

```
> restart;with(linalg):with(plots):
```

Code written by Dr. Venkat R. Subramanian, current and past MAPLE group members at Washington University St. Louis. If you find this code useful, please consider citing the following papers.

1. V. R. Subramanian and R. E. White, "Semianalytical Method of Lines for Solving Elliptic Partial Differential Equations," Chem. Eng. Sci., 59(4), 781 (2004).

2. V. R. Subramanian and R. E. White, "A Semianalytical Method for Predicting Primary and Secondary Current Density Distributions: Linear and Nonlinear Boundary Conditions," J. Electrochem. Soc., 147 (5), 1636-1644 (2000).

3. V. R. Subramanian and R. E. White, "Simulating Shape Changes during Electrodeposition - Primary and Secondary Current Distribution," J. Electrochem. Soc., 149(10), C498 (2002).

4. V. R. Subramanian, and R. E. White, "Symbolic solutions for boundary value problems using Maple," Comp. Chem. Engng., 24(11), 2405-2416 (2000).

The code given below can be used to reproduce example 1 in the first paper by changing N to 10.

```
> ge:=diff(u(x,y),y$2)=-epsilon^2*diff(u(x,y),x$2);
```

$$ge := \frac{\partial^2}{\partial y^2} u(x, y) = -\varepsilon^2 \left( \frac{\partial^2}{\partial x^2} u(x, y) \right)$$

```
> bc1:=u(x,y)-0;
```

$$bc1 := u(x, y)$$

```
> bc2:=u(x,y)-0;
```

$$bc2 := u(x, y)$$

```
> bc3:=u(x,y)-0;
```

$$bc3 := u(x, y)$$

```
> bc4:=u(x,y)-1;
```

$$bc4 := u(x, y) - 1$$

```
> epsilon:=1;
```

$$\varepsilon := 1$$

```
> fd1:=1/2/h*(-u[m+2](zeta)-3*u[m](zeta)+4*u[m+1](zeta));
```

$$fd1 := \frac{1}{2} \frac{-u_{m+2}(\zeta) - 3 u_m(\zeta) + 4 u_{m+1}(\zeta)}{h}$$

```
> bd1:=1/2/h*(u[m-2](zeta)+3*u[m](zeta)-4*u[m-1](zeta));
```

$$bd1 := \frac{1}{2} \frac{u_{m-2}(\zeta) + 3 u_m(\zeta) - 4 u_{m-1}(\zeta)}{h}$$

```
> cd1:=1/2/h*(u[m+1](zeta)-u[m-1](zeta));
```

$$cd1 := \frac{1}{2} \frac{u_{m+1}(\zeta) - u_{m-1}(\zeta)}{h}$$

> cd2:=1/h^2\*(u[m-1](zeta)-2\*u[m](zeta)+u[m+1](zeta));

$$cd2 := \frac{u_{m-1}(\zeta) - 2u_m(\zeta) + u_{m+1}(\zeta)}{h^2}$$

> bc1:=subs(diff(u(x,y),x)=subs(m=0,fd1),u(x,y)=u[0](zeta),x=0,bc1);

$$bc1 := u_0(\zeta)$$

> bc2:=subs(diff(u(x,y),x)=subs(m=N+1,bd1),u(x,y)=u[N+1](zeta),x=1,bc2);

$$bc2 := u_{N+1}(\zeta)$$

> N:=4;

$$N := 4$$

> eq[0]:=bc1;

$$eq_0 := u_0(\zeta)$$

> eq[N+1]:=bc2;

$$eq_5 := u_5(\zeta)$$

> for i from 1 to N do eq[N+1+i]:=diff(u[N+1+i](zeta),zeta)=  
subs(diff(u(x,y),x\$2) = subs(m=i,cd2),diff(u(x,y),x) =  
subs(m=i,cd1),u(x,y)=u[i](zeta),x=i\*h,rhs(h^2/epsilon^2\*ge));od;

$$eq_6 := \frac{d}{d\zeta} u_6(\zeta) = -u_0(\zeta) + 2u_1(\zeta) - u_2(\zeta)$$

$$eq_7 := \frac{d}{d\zeta} u_7(\zeta) = -u_1(\zeta) + 2u_2(\zeta) - u_3(\zeta)$$

$$eq_8 := \frac{d}{d\zeta} u_8(\zeta) = -u_2(\zeta) + 2u_3(\zeta) - u_4(\zeta)$$

$$eq_9 := \frac{d}{d\zeta} u_9(\zeta) = -u_3(\zeta) + 2u_4(\zeta) - u_5(\zeta)$$

> u[0](zeta):=(solve(eq[0],u[0](zeta)));

$$u_0(\zeta) := 0$$

> u[N+1](zeta):=solve(eq[N+1],u[N+1](zeta));

$$u_5(\zeta) := 0$$

> for i from 1 to N do eq[i]:=diff(u[i](zeta),zeta)=  
u[N+1+i](zeta);od;

$$eq_1 := \frac{d}{d\zeta} u_1(\zeta) = u_6(\zeta)$$

$$eq_2 := \frac{d}{d\zeta} u_2(\zeta) = u_7(\zeta)$$

$$eq_3 := \frac{d}{d\zeta} u_3(\zeta) = u_8(\zeta)$$

$$eq_4 := \frac{d}{d\zeta} u_4(\zeta) = u_9(\zeta)$$

> for i from 1 to N do eq[i]:=eval(eq[i]);od;for i from 1 to N do eq[N+1+i]:=eval(eq[N+1+i]);od;

$$eq_1 := \frac{d}{d\zeta} u_1(\zeta) = u_6(\zeta)$$

$$eq_2 := \frac{d}{d\zeta} u_2(\zeta) = u_7(\zeta)$$

$$eq_3 := \frac{d}{d\zeta} u_3(\zeta) = u_8(\zeta)$$

$$eq_4 := \frac{d}{d\zeta} u_4(\zeta) = u_9(\zeta)$$

$$eq_6 := \frac{d}{d\zeta} u_6(\zeta) = 2 u_1(\zeta) - u_2(\zeta)$$

$$eq_7 := \frac{d}{d\zeta} u_7(\zeta) = -u_1(\zeta) + 2 u_2(\zeta) - u_3(\zeta)$$

$$eq_8 := \frac{d}{d\zeta} u_8(\zeta) = -u_2(\zeta) + 2 u_3(\zeta) - u_4(\zeta)$$

$$eq_9 := \frac{d}{d\zeta} u_9(\zeta) = -u_3(\zeta) + 2 u_4(\zeta)$$

> eqns:=[seq(rhs(eq[j]),j=1..N),seq(rhs(eq[N+1+j]),j=1..N)];

eqns := [u<sub>6</sub>(ζ), u<sub>7</sub>(ζ), u<sub>8</sub>(ζ), u<sub>9</sub>(ζ), 2 u<sub>1</sub>(ζ) - u<sub>2</sub>(ζ), -u<sub>1</sub>(ζ) + 2 u<sub>2</sub>(ζ) - u<sub>3</sub>(ζ),  
-u<sub>2</sub>(ζ) + 2 u<sub>3</sub>(ζ) - u<sub>4</sub>(ζ), -u<sub>3</sub>(ζ) + 2 u<sub>4</sub>(ζ)]

> Y:=[seq(u[i](zeta),i=1..N),seq(u[N+1+i](zeta),i=1..N)];

Y := [u<sub>1</sub>(ζ), u<sub>2</sub>(ζ), u<sub>3</sub>(ζ), u<sub>4</sub>(ζ), u<sub>6</sub>(ζ), u<sub>7</sub>(ζ), u<sub>8</sub>(ζ), u<sub>9</sub>(ζ)]

> A:=genmatrix(eqns,Y,'bl');

$$A := \begin{bmatrix} 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 2 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 2 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 2 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & -1 & 2 & 0 & 0 & 0 & 0 \end{bmatrix}$$

> A:=map(evalf,A):

> b:=matrix(2\*N,1):for i from 1 to 2\*N do

```
b[i,1]:=-b1[i];od:evalm(b);
```

$$\begin{bmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{bmatrix}$$

```
> h:=eval(1/(N+1));
```

$$h := \frac{1}{5}$$

```
> J:=jordan(A,S);
```

$$J := \begin{bmatrix} -1.902113033, 0, 0, 0, 0, 0, 0, 0, 0 \\ 0, -1.618033989, 0, 0, 0, 0, 0, 0, 0 \\ 0, 0, -1.175570505, 0, 0, 0, 0, 0, 0 \\ 0, 0, 0, -0.6180339886, 0, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0.6180339886, 0, 0, 0, 0 \\ 0, 0, 0, 0, 0, 1.175570505, 0, 0, 0 \\ 0, 0, 0, 0, 0, 0, 1.618033989, 0, 0 \\ 0, 0, 0, 0, 0, 0, 0, 1.902113033, 0 \end{bmatrix}$$

```
> mat:=evalm(S*exponential(J,zeta)*inverse(S));
```

```
> mat:=map(convert,mat,trig):
```

```
> mat:=map(combine,mat,trig):
```

```
> mat:=map(eval,mat):
```

```
> mat:=map(simplify,mat);
```

*mat* :=

$$\begin{aligned} & [0.1381965986 \cosh(1.902113033 \zeta) + 0.3900000000 10^{-9} \sinh(1.902113033 \zeta) \\ & + 0.3618033877 \cosh(1.618033989 \zeta) + 0.1090000000 10^{-7} \sinh(1.618033989 \zeta) \\ & + 0.3618034110 \cosh(1.175570505 \zeta) - 0.1380000000 10^{-7} \sinh(1.175570505 \zeta) \\ & + 0.1381965961 \cosh(0.6180339886 \zeta) + 0.9720000000 10^{-8} \sinh(0.6180339886 \zeta), \\ & -0.2236068000 \cosh(1.902113033 \zeta) + 0.2000000000 10^{-8} \sinh(1.902113033 \zeta) \\ & - 0.2236067869 \cosh(1.618033989 \zeta) - 0.1470000000 10^{-7} \sinh(1.618033989 \zeta) \\ & + 0.2236067821 \cosh(1.175570505 \zeta) + 0.2070000000 10^{-7} \sinh(1.175570505 \zeta) \\ & + 0.2236068074 \cosh(0.6180339886 \zeta) - 0.1080000000 10^{-7} \sinh(0.6180339886 \zeta), \\ & 0.2236067980 \cosh(1.902113033 \zeta) + 0.4000000000 10^{-9} \sinh(1.902113033 \zeta) \\ & - 0.2236067986 \cosh(1.618033989 \zeta) + 0.1000000000 10^{-8} \sinh(1.618033989 \zeta) \\ & - 0.2236068063 \cosh(1.175570505 \zeta) + 0.5300000000 10^{-8} \sinh(1.175570505 \zeta) \\ & + 0.2236067996 \cosh(0.6180339886 \zeta) + 0.2000000000 10^{-9} \sinh(0.6180339886 \zeta), \end{aligned}$$

$$\begin{aligned}
& -0.1381965984 \cosh(1.902113033 \zeta) + 0.1590000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.3618034007 \cosh(1.618033989 \zeta) - 0.4700000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.3618033890 \cosh(1.175570505 \zeta) - 0.1280000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.1381966005 \cosh(0.6180339886 \zeta) + 0.2460000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) , \\
& -0.4500000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.07265425003 \sinh(1.902113033 \zeta) \\
& - 0.5200000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.2236067992 \sinh(1.618033989 \zeta) \\
& + 0.3900000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.3077683499 \sinh(1.175570505 \zeta) \\
& - 0.3000000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.2236068050 \sinh(0.6180339886 \zeta) , \\
& -0.1420000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.1175570516 \sinh(1.902113033 \zeta) \\
& + 0.9480000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966176 \sinh(1.618033989 \zeta) \\
& - 0.1305000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.1902113213 \sinh(1.175570505 \zeta) \\
& + 0.5600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618033968 \sinh(0.6180339886 \zeta) , \\
& -0.6000000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.1175570502 \sinh(1.902113033 \zeta) \\
& + 0.2890000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966092 \sinh(1.618033989 \zeta) \\
& - 0.6470000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1902112984 \sinh(1.175570505 \zeta) \\
& + 0.1600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618034020 \sinh(0.6180339886 \zeta) , \\
& -0.1130000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.07265424917 \sinh(1.902113033 \zeta) \\
& + 0.4500000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.2236067915 \sinh(1.618033989 \zeta) \\
& + 0.6900000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.3077683607 \sinh(1.175570505 \zeta) \\
& - 0.8000000000 \cdot 10^{-9} \cosh(0.6180339886 \zeta) + 0.2236068044 \sinh(0.6180339886 \zeta) ] \\
& [-0.2236067966 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& - 0.2236067828 \cosh(1.618033989 \zeta) - 0.1460000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.2236067765 \cosh(1.175570505 \zeta) + 0.2070000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.2236068056 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) , \\
& 0.3618033990 \cosh(1.902113033 \zeta) + 0.8000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& + 0.1381965918 \cosh(1.618033989 \zeta) + 0.1208000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.1381966020 \cosh(1.175570505 \zeta) - 0.8590000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618033933 \cosh(0.6180339886 \zeta) + 0.1010000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) , \\
& -0.3618034042 \cosh(1.902113033 \zeta) + 0.3600000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.1381966195 \cosh(1.618033989 \zeta) - 0.1948000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& - 0.1381966032 \cosh(1.175570505 \zeta) + 0.7910000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618034043 \cosh(0.6180339886 \zeta) - 0.8500000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) , \\
& 0.2236067906 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta)
\end{aligned}$$

$$\begin{aligned}
& -0.2236067963 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& -0.2236068005 \cosh(1.175570505 \zeta) + 0.5300000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.2236067986 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta) , \\
& -0.1420000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.1175570440 \sinh(1.902113033 \zeta) \\
& + 0.9490000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966082 \sinh(1.618033989 \zeta) \\
& - 0.1305000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.1902113151 \sinh(1.175570505 \zeta) \\
& + 0.5600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618033924 \sinh(0.6180339886 \zeta) , \\
& -0.1050000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.1902113088 \sinh(1.902113033 \zeta) \\
& - 0.2270000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.08541020339 \sinh(1.618033989 \zeta) \\
& - 0.2660000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.1175570492 \sinh(1.175570505 \zeta) \\
& - 0.1200000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.5854102040 \sinh(0.6180339886 \zeta) , \\
& -0.2530000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.1902112996 \sinh(1.902113033 \zeta) \\
& + 0.1389000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.08541018539 \sinh(1.618033989 \zeta) \\
& - 0.6000000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1175570404 \sinh(1.175570505 \zeta) \\
& + 0.4800000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.5854101938 \sinh(0.6180339886 \zeta) , \\
& -0.6000000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.1175570472 \sinh(1.902113033 \zeta) \\
& + 0.2900000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966025 \sinh(1.618033989 \zeta) \\
& - 0.6470000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1902112925 \sinh(1.175570505 \zeta) \\
& + 0.1600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618034034 \sinh(0.6180339886 \zeta) ] \\
& [0.2236067956 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& - 0.2236067997 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.2236068001 \cosh(1.175570505 \zeta) + 0.5300000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.2236067966 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta) , \\
& -0.3618034056 \cosh(1.902113033 \zeta) + 0.3600000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.1381966227 \cosh(1.618033989 \zeta) - 0.1948000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& - 0.1381966028 \cosh(1.175570505 \zeta) + 0.7910000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618034021 \cosh(0.6180339886 \zeta) - 0.8500000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) , \\
& 0.3618034022 \cosh(1.902113033 \zeta) + 0.8000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& + 0.1381965915 \cosh(1.618033989 \zeta) + 0.1208000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.1381966023 \cosh(1.175570505 \zeta) - 0.8590000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618033903 \cosh(0.6180339886 \zeta) + 0.1010000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) , \\
& -0.2236067944 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& - 0.2236067842 \cosh(1.618033989 \zeta) - 0.1460000000 \cdot 10^{-7} \sinh(1.618033989 \zeta)
\end{aligned}$$

$$\begin{aligned}
& + 0.2236067769 \cosh(1.175570505 \zeta) + 0.2070000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.2236068044 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta), \\
& - 0.6000000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.1175570468 \sinh(1.902113033 \zeta) \\
& + 0.2900000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966024 \sinh(1.618033989 \zeta) \\
& - 0.6470000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1902112940 \sinh(1.175570505 \zeta) \\
& + 0.1600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618033954 \sinh(0.6180339886 \zeta), \\
& - 0.2530000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.1902113064 \sinh(1.902113033 \zeta) \\
& + 0.1390000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.08541018702 \sinh(1.618033989 \zeta) \\
& - 0.6000000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1175570415 \sinh(1.175570505 \zeta) \\
& + 0.4800000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.5854101902 \sinh(0.6180339886 \zeta), \\
& - 0.1050000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.1902113044 \sinh(1.902113033 \zeta) \\
& - 0.2270000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.08541020359 \sinh(1.618033989 \zeta) \\
& - 0.2660000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.1175570481 \sinh(1.175570505 \zeta) \\
& - 0.1200000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.5854101992 \sinh(0.6180339886 \zeta), \\
& - 0.1420000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.1175570459 \sinh(1.902113033 \zeta) \\
& + 0.9490000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966112 \sinh(1.618033989 \zeta) \\
& - 0.1305000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.1902113136 \sinh(1.175570505 \zeta) \\
& + 0.5600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618033952 \sinh(0.6180339886 \zeta) ] \\
& [-0.1381966021 \cosh(1.902113033 \zeta) + 0.1590000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.3618034037 \cosh(1.618033989 \zeta) - 0.4700000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.3618033884 \cosh(1.175570505 \zeta) - 0.1280000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.1381966006 \cosh(0.6180339886 \zeta) + 0.2460000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta), \\
& 0.2236068000 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& - 0.2236068021 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.2236068057 \cosh(1.175570505 \zeta) + 0.5300000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.2236068006 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta), \\
& - 0.2236068030 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& - 0.2236067848 \cosh(1.618033989 \zeta) - 0.1460000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.2236067827 \cosh(1.175570505 \zeta) + 0.2070000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.2236068076 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta), \\
& 0.1381965979 \cosh(1.902113033 \zeta) + 0.3900000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& + 0.3618033871 \cosh(1.618033989 \zeta) + 0.1090000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.3618034116 \cosh(1.175570505 \zeta) - 0.1380000000 \cdot 10^{-7} \sinh(1.175570505 \zeta)
\end{aligned}$$

$$\begin{aligned}
& + 0.1381965967 \cosh(0.6180339886 \zeta) + 0.9720000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) , \\
& - 0.1130000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.07265424925 \sinh(1.902113033 \zeta) \\
& + 0.4500000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.2236067899 \sinh(1.618033989 \zeta) \\
& + 0.6900000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.3077683631 \sinh(1.175570505 \zeta) \\
& - 0.8000000000 \cdot 10^{-9} \cosh(0.6180339886 \zeta) + 0.2236068018 \sinh(0.6180339886 \zeta) , \\
& - 0.6000000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.1175570549 \sinh(1.902113033 \zeta) \\
& + 0.2880000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966108 \sinh(1.618033989 \zeta) \\
& - 0.6470000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1902113002 \sinh(1.175570505 \zeta) \\
& + 0.1600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618034034 \sinh(0.6180339886 \zeta) , \\
& - 0.1420000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.1175570495 \sinh(1.902113033 \zeta) \\
& + 0.9480000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.1381966169 \sinh(1.618033989 \zeta) \\
& - 0.1305000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.1902113195 \sinh(1.175570505 \zeta) \\
& + 0.5600000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.3618033974 \sinh(0.6180339886 \zeta) , \\
& - 0.4500000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.07265425155 \sinh(1.902113033 \zeta) \\
& - 0.5200000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.2236068024 \sinh(1.618033989 \zeta) \\
& + 0.3900000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.3077683475 \sinh(1.175570505 \zeta) \\
& - 0.3000000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.2236068088 \sinh(0.6180339886 \zeta) ] \\
& [ 0.5000000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.2628655553 \sinh(1.902113033 \zeta) \\
& - 0.1980000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.5854102116 \sinh(1.618033989 \zeta) \\
& + 0.2050000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.4253253855 \sinh(1.175570505 \zeta) \\
& - 0.1125000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.08541021735 \sinh(0.6180339886 \zeta) , \\
& - 0.1700000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.4253254089 \sinh(1.902113033 \zeta) \\
& + 0.2120000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) - 0.3618034236 \sinh(1.618033989 \zeta) \\
& - 0.2330000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.2628655919 \sinh(1.175570505 \zeta) \\
& + 0.1230000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.1381965885 \sinh(0.6180339886 \zeta) , \\
& 0.1300000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.4253254103 \sinh(1.902113033 \zeta) \\
& - 0.8200000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.3618033874 \sinh(1.618033989 \zeta) \\
& - 0.6900000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.2628655585 \sinh(1.175570505 \zeta) \\
& - 0.1440000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.1381966096 \sinh(0.6180339886 \zeta) , \\
& - 0.1700000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.2628655499 \sinh(1.902113033 \zeta) \\
& + 0.5900000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.5854101815 \sinh(1.618033989 \zeta) \\
& + 0.2050000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) - 0.4253254341 \sinh(1.175570505 \zeta) \\
& - 0.3170000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.08541020393 \sinh(0.6180339886 \zeta) ,
\end{aligned}$$



$$\begin{aligned}
& 0.1381965966 \cosh(1.902113033 \zeta) + 0.3900000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& + 0.3618033793 \cosh(1.618033989 \zeta) + 0.1090000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.3618034168 \cosh(1.175570505 \zeta) - 0.1380000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.1381966008 \cosh(0.6180339886 \zeta) + 0.9720000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) , \\
& -0.2236068078 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& - 0.2236067928 \cosh(1.618033989 \zeta) - 0.1460000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.2236067873 \cosh(1.175570505 \zeta) + 0.2070000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.2236068158 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) , \\
& 0.2236067988 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& - 0.2236068055 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.2236068089 \cosh(1.175570505 \zeta) + 0.5300000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.2236068082 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta) , \\
& -0.1381965999 \cosh(1.902113033 \zeta) + 0.1590000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.3618033981 \cosh(1.618033989 \zeta) - 0.4700000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.3618033914 \cosh(1.175570505 \zeta) - 0.1280000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.1381966070 \cosh(0.6180339886 \zeta) + 0.2460000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) ] \\
& [-0.1700000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.4253253967 \sinh(1.902113033 \zeta) \\
& + 0.2120000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) - 0.3618034186 \sinh(1.618033989 \zeta) \\
& - 0.2340000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.2628655776 \sinh(1.175570505 \zeta) \\
& + 0.1230000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.1381965779 \sinh(0.6180339886 \zeta) , \\
& 0.1800000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.6881909602 \sinh(1.902113033 \zeta) \\
& - 0.2790000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.2236068295 \sinh(1.618033989 \zeta) \\
& + 0.1365000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.1624598194 \sinh(1.175570505 \zeta) \\
& - 0.1280000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.2236068100 \sinh(0.6180339886 \zeta) , \\
& -0.3400000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.6881909602 \sinh(1.902113033 \zeta) \\
& + 0.2720000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.2236067676 \sinh(1.618033989 \zeta) \\
& - 0.2810000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1624598333 \sinh(1.175570505 \zeta) \\
& + 0.9000000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.2236067748 \sinh(0.6180339886 \zeta) , \\
& 0.1300000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.4253253903 \sinh(1.902113033 \zeta) \\
& - 0.8200000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.3618033850 \sinh(1.618033989 \zeta) \\
& - 0.6800000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.2628655440 \sinh(1.175570505 \zeta) \\
& - 0.1440000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.1381965994 \sinh(0.6180339886 \zeta) , \\
& -0.2236067902 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta)
\end{aligned}$$

$$\begin{aligned}
& - 0.2236067786 \cosh(1.618033989 \zeta) - 0.1460000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.2236067736 \cosh(1.175570505 \zeta) + 0.2060000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.2236067978 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) , \\
& 0.3618034066 \cosh(1.902113033 \zeta) + 0.8000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& + 0.1381965960 \cosh(1.618033989 \zeta) + 0.1210000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.1381966012 \cosh(1.175570505 \zeta) - 0.8640000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618033821 \cosh(0.6180339886 \zeta) + 0.1010000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) , \\
& -0.3618034004 \cosh(1.902113033 \zeta) + 0.3600000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.1381966243 \cosh(1.618033989 \zeta) - 0.1946000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& - 0.1381966008 \cosh(1.175570505 \zeta) + 0.7970000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618033931 \cosh(0.6180339886 \zeta) - 0.8500000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta) , \\
& 0.2236067900 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& - 0.2236067955 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.2236067955 \cosh(1.175570505 \zeta) + 0.5300000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.2236067936 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta) ] \\
& [ 0.1300000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.4253253989 \sinh(1.902113033 \zeta) \\
& - 0.8200000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.3618033908 \sinh(1.618033989 \zeta) \\
& - 0.6800000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.2628655434 \sinh(1.175570505 \zeta) \\
& - 0.1440000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.1381966032 \sinh(0.6180339886 \zeta) , \\
& -0.3400000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.6881909614 \sinh(1.902113033 \zeta) \\
& + 0.2720000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.2236067730 \sinh(1.618033989 \zeta) \\
& - 0.2810000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.1624598328 \sinh(1.175570505 \zeta) \\
& + 0.9000000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.2236067816 \sinh(0.6180339886 \zeta) , \\
& 0.1800000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.6881909648 \sinh(1.902113033 \zeta) \\
& - 0.2790000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.2236068293 \sinh(1.618033989 \zeta) \\
& + 0.1365000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.1624598199 \sinh(1.175570505 \zeta) \\
& - 0.1280000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.2236068162 \sinh(0.6180339886 \zeta) , \\
& -0.1700000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.4253253917 \sinh(1.902113033 \zeta) \\
& + 0.2120000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) - 0.3618034208 \sinh(1.618033989 \zeta) \\
& - 0.2340000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.2628655782 \sinh(1.175570505 \zeta) \\
& + 0.1230000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.1381965821 \sinh(0.6180339886 \zeta) , \\
& 0.2236067888 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& - 0.2236067955 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta)
\end{aligned}$$

$$\begin{aligned}
& -0.2236067972 \cosh(1.175570505 \zeta) + 0.5400000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.2236067968 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta), \\
& -0.3618034126 \cosh(1.902113033 \zeta) + 0.3600000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.1381966270 \cosh(1.618033989 \zeta) - 0.1947000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& - 0.1381966021 \cosh(1.175570505 \zeta) + 0.7960000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618034041 \cosh(0.6180339886 \zeta) - 0.8500000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta), \\
& 0.3618033974 \cosh(1.902113033 \zeta) + 0.8000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\
& + 0.1381965964 \cosh(1.618033989 \zeta) + 0.1210000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.1381965999 \cosh(1.175570505 \zeta) - 0.8650000000 \cdot 10^{-8} \sinh(1.175570505 \zeta) \\
& + 0.3618033921 \cosh(0.6180339886 \zeta) + 0.1010000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta), \\
& -0.2236067934 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& - 0.2236067835 \cosh(1.618033989 \zeta) - 0.1470000000 \cdot 10^{-7} \sinh(1.618033989 \zeta) \\
& + 0.2236067719 \cosh(1.175570505 \zeta) + 0.2070000000 \cdot 10^{-7} \sinh(1.175570505 \zeta) \\
& + 0.2236068074 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta) ] \\
& [-0.1700000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.2628655523 \sinh(1.902113033 \zeta) \\
& + 0.5900000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) + 0.5854101893 \sinh(1.618033989 \zeta) \\
& + 0.2050000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) - 0.4253254273 \sinh(1.175570505 \zeta) \\
& - 0.3170000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.08541020275 \sinh(0.6180339886 \zeta), \\
& 0.1300000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) + 0.4253254063 \sinh(1.902113033 \zeta) \\
& - 0.8200000000 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.3618033950 \sinh(1.618033989 \zeta) \\
& - 0.6900000000 \cdot 10^{-8} \cosh(1.175570505 \zeta) - 0.2628655539 \sinh(1.175570505 \zeta) \\
& - 0.1440000000 \cdot 10^{-8} \cosh(0.6180339886 \zeta) + 0.1381966081 \sinh(0.6180339886 \zeta), \\
& -0.1700000000 \cdot 10^{-8} \cosh(1.902113033 \zeta) - 0.4253254069 \sinh(1.902113033 \zeta) \\
& + 0.2120000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) - 0.3618034222 \sinh(1.618033989 \zeta) \\
& - 0.2330000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.2628655889 \sinh(1.175570505 \zeta) \\
& + 0.1230000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.1381965867 \sinh(0.6180339886 \zeta), \\
& 0.5000000000 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.2628655493 \sinh(1.902113033 \zeta) \\
& - 0.1980000000 \cdot 10^{-7} \cosh(1.618033989 \zeta) + 0.5854102138 \sinh(1.618033989 \zeta) \\
& + 0.2050000000 \cdot 10^{-7} \cosh(1.175570505 \zeta) + 0.4253253803 \sinh(1.175570505 \zeta) \\
& - 0.1125000000 \cdot 10^{-7} \cosh(0.6180339886 \zeta) + 0.08541021653 \sinh(0.6180339886 \zeta), \\
& -0.1381965976 \cosh(1.902113033 \zeta) + 0.1590000000 \cdot 10^{-8} \sinh(1.902113033 \zeta) \\
& + 0.3618033971 \cosh(1.618033989 \zeta) - 0.4700000000 \cdot 10^{-8} \sinh(1.618033989 \zeta) \\
& - 0.3618033892 \cosh(1.175570505 \zeta) - 0.1280000000 \cdot 10^{-7} \sinh(1.175570505 \zeta)
\end{aligned}$$

$+ 0.1381966033 \cosh(0.6180339886 \zeta) + 0.2460000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta),$   
 $0.2236068038 \cosh(1.902113033 \zeta) + 0.4000000000 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $- 0.2236068093 \cosh(1.618033989 \zeta) + 0.1100000000 \cdot 10^{-8} \sinh(1.618033989 \zeta)$   
 $- 0.2236068079 \cosh(1.175570505 \zeta) + 0.5300000000 \cdot 10^{-8} \sinh(1.175570505 \zeta)$   
 $+ 0.2236068058 \cosh(0.6180339886 \zeta) + 0.2000000000 \cdot 10^{-9} \sinh(0.6180339886 \zeta),$   
 $-0.2236067996 \cosh(1.902113033 \zeta) + 0.2000000000 \cdot 10^{-8} \sinh(1.902113033 \zeta)$   
 $- 0.2236067928 \cosh(1.618033989 \zeta) - 0.1460000000 \cdot 10^{-7} \sinh(1.618033989 \zeta)$   
 $+ 0.2236067821 \cosh(1.175570505 \zeta) + 0.2070000000 \cdot 10^{-7} \sinh(1.175570505 \zeta)$   
 $+ 0.2236068130 \cosh(0.6180339886 \zeta) - 0.1080000000 \cdot 10^{-7} \sinh(0.6180339886 \zeta),$   
 $0.1381965970 \cosh(1.902113033 \zeta) + 0.3900000000 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $+ 0.3618033863 \cosh(1.618033989 \zeta) + 0.1090000000 \cdot 10^{-7} \sinh(1.618033989 \zeta)$   
 $+ 0.3618034090 \cosh(1.175570505 \zeta) - 0.1380000000 \cdot 10^{-7} \sinh(1.175570505 \zeta)$   
 $+ 0.1381966012 \cosh(0.6180339886 \zeta) + 0.9720000000 \cdot 10^{-8} \sinh(0.6180339886 \zeta)]$

[ > **mat1:=evalm(subs(zeta=zeta-zeta1,evalm(mat))):**

[ > **b2:=evalm(subs(zeta=zeta1,evalm(b))):**

[ > **mat2:=evalm(mat1\*b2):**

[ > **mat2:=map(expand,mat2):**

[ > **mat2:=map(convert,mat2,trig):**

[ > **mat3:=map(int,mat2,zeta1=0..zeta):**

[ > **mat3:=map(convert,mat3,trig):**

[ > **Y0:=matrix(2\*N,1);**

$Y0 := \text{array}(1 .. 8, 1 .. 1, [ \ ])$

[ > **for i to N do Y0[i,1]:=p[i];od:**

[ > **for i to N do Y0[N+i,1]:=c[i];od:**

[ > **evalm(Y0);**

$$\begin{bmatrix} p_1 \\ p_2 \\ p_3 \\ p_4 \\ c_1 \\ c_2 \\ c_3 \\ c_4 \end{bmatrix}$$

[ > **Y:=evalm(mat\*Y0+mat3):**

[ > **sol0:=map(eval,evalm(subs(zeta=0,evalm(Y)))):**

[ > **sol1:=map(eval,evalm(subs(zeta=epsilon/h,evalm(Y)))):**

[ > **for i to N do**

```
Eq[i]:=subs(diff(u(x,y),y)=c[i],u(x,y)=p[i],x=i*h,bc3);od;
```

$$Eq_1 := p_1$$

$$Eq_2 := p_2$$

$$Eq_3 := p_3$$

$$Eq_4 := p_4$$

```
> for i to N do
```

```
Eq[N+i]:=evalf(subs(diff(u(x,y),y)=sol1[N+i,1],u(x,y)=sol1[i,1],bc4));od;
```

$$Eq_5 := 1589.196278 p_1 - 1831.875702 p_2 + 1107.341393 p_3 - 405.8619811 p_4 + 912.5999059 c_1 - 981.0963000 c_2 + 538.2018271 c_3 - 178.2356315 c_4 - 1.$$

$$Eq_6 := -1831.875673 p_1 + 2696.537690 p_2 - 2237.737714 p_3 + 1107.341348 p_4 - 981.0962341 c_1 + 1450.801813 c_2 - 1159.331904 c_3 + 538.2018189 c_4 - 1.$$

$$Eq_7 := 1107.341376 p_1 - 2237.737719 p_2 + 2696.537712 p_3 - 1831.875660 p_4 + 538.2018160 c_1 - 1159.331948 c_2 + 1450.801783 c_3 - 981.0962524 c_4 - 1.$$

$$Eq_8 := -405.8620011 p_1 + 1107.341401 p_2 - 1831.875718 p_3 + 1589.196273 p_4 - 178.2356351 c_1 + 538.2018560 c_2 - 981.0962853 c_3 + 912.5999210 c_4 - 1.$$

```
> csol:=solve({seq(Eq[i],i=1..2*N)},{seq(c[i],i=1..N),seq(p[i],i=1..N)});
```

$$csol := \{c_1 = 0.04104983789, c_2 = 0.06580695264, c_3 = 0.06580695484, c_4 = 0.04104983789, p_1 = 0., p_2 = 0., p_3 = 0., p_4 = 0.\}$$

```
> assign(csol);
```

```
> Y:=map(eval,Y):
```

```
> Y:=map(convert,Y,trig):
```

```
> Y:=map(combine,Y,trig);
```

```
Y:=
```

$$\begin{aligned} & [0.3178206785 \cdot 10^{-9} \cosh(0.6180339886 \zeta) + 0.06597640531 \sinh(0.6180339886 \zeta) \\ & - 0.1977887895 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.203 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\ & + 0.7852971240 \cdot 10^{-9} \cosh(1.618033989 \zeta) + 0.000169449110 \sinh(1.618033989 \zeta) \\ & - 0.8412134806 \cdot 10^{-9} \cosh(1.175570505 \zeta) + 0.65 \cdot 10^{-9} \sinh(1.175570505 \zeta)] \end{aligned}$$

$$\begin{aligned} & [0.5324638728 \cdot 10^{-9} \cosh(0.6180339886 \zeta) + 0.1067520654 \sinh(0.6180339886 \zeta) \\ & - 0.3185095685 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.32 \cdot 10^{-9} \sinh(1.902113033 \zeta) \\ & + 0.1273284312 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.000104727072 \sinh(1.618033989 \zeta) \\ & - 0.1371181059 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.1249 \cdot 10^{-8} \sinh(1.175570505 \zeta)] \end{aligned}$$

$$\begin{aligned} & [0.5324638597 \cdot 10^{-9} \cosh(0.6180339886 \zeta) + 0.1067520646 \sinh(0.6180339886 \zeta) \\ & - 0.3185095653 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.33 \cdot 10^{-9} \sinh(1.902113033 \zeta) \end{aligned}$$

$+ 0.1273942346 \cdot 10^{-8} \cosh(1.618033989 \zeta) - 0.000104727069 \sinh(1.618033989 \zeta)$   
 $- 0.1371181051 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.1494 \cdot 10^{-8} \sinh(1.175570505 \zeta)$   
 $[0.3178206873 \cdot 10^{-9} \cosh(0.6180339886 \zeta) + 0.06597640548 \sinh(0.6180339886 \zeta)$   
 $- 0.1977887914 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.190 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $+ 0.7846390690 \cdot 10^{-9} \cosh(1.618033989 \zeta) + 0.000169449117 \sinh(1.618033989 \zeta)$   
 $- 0.8412134952 \cdot 10^{-9} \cosh(1.175570505 \zeta) + 0.105 \cdot 10^{-8} \sinh(1.175570505 \zeta)]$   
 $[0.04077566264 \cosh(0.6180339886 \zeta) - 0.1975666720 \cdot 10^{-9} \sinh(0.6180339886 \zeta)$   
 $- 0.233 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.2392153663 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $+ 0.00027417634 \cosh(1.618033989 \zeta) - 0.6338848633 \cdot 10^{-9} \sinh(1.618033989 \zeta)$   
 $- 0.88 \cdot 10^{-9} \cosh(1.175570505 \zeta) + 0.6190550931 \cdot 10^{-9} \sinh(1.175570505 \zeta)]$   
 $[0.06597640301 \cosh(0.6180339886 \zeta) - 0.3298371760 \cdot 10^{-9} \sinh(0.6180339886 \zeta)$   
 $- 0.399 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.3880702105 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $- 0.000169448858 \cosh(1.618033989 \zeta) - 0.1038512025 \cdot 10^{-8} \sinh(1.618033989 \zeta)$   
 $- 0.1177 \cdot 10^{-8} \cosh(1.175570505 \zeta) + 0.1019100161 \cdot 10^{-8} \sinh(1.175570505 \zeta)]$   
 $[0.06597640492 \cosh(0.6180339886 \zeta) - 0.3298371351 \cdot 10^{-9} \sinh(0.6180339886 \zeta)$   
 $- 0.400 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.3880702043 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $- 0.000169448854 \cosh(1.618033989 \zeta) - 0.1043275009 \cdot 10^{-8} \sinh(1.618033989 \zeta)$   
 $- 0.878 \cdot 10^{-9} \cosh(1.175570505 \zeta) + 0.1025993952 \cdot 10^{-8} \sinh(1.175570505 \zeta)]$   
 $[0.04077566217 \cosh(0.6180339886 \zeta) - 0.1975666963 \cdot 10^{-9} \sinh(0.6180339886 \zeta)$   
 $- 0.236 \cdot 10^{-9} \cosh(1.902113033 \zeta) + 0.2392153699 \cdot 10^{-9} \sinh(1.902113033 \zeta)$   
 $+ 0.00027417633 \cosh(1.618033989 \zeta) - 0.6338848980 \cdot 10^{-9} \sinh(1.618033989 \zeta)$   
 $- 0.39 \cdot 10^{-9} \cosh(1.175570505 \zeta) + 0.6190551260 \cdot 10^{-9} \sinh(1.175570505 \zeta)]$

```

> for i from 1 to N do u[i](zeta):=eval((Y[i,1]));od:
> for i from 0 to N+1 do u[i](zeta):=eval(u[i](zeta));od:
> for i from 0 to N+1 do
  u[i](y):=eval(subs(zeta=epsilon*y/h,u[i](zeta)));od;

```

$$u_0(y) := 0$$

$$\begin{aligned}
u_1(y) := & 0.3178206785 \cdot 10^{-9} \cosh(3.090169943 y) + 0.06597640531 \sinh(3.090169943 y) \\
& - 0.1977887895 \cdot 10^{-9} \cosh(9.510565165 y) + 0.203 \cdot 10^{-9} \sinh(9.510565165 y) \\
& + 0.7852971240 \cdot 10^{-9} \cosh(8.090169945 y) + 0.000169449110 \sinh(8.090169945 y) \\
& - 0.8412134806 \cdot 10^{-9} \cosh(5.877852525 y) + 0.65 \cdot 10^{-9} \sinh(5.877852525 y)
\end{aligned}$$

$$\begin{aligned}
u_2(y) := & 0.5324638728 \cdot 10^{-9} \cosh(3.090169943 y) + 0.1067520654 \sinh(3.090169943 y) \\
& - 0.3185095685 \cdot 10^{-9} \cosh(9.510565165 y) + 0.32 \cdot 10^{-9} \sinh(9.510565165 y) \\
& + 0.1273284312 \cdot 10^{-8} \cosh(8.090169945 y) - 0.000104727072 \sinh(8.090169945 y)
\end{aligned}$$

$$- 0.1371181059 \cdot 10^{-8} \cosh(5.877852525 y) + 0.1249 \cdot 10^{-8} \sinh(5.877852525 y)$$

$$u_3(y) := 0.5324638597 \cdot 10^{-9} \cosh(3.090169943 y) + 0.1067520646 \sinh(3.090169943 y)$$

$$- 0.3185095653 \cdot 10^{-9} \cosh(9.510565165 y) + 0.33 \cdot 10^{-9} \sinh(9.510565165 y)$$

$$+ 0.1273942346 \cdot 10^{-8} \cosh(8.090169945 y) - 0.000104727069 \sinh(8.090169945 y)$$

$$- 0.1371181051 \cdot 10^{-8} \cosh(5.877852525 y) + 0.1494 \cdot 10^{-8} \sinh(5.877852525 y)$$

$$u_4(y) := 0.3178206873 \cdot 10^{-9} \cosh(3.090169943 y) + 0.06597640548 \sinh(3.090169943 y)$$

$$- 0.1977887914 \cdot 10^{-9} \cosh(9.510565165 y) + 0.190 \cdot 10^{-9} \sinh(9.510565165 y)$$

$$+ 0.7846390690 \cdot 10^{-9} \cosh(8.090169945 y) + 0.000169449117 \sinh(8.090169945 y)$$

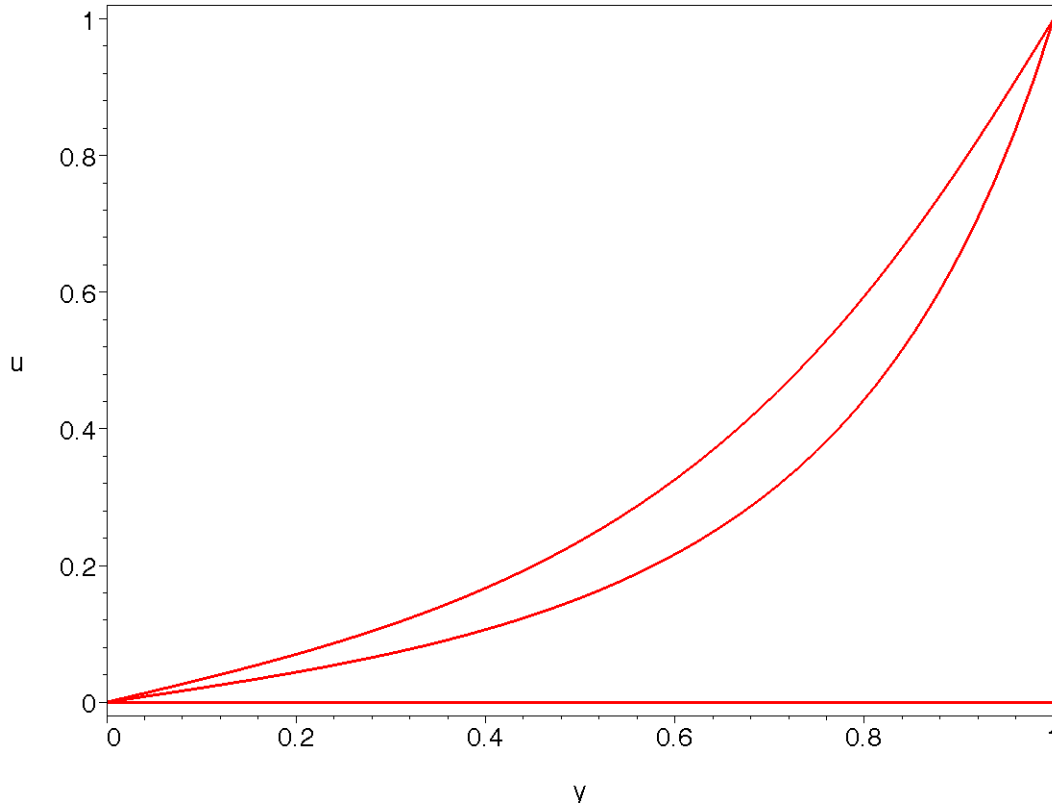
$$- 0.8412134952 \cdot 10^{-9} \cosh(5.877852525 y) + 0.105 \cdot 10^{-8} \sinh(5.877852525 y)$$

$$u_5(y) := 0$$

>

> for i from 0 to N+1 do p[i]:=plot(u[i](y),y=0..1.,thickness=3);od:

> display({seq(p[i],i=0..N+1)},axes=boxed,labels=[y,"u"]);



> M:=20;

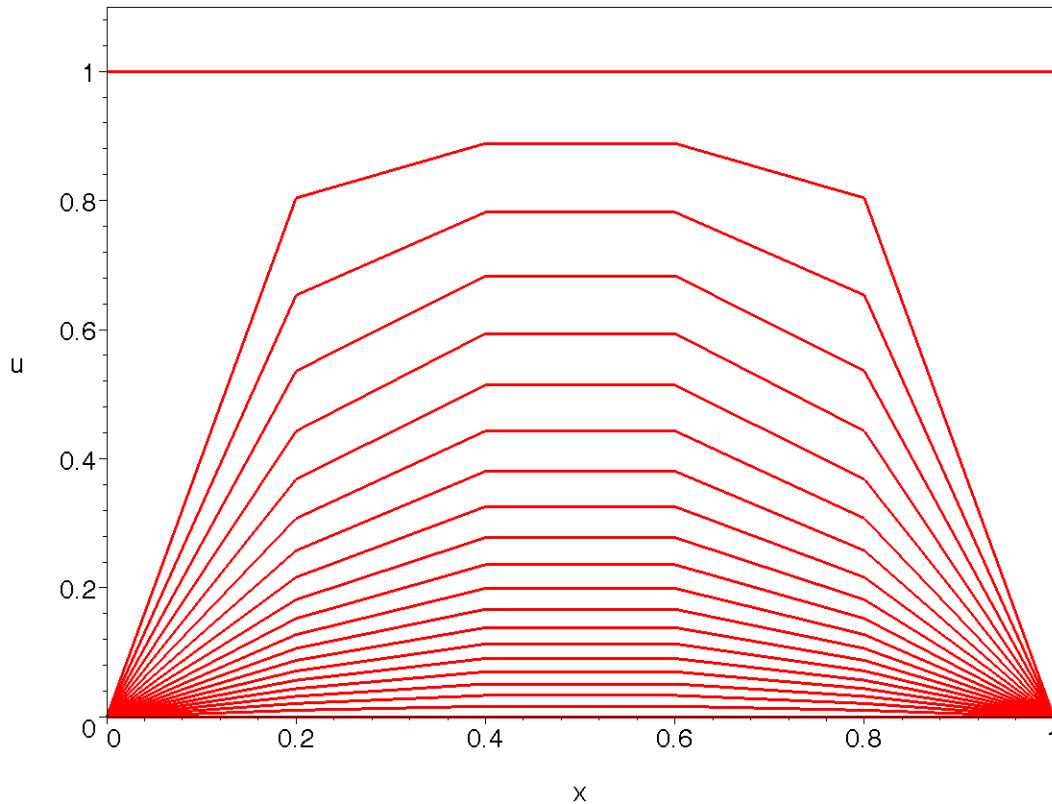
M := 20

> T1:=[seq(evalf(i/M),i=0..M)];

T1 := [0., 0.0500000000, 0.1000000000, 0.1500000000, 0.2000000000, 0.2500000000,  
0.3000000000, 0.3500000000, 0.4000000000, 0.4500000000, 0.5000000000, 0.5500000000,  
0.6000000000, 0.6500000000, 0.7000000000, 0.7500000000, 0.8000000000, 0.8500000000,

```
0.9000000000, 0.9500000000, 1.]
```

```
> P[1]:=plot([seq([h*i,evalf(subs(x=i*h,0))],i=0..N+1)],style=line,thickness=3,axes=boxed):  
> for j from 2 to M do  
  P[j]:=plot([seq([h*i,evalf(subs(y=T1[j],evalf(u[i](y))))],i=0..N+1)],style=line,thickness=3,axes=boxed,view=[0..1,0..1.1]):od:  
> P[M+1]:=plot([seq([h*i,evalf(subs(x=i*h,1))],i=0..N+1)],style=line,thickness=3,axes=boxed):  
> display({seq(P[i],i=1..M+1)},labels=[x,u]);
```



```
> Ny:=30;
```

```
Ny := 30
```

```
> PP:=matrix(N+2,Ny);
```

```
PP := array(1..6, 1..30, [ ])
```

```
> for i to Ny do PP[1,i]:=0;PP[N+2,i]:=0;od:
```

```
> for i to N+2 do PP[i,1]:=0;PP[i,Ny]:=1;od:
```

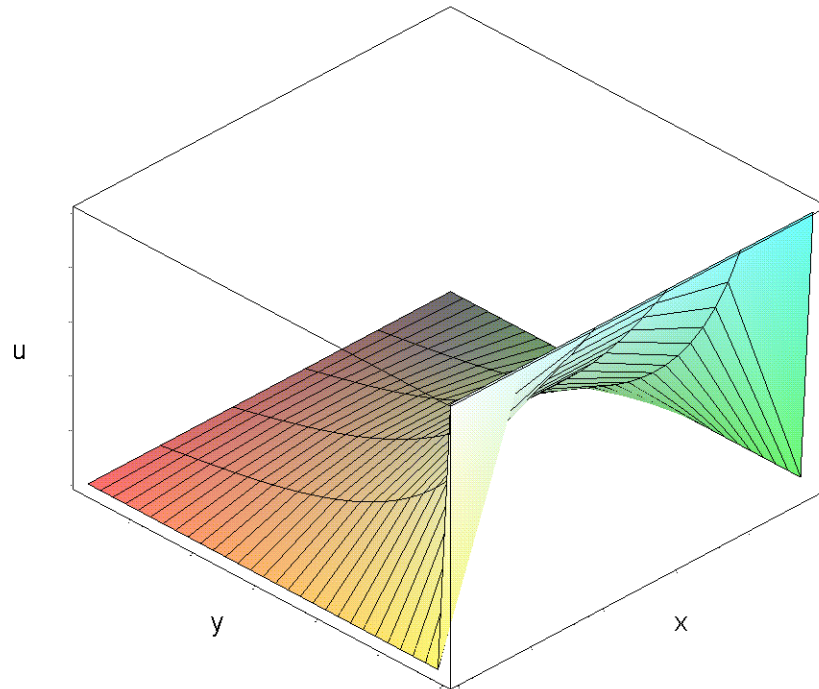
```
> for i from 2 to N+1 do for j from 2 to Ny-1 do
```

```
  PP[i,j]:=evalf(subs(y=(j-1)/(Ny-1),u[i-1](y)));od;od:
```

```
> evalm(PP):
```

```
> matrixplot(PP,style=patch,axes=boxed,axesfont=[TIMES,ROMAN,1],labels=[x,y,u]);
```

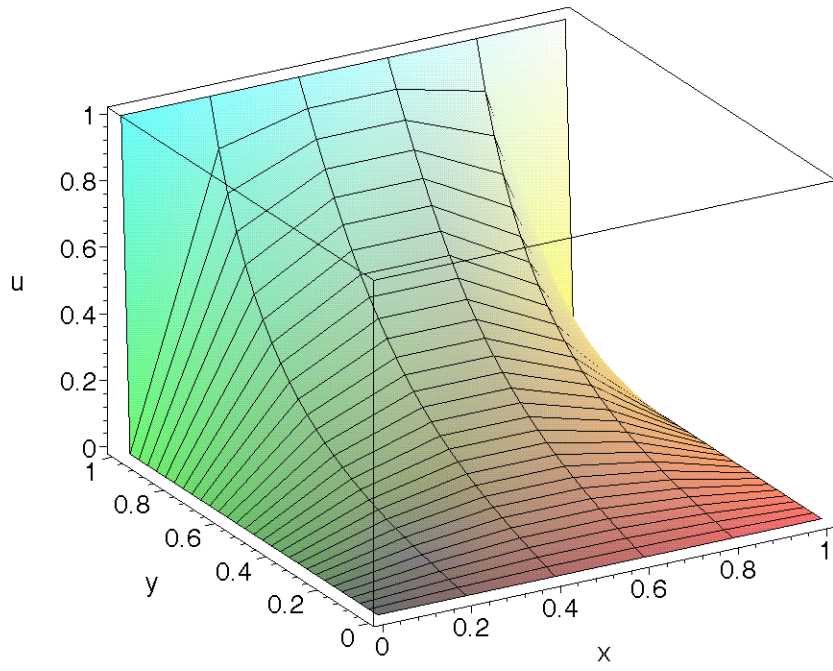




```

> DD:=matrix((N+2)*Ny,3);
                                DD := array(1 .. 180, 1 .. 3, [ ])
> for i to N+2 do for j to Ny do
  DD[i+(j-1)*(N+2),1]:=evalf((i-1)/(N+1));DD[i*j,2]:=evalf((j-1)/(Ny
-1));DD[i*j,3]:=PP[i,j];od:od:
> for i to N+2 do for j to Ny do
  DD[(i-1)*(Ny)+j,1]:=evalf((i-1)/(N+1));DD[(i-1)*(Ny)+j,2]:=evalf((
j-1)/(Ny-1));DD[(i-1)*(Ny)+j,3]:=evalf(PP[i,j]);od;od:
> pd := [seq([DD[i,1],DD[i,2],DD[i,3]], i=1..(N+2)*Ny)]:
> cosdata := [seq([ seq([(i-1)/(N+1),(j-1)/(Ny-1),PP[i,j]],
i=1..N+2)], j=1..Ny)]:
> surfdata( cosdata, axes=boxed,
  labels=[x,y,u],orientation=[-120,60] );

```



[ >