state for test problem P3 for *Pe=100, N=21*

It is important to note that if bounds of 0...1 (physically meaningful bound) were provided for this problem, only SDM method works and other methods fail because of the observed oscillations for internal stages. In addition, for complicated large scale DAE system, one might encounter 'square root' or other terms which make the equation undefined outside a meaningful range for concentration or mole fraction. In these cases, SDM guarantees same order of accuracy for all the discretized state variables, and converges. For example, in P3 if there was an additional variable defined as $\frac{dz}{dt} = \left(\sqrt{1 - y_1(t)}\right)u(t)$, then SDM will work and other methods may fail.

IV. CONCLUSION

The aim of this work is to examine the performance of SDM for order of convergence applied to optimal control problems. From computational results, we have seen that for the optimal control problems that are easily solved SDM and LobattoIIIA give fourth order convergence for states and second order convergence for controls, whereas RadauIIA gives third order convergence in states and second order convergence in controls as expected. Though the order of convergence of controls is same for LobattoIIIA, SDM and RadauIIA methods, SDM gives an advantage over the number of discretized variables which will help to reduce RAM size and CPU time for non-sparse type of optimizers.

In the case of models that are known to cause order reduction in typical IRK methods; with SDM, it is possible to achieve fourth or third order convergence in states and second order convergence in controls. For certain optimal control problems where system dynamics are described by stiff ODEs, IRK methods have state variables oscillate at internal node points because of lower stage accuracy; in such cases SDM is competitive. For stiff BVP type problems with terminal constraints requiring exhibiting boundary layers (requiring large number of nodes), LobattoIIIA and RadauIIA oscillate for state variables while SDM has no oscillations and gives fourth order convergence for states for the same number of node points.

In particular, for regular non-sparse optimizers, SDM is more useful and efficient because of the lower number of optimization variables. A Maple code is provided to take a general system of ODEs with bounds as input. The code calculates the second derivative and its non-sparse optimizer is then used to solve the optimal control problem. The code will be posted online at *http://depts.washington.edu/maple/*.

Current work includes applying these methods for index-1 and higher orders DAEs optimal control problems. For index-1 DAEs, without differentiating algebraic constraints, by using approximations as used for control variable in this paper, one can obtain second order accuracy for the algebraic variables. By adding derivatives of algebraic constraints as additional equations, and including time derivatives for algebraic variables as additional decision variables, one can obtain 4th order accuracy for algebraic variables. A more detailed analysis of this will be reported in future.

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