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Variational formulations, instabilities and critical loadings of space curved beams

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A B S T R A C T

Beam theories have been extensively studied for applications in structural engineering. Space curved beams with large displacements, however, have been explored to a much less extent, not to mention explicit solutions concerning instabilities and critical loadings. In this paper, by carefully accounting for geometric nonlinearity and different scalings of kinematic variables, we present a variational framework for large-displacement space curved beams. We show that the variational formulation is consistent with the classic field equations, derive the appropriate boundary value problems for a variety of loading conditions and kinematic constraints, and generalize the Kirchhoff's helical solutions. Explicit planar solutions for semi-circular arches are obtained upon linearization. Further, two nonlinear asymptotic theories are proposed to address ribbon-like and moderately deformed curved beams, respectively. Based on the method of trial solutions, we obtain explicit approximate solutions to critical loadings for semi-circular arches losing stabilities due to twisting and out-of-plane displacement. The variational framework, nonlinear asymptotic theories, stability analysis and explicit solutions are anticipated to have novel applications in stretchable electronics and biological macromolecules.

1. Introduction

Upon making suitable kinematic hypotheses, there are two approaches to formulating an effective theory for lower dimensional elastic bodies. In the field-equation approach, the concept of forces/moments is primitive, the balance laws relating internal forces/moments and external forces/moments are derived by free-body-diagram analysis, and finally, constitutive laws relating internal forces/moments and kinematic variables are postulated to close the system. In the variational approach, the concept of free/strain energy is primitive, and upon postulating the functional dependence of strain energy on kinematic variables, the field equations follow as the Euler–Lagrange equations of the variational principles, e.g., the Hamilton's principle or the principle of minimum free energy. The two approaches shall always yield equivalent, though sometimes not obvious, boundary value problems for self-consistency if the kinematic and constitutive hypotheses are the same in these two approaches.

In this paper we formulate nonlinear variational theories for curved beams, which are motivated by novel applications in stretchable electronics and biological macromolecules. To achieve high electrical performance and mechanical reliability, stretchable electronics have to leverage intrinsically stiff but well established inorganic materials like metal and silicon. A reliable way to build continuous, stretchable structure out of stiff materials is the serpentine design, i.e., meandering ribbons or wires (Fig. 1). When stretched end-to-end, serpentine ribbons or wires can rotate in plane as well as buckle out of plane to accommodate the applied displacement, resulting in greatly reduced local elastic strains and much lower effective stiffness (Li et al., 2005; Su et al., 2012; Widlund et al., 2014; Zhang et al., 2014). These features enable applications ranging from tissue-like bio-integrated electrodes (Kim et al., 2011; Yeo et al., 2013), micro-heaters (Yu et al., 2013), deformable solar cells (Tang et al., 2014), transparent stretchable conductors (Yang et al., 2015), soft nanogenerators (Ma et al., 2013) to deployable sensor networks (Lanzara et al., 2010) and coronary stents (Mani et al., 2007). However, in spite of recent efforts in plane

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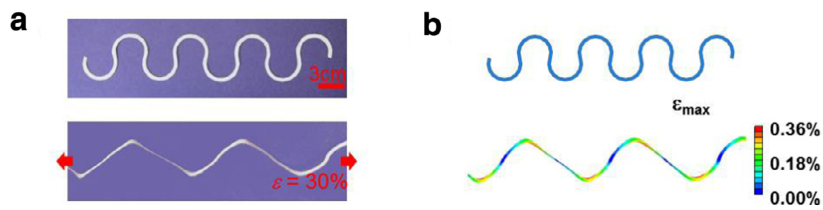

Fig. 1. A serpentine ribbon buckles out of plane when stretched end-to-end: (a) experimental observation of a paper ribbon with 30% end-to-end elongation, and (b) finite element model (FEM) results showing the maximum principle strain in the corresponding ribbon.

strain modeling of freestanding serpentines (Widlund et al., 2014), buckling analysis of thin freestanding serpentine ribbons (Zhang et al., 2013), and analytical and numerical modeling of self-similar serpentines (Su et al., 2015; Zhang et al., 2014), the designs of the serpentine shape are still largely empirical, particularly for serpentines of extreme cross-sectional aspect ratios and undergoing large out-of-plane deformations. A general, preferably variational, framework will be convenient for stability analysis and rational design of high-performance serpentines for stretchable electronics. Meanwhile, it has been a standard practice to model macromolecules such as DNA, polymers and proteins, as an elastic rod for their mechanical behaviors, see e.g. the textbook of Doi and Edwards (1989), review article of Manning (1985), series of works of Zhou et al. (1999, 2000, 2002), and references therein. Though it has been shown that the worm-like-chain (WLC) model, i.e., a uniform circular elastic rod under bending, predicts reasonable force-versus-extension relation of DNA strands beyond a few kilobase-pair range (Smith et al., 1996, 1992). At a lengthscale of tens of base pairs, a more precise description of DNA is necessary to account for the anisotropy, twisting and kinks of DNA structures (Hoffman, 2004; Noy and Golestanian, 2012; Wiggins et al., 2005). Moreover, depending on the salt concentration of the ambient solution, the natural (i.e., stress-free or ground) state of the DNA is not a straight chain, but admits a variety of supercoiling configurations (Manning, 1985). It is of great interest to include effects of charge screening and electrostatic interactions and to carry out statistical mechanics analysis for DNA. These purposes demand a variational framework, i.e., a Hamiltonian in terms of reasonable set of kinematic variables.

Though many of the essential components of a general 3D curved beam theory have been investigated more than 150 years ago in the works of Kirchhoff (Love, 1944), our variational framework accounting for the geometric nonlinearity of large displacements is simple, self-contained and ready for novel applications in stretchable electronics and biological macromolecules. We systematically derive general boundary conditions and find some inconsistency in earlier works. The variational formulation is particularly convenient for rigorous analysis by the direct method of calculus of variations and for investigating beams with extreme cross-sectional aspect ratios (i.e., ribbons). However, the fully nonlinear theory is not prone to explicit solution on one hand, on the other hand, the linearized theory cannot address instabilities due to twisting and out-of-plane displacement. Therefore, we propose some simplified nonlinear theories and explicitly calculate the critical loadings by the method of trial solutions. More accurate solutions on the critical loadings and stabilities of equilibrium states can be achieved by numerical methods.

For classical applications in structural engineering, there are many works on elastic theories of rods in the literature which are too voluminous to recount here. For historical references, the reader may consult Love's treatise (Love, 1944) and Antman's survey (Antman and Truesdell, 1973). As for space curved beams, Reissner (1973); 1981) pioneered a finite strain theory that was later refined by subsequent works of Simo (1985), Simo and Vu-Quoc (1986), and Iura and Atluri (1988, 1989). The numerical aspect of space-curved beam models has been a particularly active research area in the last thirty years with contributions from, e.g., Petrov and Geradin (1998), Ishaquddin et al. (2012), Saje et al. (2012) and references therein. Alternatively, the theory of an elastic rod can be reformulated as a one-dimensional Cosserat or micropolar theory (Cosserat and Cosserat, 1909); kinematic relations and balance laws can be conveniently explored using Clifford or geometric algebra (McRobie and Lasenby, 1999). In this model, each material point admits rotational degrees of freedom represented by a triad of orthonormal vectors in additional to the usual translational degrees of freedom. Some of the fundamental questions such as the existence, uniqueness and stability of a solution may be more conveniently addressed in the Cosserat framework (James, 1981; Steigmann and Faulkner, 1993).

The paper is organized as follows. We begin with the kinematic hypotheses and calculate the strain energy in Section 2.1. We formulate the variational principle and derive the associated Euler–Lagrange equations and boundary conditions in Section 2.2, and find that Kirchhoff's helical solutions can be applied to more general boundary conditions in Section 2.3. In Section 2.4 and 2.5, the geometrically nonlinear theory is linearized and solved for semi-circular arches with clamped supports, simple supports and cantilever. We propose two simplified nonlinear theories in Section 3.1, and obtain explicit solutions to critical loadings in Section 3.2. We conclude and summarize in Section 4. In the Appendix, we show our variational formulation is consistent with the existing field-equation approach.

Notation. We employ direct notation for brevity if possible. Vectors are denoted by bold symbols such as **e, u**, etc. When index notations are in use, the convention of summation over repeated index is followed. The inner (or dot) product of two vectors $\mathbf{a}, \mathbf{b} \in \mathbb{R}^3$ is defined as $\mathbf{a} \cdot \mathbf{b} := (\mathbf{a})_i(\mathbf{b})_i$, and the cross product $(\mathbf{a} \times \mathbf{b})_i := \mathcal{E}_{ijk}(\mathbf{a})_j(\mathbf{b})_k$, where $\mathcal{E}_{ijk}$ is the Levi-Civita symbol.

2. A variational formulation for space curved beams

2.1. Kinematics and strain energy

Consider a curved beam in space as illustrated in Fig. 2. In the reference configuration (Fig. 2(a)), the centroid line of the beam is a space curve with arc-length parameterization given by $\{\mathbf{c}_0(\xi^1) : 0 \le \xi^1 \le L\} \subset \mathbb{R}^3$. For simplicity, we assume the centroid curve remains to be of C^3-class (continuously differentiable up to the third order) with nonzero curvature in Sections 2-3 and postpone our discussion about less regular curves to Section 4. Let

$$\tilde{\mathbf{e}}_1(\xi^1) = \mathbf{c}_0{}'(\xi^1), \quad \tilde{\mathbf{e}}_2(\xi^1) = \frac{\mathbf{c}_0{}''(\xi^1)}{|\mathbf{c}_0{}''(\xi^1)|},$$
$$\tilde{\mathbf{e}}_3(\xi^1) = \tilde{\mathbf{e}}_1(\xi^1) \times \tilde{\mathbf{e}}_2(\xi^1)$$

be the local orthogonal Frenet frame ($' = d/d\xi^1$), and

$$\kappa_0(\xi^1) = \tilde{\mathbf{e}}_2(\xi^1) \cdot \tilde{\mathbf{e}}_1'(\xi^1) \qquad (\text{resp. } \tau_0(\xi^1) = \tilde{\mathbf{e}}_2' \cdot \tilde{\mathbf{e}}_3(\xi^1))$$

be the curvature (resp. torsion) of the space curve. Denote by $\mathcal{A}_0(\xi^1)$ the cross-sectional area normal to $\tilde{\mathbf{e}}_1(\xi^1)$ and $\mathcal{B}_0$ the reference, stress-free and undeformed elastic body of the beam. In the

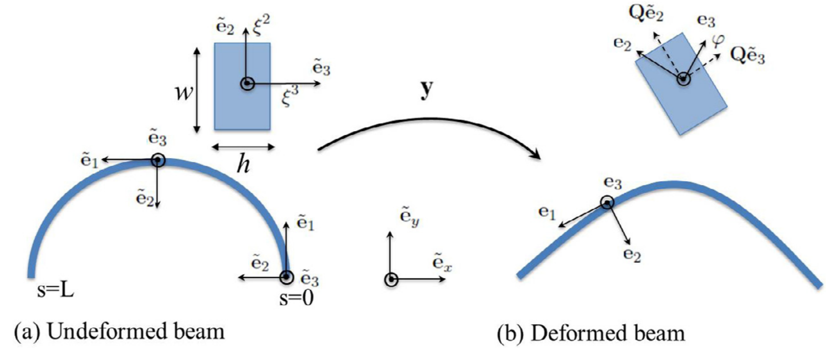

Fig. 2. Frenet frames associated with the reference and deformed centroid lines: The top insets show the cross-sectional areas of (a) undeformed / reference beam, (b) deformed beam and relative twist angle.

local curvilinear frame $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, we represent a material point $\mathbf{x} \in \mathcal{B}_0$ by its local coordinates (ξ^1, ξ^2, ξ^3) such that

$$\mathbf{x} = \mathbf{c}_0(\xi^1) + \xi^2\tilde{\mathbf{e}}_2 + \xi^3\tilde{\mathbf{e}}_3.$$

By the *Frenet equations*:

$$\tilde{\mathbf{e}}_1' = \kappa_0\tilde{\mathbf{e}}_2, \qquad \tilde{\mathbf{e}}_2' = -\kappa_0\tilde{\mathbf{e}}_1 + \tau_0\tilde{\mathbf{e}}_3, \qquad \tilde{\mathbf{e}}_3' = -\tau_0\tilde{\mathbf{e}}_2, \tag{1}$$

we have

$$\frac{\partial \mathbf{x}}{\partial \xi^1} = (1 - \kappa_0\xi^2)\tilde{\mathbf{e}}_1 - \xi^3\tau_0\tilde{\mathbf{e}}_2 + \tau_0\xi^2\tilde{\mathbf{e}}_3, \quad \frac{\partial \mathbf{x}}{\partial \xi^2} = \tilde{\mathbf{e}}_2, \quad \frac{\partial \mathbf{x}}{\partial \xi^3} = \tilde{\mathbf{e}}_3.$$

Therefore, the distance between two material points (ξ^1, ξ^2, ξ^3) and $(\xi^1, \xi^2, \xi^3) + d(\xi^1, \xi^2, \xi^3)$ in the *reference configuration* is given by

$$ds_0^2 = \tilde{g}_{ij} d\xi^i d\xi^j, \tag{2}$$

where the metric tensor

$$\tilde{g}_{ij} = \frac{\partial \mathbf{x}}{\partial \xi^i} \cdot \frac{\partial \mathbf{x}}{\partial \xi^j} = \begin{bmatrix} |1 - \kappa_0\xi^2|^2 + |\xi^2\tau_0|^2 + |\xi^3\tau_0|^2 & -\xi^3\tau_0 & \tau_0\xi^2 \\ -\xi^3\tau_0 & 1 & 0 \\ \tau_0\xi^2 & 0 & 1 \end{bmatrix}. \tag{3}$$

We now consider deformations of the beam (Fig. 2(b)). Let $\{\mathbf{c}(\xi^1)\colon 0 \le \xi^1 \le L\}$ be the parametrization of the centroid line of the deformed beam. Similarly, we can establish the local orthogonal Frenet frame associated with the deformed centroid line:

$$\mathbf{e}_1(\xi^1) = \frac{\mathbf{c}'(\xi^1)}{|\mathbf{c}'(\xi^1)|}, \quad \mathbf{e}_2(\xi^1) = \frac{\mathbf{e}_1'(\xi^1)}{|\mathbf{e}_1'(\xi^1)|},$$

$$\mathbf{e}_3(\xi^1) = \mathbf{e}_1(\xi^1) \times \mathbf{e}_2(\xi^1).$$

Let

$$s = \gamma(\xi^1) = \int_0^{\xi^1} |\mathbf{c}'(t)| dt$$

be the arc-length parameter of the deformed centroid line,

$$\kappa(\xi^1) = \frac{1}{\gamma'}\mathbf{e}_2(\xi^1) \cdot \mathbf{e}_1'(\xi^1) \quad \left(\text{resp}\,.\ \tau(\xi^1) = \frac{1}{\gamma'}\mathbf{e}_2' \cdot \mathbf{e}_3(\xi^1) \right)$$

be the curvature (resp. torsion) of the deformed centroid curve, and denote by $\mathcal{A}(\xi^1)$ the cross-section area normal to $\mathbf{e}_1(\xi^1)$ and $\mathcal{B}$ the deformed elastic body of the beam. The *Frenet equations* for the deformed curve read

$$\frac{d\mathbf{e}_1}{d\xi^1} = \gamma'\kappa\mathbf{e}_2, \quad \frac{d\mathbf{e}_2}{d\xi^1} = \gamma'(-\kappa\mathbf{e}_1 + \tau\mathbf{e}_3), \quad \frac{d\mathbf{e}_3}{d\xi^1} = -\gamma'\tau\mathbf{e}_2. \tag{4}$$

To establish the one-to-one correspondence of material points between the reference stress-free configuration and the current deformed configuration, we postulate the following kinematic hypotheses:

(H1) The point $\mathbf{c}_0(\xi^1)$ on the centroid line in the reference configuration moves to the point $\mathbf{c}(\xi^1)$ in the deformed configuration. That is, the mapping $\gamma\colon [0, L] \to [0, l]$ characterizes the stretching of the centroid line (l is the length of the deformed centroid curve).

(H2) As for the conventional theories of straight beams or curved planar beams, we assume that each reference material cross-sectional area $\mathcal{A}_0(\xi^1)$ together with the normal $\tilde{\mathbf{e}}_1(\xi^1)$, aside from the lateral deformation due to the Poisson's effect, moves as a 'rigid body' and becomes $\mathcal{A}(\xi^1)$ with the normal $\mathbf{e}_1(\xi^1)$. This is referred to as the Bernoulli–Euler hypothesis.[1]

(H3) The lateral normal Cauchy stress is negligible in the beam, i.e., $\sigma_{22} \approx \sigma_{33} \approx 0$ in $\mathcal{B}$.

Subsequently, we refer to the above kinematic hypotheses as the *Bernoulli–Euler kinematics*. The hypothesis (**H2**) implies that upon deformation, the material frame $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$ is transformed into a new orthogonal frame

$$\mathbf{f}_i = \mathbf{Q}\tilde{\mathbf{e}}_i \quad (i = 1, 2, 3)$$

for some rigid rotation $\mathbf{Q} \in \mathbb{R}^{3\times3}$ satisfying

$$\mathbf{f}_1 = \mathbf{Q}\tilde{\mathbf{e}}_1 = \mathbf{e}_1, \quad \mathbf{Q}^{-1} = \mathbf{Q}^T, \quad \det\mathbf{Q} = 1. \tag{5}$$

Therefore, a material point $\mathbf{x} = \mathbf{c}_0(\xi^1) + \xi^2\tilde{\mathbf{e}}_2 + \xi^3\tilde{\mathbf{e}}_3$ on $\mathcal{A}_0(\xi^1)$ in the reference configuration moves to the point [2]

$$\mathbf{y} = \mathbf{c}(\xi^1) + \mathbf{Q}(\xi^2\tilde{\mathbf{e}}_2 + \xi^3\tilde{\mathbf{e}}_3) \in \mathcal{A}(\xi^1). \tag{6}$$

Moreover, since $\{\mathbf{Q}\tilde{\mathbf{e}}_2, \mathbf{Q}\tilde{\mathbf{e}}_3\}$ and $\{\mathbf{e}_2, \mathbf{e}_3\}$ are two orthogonal bases for the plane normal to $\mathbf{e}_1$, we can define the *relative* twist angle φ such that (see Fig. 2 (b))

$$\begin{cases} \mathbf{f}_2 = \mathbf{Q}\tilde{\mathbf{e}}_2 = \cos\varphi\mathbf{e}_2 + \sin\varphi\mathbf{e}_3, \\ \mathbf{f}_3 = \mathbf{Q}\tilde{\mathbf{e}}_3 = -\sin\varphi\mathbf{e}_2 + \cos\varphi\mathbf{e}_3. \end{cases} \tag{7}$$

We remark that the *relative* twist angle $\varphi = \varphi(\xi^1)$ is different from the elastic twist angle if the deformed curve is a space curve with nonzero torsion.

Inserting (7) into (6) we write the deformation $\mathbf{y}: \mathcal{B}_0 \to \mathcal{B}$ in terms of curvilinear coordinates (ξ^1, ξ^2, ξ^3) as

$$\mathbf{y}(\xi^1, \xi^2, \xi^3) = \mathbf{c}(\xi^1) + \xi^2(\cos\varphi(\xi^1)\mathbf{e}_2(\xi^1) + \sin\varphi(\xi^1)\mathbf{e}_3(\xi^1)) + \xi^3(-\sin\varphi(\xi^1)\mathbf{e}_2(\xi^1) + \cos\varphi(\xi^1)\mathbf{e}_3(\xi^1)). \tag{8}$$

[1] This kinematic hypothesis neglects the shearing of cross-sections, see (Simo, 1985).

[2] The lateral normal strains will be accounted for later by (**H3**).

By the chain rule and Frenet Eq. (4) we find that

$$\begin{cases} \dfrac{\partial \mathbf{y}}{\partial \xi^1} = \zeta_1 \mathbf{e}_1 + \zeta_2 \mathbf{e}_2 + \zeta_3 \mathbf{e}_3, \\ \dfrac{\partial \mathbf{y}}{\partial \xi^2} = (\cos\varphi \mathbf{e}_2 + \sin\varphi \mathbf{e}_3), \qquad \text{where} \\ \dfrac{\partial \mathbf{y}}{\partial \xi^3} = (-\sin\varphi \mathbf{e}_2 + \cos\varphi \mathbf{e}_3), \end{cases}$$

$$\begin{cases} \zeta_1 = \gamma'(1 - \xi^2 \kappa \cos\varphi + \xi^3 \kappa \sin\varphi), \\ \zeta_2 = (\varphi' + \tau\gamma')(-\xi^2 \sin\varphi - \xi^3 \cos\varphi), \\ \zeta_3 = (\varphi' + \tau\gamma')(\xi^2 \cos\varphi - \xi^3 \sin\varphi). \end{cases}$$

Therefore, for two material points with coordinates (ξ^1, ξ^2, ξ^3) and $(\xi^1, \xi^2, \xi^3) + d(\xi^1, \xi^2, \xi^3)$, the distance between them in the *deformed configuration* is given by

$$ds^2 = g_{ij} d\xi^i d\xi^j, \tag{9}$$

where the metric tensor is given by

$$[g_{ij}] = \left[\frac{\partial \mathbf{y}}{\partial \xi^i} \cdot \frac{\partial \mathbf{y}}{\partial \xi^j} \right] = \begin{bmatrix} \zeta_1^2 + \zeta_2^2 + \zeta_3^2 & \zeta_2 \cos\varphi + \zeta_3 \sin\varphi & -\zeta_2 \sin\varphi + \zeta_3 \cos\varphi \\ \zeta_2 \cos\varphi + \zeta_3 \sin\varphi & 1 & 0 \\ -\zeta_2 \sin\varphi + \zeta_3 \cos\varphi & 0 & 1 \end{bmatrix}. \tag{10}$$

We now proceed to calculate the linearized elastic strain, stress and energy. For simplicity, we restrict ourselves to small *elastic* strains in the sense that $|ds - ds_0| \ll ds_0$, i.e.,

$$\gamma'(\xi^1) - 1 \sim \eta \ll 1, \quad \xi^2\kappa \sim \xi^3\kappa \sim \xi^2\tau \sim \xi^3\tau \sim \eta \ll 1.$$

By (2)-(3) and (9)-(10), to the leading order $O(\eta)$ we find that

$$\frac{ds - ds_0}{ds_0} \approx \frac{ds^2 - ds_0^2}{2|ds_0|^2} = \frac{1}{2}(g_{ij} - \tilde{g}_{ij}) \frac{d\xi^i}{ds_0} \frac{d\xi^j}{ds_0} = \epsilon_{ij}(\xi^1, \xi^2, \xi^3) \frac{d\xi^i}{ds_0} \frac{d\xi^j}{ds_0}, \tag{11}$$

where

$$\epsilon_{ij} \approx \begin{bmatrix} \epsilon_{11} & \xi^3(\tau_0 - \varphi' - \tau\gamma')/2 & \xi^2(\varphi' + \tau\gamma' - \tau_0)/2 \\ \xi^3(\tau_0 - \varphi' - \tau\gamma')/2 & 0 & 0 \\ \xi^2(\varphi' + \tau\gamma' - \tau_0)/2 & 0 & 0 \end{bmatrix},$$
$$\epsilon_{11} = \gamma' - 1 + \xi^2(-\kappa\gamma' \cos\varphi + \kappa_0) + \xi^3 \kappa\gamma' \sin\varphi. \tag{12}$$

We identify the above tensor ϵ_{ij} as the usual linearized strain tensor since the tensor $(\frac{d\xi^i}{ds_0} \frac{d\xi^j}{ds_0}) = (\tilde{g}_{ij})^{-1} = \delta_{ij} + O(\eta)$. ($\delta_{ij}$ is the Kronecker delta) For isotropic materials with Young's modulus E and Poisson's ration ν (shear modulus $G = \frac{E}{2(1+\nu)}$), it is clear that the stress

$$\sigma_{ij} = \frac{E}{1+\nu} \epsilon_{ij} + \frac{E\nu}{(1+\nu)(1-2\nu)} \delta_{ij} \tag{13}$$

violates the hypothesis (**H3**); we shall correct the above linearized strain tensor by setting the lateral normal strains $\epsilon_{22} = \epsilon_{33} = -\nu\epsilon_{11}$ according to the familiar Poisson's effects.[3]

From the above discussion, we can now write the strain energy in terms of the deformed centroid line parametrization $\mathbf{c} : (0, L) \to \mathbb{R}^3$ and the relative twist angle $\varphi : (0, L) \to \mathbb{R}$ as

$$\begin{aligned} U_e[\mathbf{c}, \varphi; \mathbf{c}_0] &= \int_{\mathcal{B}_0} \frac{1}{2} \epsilon_{ij} \sigma_{ij} \\ &= \int_0^L \int_{\mathcal{A}_0(\xi^1)} \frac{1}{2} \{ E|\epsilon_{11}|^2 + G[(|\xi^2|^2 + |\xi^3|^2) \\ &\quad \times (\tau_0 - \varphi' - \tau\gamma')^2] \} d\xi^2 d\xi^3 d\xi^1 \\ &= \frac{1}{2} \int_0^L \Bigg\{ EA(\gamma' - 1)^2 + GJ(\varphi' + \tau\gamma' - \tau_0)^2 \\ &\quad + \begin{bmatrix} -\kappa\gamma' \cos\varphi + \kappa_0 \\ \kappa\gamma' \sin\varphi \end{bmatrix} \cdot E\mathbf{I} \begin{bmatrix} -\kappa\gamma' \cos\varphi + \kappa_0 \\ \kappa\gamma' \sin\varphi \end{bmatrix} \Bigg\} d\xi^1, \end{aligned} \tag{14}$$

where $A = A(\xi^1)$ is the area of cross-section $\mathcal{A}_0(\xi^1)$ and the moment of inertia tensor $\mathbf{I}$ (resp. polar moment of inertia J) is defined as

$$\mathbf{I}(\xi^1) = \int_{\mathcal{A}_0(\xi^1)} \begin{bmatrix} |\xi^2|^2 & \xi^3\xi^2 \\ \xi^3\xi^2 & |\xi^3|^2 \end{bmatrix} d\xi^2 d\xi^3 \quad (\text{resp. } J = \ \mathrm{Tr}\mathbf{I}). \tag{15}$$

We remark that the approximations "≈" in (11) follow from our assumption of *small elastic strain* (i.e., $|ds - ds_0| \ll ds_0$) instead of small displacements. The strain energy functional (14) does apply to space curved beams with large displacements and is suitable for post-buckling analysis (Su et al., 2012).

2.2. Variational principles and boundary value problems

For simplicity, from now on we assume inextensible line of centroid, i.e., $\gamma'(\xi^1) \equiv 1$ ($s \equiv \xi^1$), and $\mathbf{I} = \ \mathrm{diag}[I_3, I_2]$. By (14) we write the strain energy functional as

$$U_e[\mathbf{c}, \varphi; \mathbf{c}_0] = \int_0^L \left\{ \frac{EI_3}{2}(-\kappa\cos\varphi + \kappa_0)^2 + \frac{EI_2}{2}(\kappa\sin\varphi)^2 + \frac{GJ}{2}(\varphi' + \tau - \tau_0)^2 \right\} d\xi^1. \tag{16}$$

The free energy of the system depends on the external loadings and boundary conditions. For the moment, we consider only distributed "dead" force $\mathbf{q} : (0, L) \to \mathbb{R}^3$ acting on the centroid line that is independent of deformation of the beam with **clamped** boundary conditions:

$$\begin{aligned} &(i) \ \ \mathbf{c}(0) = \mathbf{c}_0(0) + \mathbf{u}_0, \ \mathbf{c}(L) = \mathbf{c}_0(L) + \mathbf{u}_L, \\ &(ii) \ \ (\mathbf{c} - \mathbf{c}_0)' = 0 \ \ \text{at } s = 0 \,\&\, L, \\ &(iii) \ \ \sin\varphi = \tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3, \end{aligned} \tag{17}$$

where constant vectors $\mathbf{u}_0, \mathbf{u}_L \in \mathbb{R}^3$ can be interpreted as the displacements of the ends of the beam. Also, being clamped the rigid rotation defined by (5) and (7) shall be such that the deformed material frame $\{\mathbf{Q}\tilde{\mathbf{e}}_1, \mathbf{Q}\tilde{\mathbf{e}}_2, \mathbf{Q}\tilde{\mathbf{e}}_3\}$ coincides with the undeformed frame $\{\tilde{\mathbf{e}}_1, \tilde{\mathbf{e}}_2, \tilde{\mathbf{e}}_3\}$, meaning that $\tilde{\mathbf{e}}_1 = \mathbf{e}_1$,

$$\begin{cases} \tilde{\mathbf{e}}_2 = \cos\varphi \mathbf{e}_2 + \sin\varphi \mathbf{e}_3 \\ \tilde{\mathbf{e}}_3 = -\sin\varphi \mathbf{e}_2 + \cos\varphi \mathbf{e}_3 \end{cases} \quad \text{at} \quad s = 0 \,\&\, L,$$

and hence part (ii) & (iii). Other boundary conditions, e.g., applied point forces, simple supports and a free end will be discussed later.

In account of the external distributed load $\mathbf{q} : (0, L) \to \mathbb{R}^3$, the total free energy of the system is given by

$$F[\mathbf{c}, \varphi; \mathbf{c}_0] = U_e[\mathbf{c}, \varphi; \mathbf{c}_0] - \int_0^L \mathbf{q} \cdot \mathbf{c} ds. \tag{18}$$

By the principle of minimum free energy, the equilibrium configuration is determined by the minimization problem

$$\min\{F[\mathbf{c}, \varphi; \mathbf{c}_0] : \mathbf{c}, \varphi \text{ satisfy boundary conditions in (17)}\}, \tag{19}$$

[3] With this 'correction', the linearized strain tensor will generally violate the compatibility condition, i.e., it cannot be written as $\frac{1}{2}(\nabla\mathbf{u} + (\nabla\mathbf{u})^T)$ for a (continuous) vector field $\mathbf{u} : \mathcal{B}_0 \to \mathbb{R}^3$. This inconsistency exists even in the classic theory for straight beams.

and a minimizing pair ($\mathbf{c}$, φ) necessarily satisfy that ($|\varepsilon| \ll 1$)

$$\frac{dF[\mathbf{c}+\varepsilon\mathbf{v},\varphi+\varepsilon\varphi_1;\mathbf{c}_0]}{d\varepsilon}\bigg|_{\varepsilon=0}=0 \quad \text{for any admissible variations } \mathbf{v} \text{ and } \varphi_1. \tag{20}$$

We now derive the Euler–Lagrange equations for the centroid line parametrization $\mathbf{c}=\mathbf{c}(s)$ and relative twist angle $\varphi=\varphi(s)$ by calculus of variations. Consider a variation of the centroid line of the deformed beam: $\mathbf{c}(s)\to\mathbf{c}(s)+\varepsilon\mathbf{v}(s)$. Then the Frenet frame of the varied centroid line is given by

$$\mathbf{e}_{1\varepsilon}=\mathbf{e}_1+\varepsilon\mathbf{v}', \quad \mathbf{e}_{2\varepsilon}=\frac{\mathbf{c}''+\varepsilon\mathbf{v}''}{|\mathbf{c}''+\varepsilon\mathbf{v}''|}, \quad \mathbf{e}_{3\varepsilon}=\mathbf{e}_{1\varepsilon}\times\mathbf{e}_{2\varepsilon}. \tag{21}$$

To ensure the parameter s to be the 'arc-length', we shall require

$$|\mathbf{e}_{1\varepsilon}|=1+\varepsilon\mathbf{v}'\cdot\mathbf{e}_1+o(\varepsilon)=1 \quad \text{i.e.,} \quad \mathbf{v}'\cdot\mathbf{e}_1=0. \tag{22}$$

By tedious but direct calculations we find that

$$\begin{aligned}
\mathbf{e}_{2\varepsilon}&=\mathbf{e}_2+\varepsilon\frac{\mathbf{v}''-(\mathbf{e}_2\cdot\mathbf{v}'')\mathbf{e}_2}{|\mathbf{c}''|}+o(\varepsilon),\\
\mathbf{e}'_{2\varepsilon}&=-\kappa\mathbf{e}_1+\tau\mathbf{e}_3+\varepsilon\left[\frac{\mathbf{v}''-(\mathbf{e}_2\cdot\mathbf{v}'')\mathbf{e}_2}{|\mathbf{c}''|}\right]'+o(\varepsilon),\\
\mathbf{e}_{3\varepsilon}&=\mathbf{e}_3+\varepsilon\left\{\frac{1}{\kappa}\mathbf{e}_1\times[\mathbf{v}''-(\mathbf{e}_2\cdot\mathbf{v}'')\mathbf{e}_2]+\mathbf{v}'\times\mathbf{e}_2\right\}+o(\varepsilon).
\end{aligned} \tag{23}$$

Therefore, the curvature and torsion of the varied centroid line are given by

$$\begin{aligned}
\kappa_\varepsilon&=\mathbf{e}_{2\varepsilon}\cdot\mathbf{e}'_{1\varepsilon}=\kappa+\varepsilon\mathbf{e}_2\cdot\mathbf{v}''+o(\varepsilon),\\
\tau_\varepsilon&=\mathbf{e}'_{2\varepsilon}\cdot\mathbf{e}_{3\varepsilon}=\tau+\varepsilon\left\{(\kappa\mathbf{e}_3+\tau\mathbf{e}_1)\cdot\mathbf{v}'+\left(\frac{\mathbf{e}_3\cdot\mathbf{v}''}{\kappa}\right)'\right\}+o(\varepsilon),
\end{aligned} \tag{24}$$

and henceforth the first variation of the strain energy with respect to the centroid line parametrization is given by

$$\begin{aligned}
&\frac{dF[\mathbf{c}+\varepsilon\mathbf{v},\varphi;\mathbf{c}_0]}{d\varepsilon}\bigg|_{\varepsilon=0}\\
&=\int_0^L\bigg\{[(EI_3\cos^2\varphi+EI_2\sin^2\varphi)\kappa-EI_3\kappa_0\cos\varphi]\mathbf{e}_2\cdot\mathbf{v}''\\
&\quad+GJ(\varphi'+\tau-\tau_0)\left[(\kappa\mathbf{e}_3+\tau\mathbf{e}_1)\cdot\mathbf{v}'+\left(\frac{\mathbf{e}_3\cdot\mathbf{v}''}{\kappa}\right)'\right]-\mathbf{q}\cdot\mathbf{v}\bigg\}ds\\
&=\int_0^L\left\{M\mathbf{e}_2\cdot\mathbf{v}''+T\left[(\kappa\mathbf{e}_3+\tau\mathbf{e}_1)\cdot\mathbf{v}'+\left(\frac{\mathbf{e}_3\cdot\mathbf{v}''}{\kappa}\right)'\right]-\mathbf{q}\cdot\mathbf{v}\right\}ds,
\end{aligned} \tag{25}$$

where, for brevity, we have introduced quantities

$$M=(EI_3\cos^2\varphi+EI_2\sin^2\varphi)\kappa-EI_3\kappa_0\cos\varphi,\ T=GJ(\varphi'+\tau-\tau_0). \tag{26}$$

The physical meanings of scalar functions M, T will be explored later (cf., (38)). By (20), we conclude that a minimizing pair ($\mathbf{c}$, φ) of (19) necessarily satisfy that

$$[M\mathbf{e}_2]''-[T(\kappa\mathbf{e}_3+\tau\mathbf{e}_1)+\lambda\mathbf{e}_1]'-\left[\frac{T'}{\kappa}\mathbf{e}_3\right]''-\mathbf{q}=0 \quad \text{on } (0,L), \tag{27}$$

where $\lambda:(0,L)\to\mathbb{R}$ is the Lagrangian multiplier associated with the constraint (22). Moreover, consider variations of relative twist angle: $\varphi\to\varphi+\varepsilon\varphi_1$. It is straightforward to find the associated first variation of the free energy:

$$\begin{aligned}
&\frac{dF[\mathbf{c},\varphi+\varepsilon\varphi_1;\mathbf{c}_0]}{d\varepsilon}\bigg|_{\varepsilon=0}\\
&=\int_0^L\{[EI_3(-\kappa\cos\varphi+\kappa_0)\kappa\sin\varphi+EI_2\kappa^2\sin\varphi\cos\varphi]\varphi_1\\
&\quad+GJ(\varphi'+\tau-\tau_0)\varphi'_1\}ds.
\end{aligned} \tag{28}$$

Together with (27), by (20) we conclude that the Euler–Lagrange equations associated with the variational principle (19) is given by

$$\begin{cases}
[M\mathbf{e}_2]''-[T(\kappa\mathbf{e}_3+\tau\mathbf{e}_1)+\lambda\mathbf{e}_1]'-\left[\dfrac{T'}{\kappa}\mathbf{e}_3\right]''-\mathbf{q}=0,\\
-T'+EI_3(-\kappa\cos\varphi+\kappa_0)\kappa\sin\varphi+EI_2\kappa^2\sin\varphi\cos\varphi=0,
\end{cases} \tag{29}$$

which shall be satisfied by an equilibrium state ($\mathbf{c}$, φ) on (0, L). Supplemented with the boundary conditions (17) we can solve the above differential equations for the unknown centroid line $\mathbf{c}$ and relative twist angle φ. It is worthwhile to notice that the kinematic boundary conditions (17) for clamped supports are exactly such that there is no boundary contribution for all admissible variations $\mathbf{v}$ and φ_1 in (20) when (25) and (28) are integrated by parts for deriving the Euler–Lagrange Eq. (29).

Further, denote by

$$\mathbf{Q}(\xi^1)=\int_0^{\xi^1}\mathbf{q}(s)ds, \quad Q_i=\mathbf{Q}\cdot\mathbf{e}_i, \quad \tilde{Q}_i=\mathbf{Q}\cdot\tilde{\mathbf{e}}_i, \quad (i=1,2,3). \tag{30}$$

With respect to the frame {$\mathbf{e}_1$, $\mathbf{e}_2$, $\mathbf{e}_3$}, by (4) we can rewrite Eq. (29) as

$$\begin{cases}
M\kappa+T\tau+Q_1+\lambda+V_1=0,\\
M'+\dfrac{T'\tau}{\kappa}-Q_2-V_2=0,\\
M\tau-T\kappa-\left(\dfrac{T'}{\kappa}\right)'-Q_3-V_3=0, & \forall s\in(0,L),\\
-T'+EI_3(-\kappa\cos\varphi+\kappa_0)\kappa\sin\varphi\\
\quad+EI_2\kappa^2\sin\varphi\cos\varphi=0
\end{cases} \tag{31}$$

where $V_i=\mathbf{V}\cdot\mathbf{e}_i$ and $\mathbf{V}\in\mathbb{R}^3$ is a constant vector arising from integration. Note that V_i generally depends on position since $\mathbf{e}_i$ ($i=1,2,3$) vary along the space curve. In the Appendix, we compare the above equilibrium equations with earlier works of Reissner (1973), Simo (1985), and Simo and Vu-Quoc (1986), and conclude that they are consistent.

The above variational framework can be applied to general boundary conditions. A key advantage of the variational framework lies in that it facilitates a systematic method of deriving consistent boundary conditions which have inspired many discussions in the literature. Besides the **clamped** boundary conditions, we consider another two types of boundary conditions frequently encountered in engineering applications.

Simple supports. By a curved beam with two ends simply supported, we mean the following kinematic boundary conditions (cf., (i) in (17)):

$$(i) \quad \mathbf{c}(0)-\mathbf{c}_0(0)=\mathbf{u}_0, \quad \mathbf{c}(L)-\mathbf{c}_0(L)=\mathbf{u}_L. \tag{32}$$

The free energy of the system remains the same as in (18). To conform with (32), we have $\mathbf{v}=0$ at $s=0$ & L. Then the first variation of the free energy with respect to change of centroid line $\mathbf{c}\to\mathbf{c}+\varepsilon\mathbf{v}$ and relative twist $\varphi\to\varphi+\varepsilon\varphi_1$ gives rise to (recall (25), (26) and (28))

$$\frac{dF[\mathbf{c}+\varepsilon\mathbf{v},\varphi+\varepsilon\varphi_1;\mathbf{c}_0]}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_0^L \boxed{\text{EL1}}\cdot\mathbf{v}+\boxed{\text{EL2}}\cdot\varphi_1 ds + \left[T\frac{\mathbf{e}_3\cdot\mathbf{v}''}{\kappa}+M\mathbf{e}_2\cdot\mathbf{v}'-\frac{T'}{\kappa}\mathbf{e}_3\cdot\mathbf{v}'+T\varphi_1\right]\bigg|_0^L, \tag{33}$$

where $\boxed{\text{EL1}}$ and $\boxed{\text{EL2}}$ stand for the first and second equation on the left hand side of (29), respectively. Since the values of φ_1, $\mathbf{v}'$ and $\mathbf{v}''$ at the boundary $s = 0\,\&\,L$ can be independently assigned, by (20) and (33) we conclude the following Neumann-type boundary conditions (κ is nonzero and finite at the two ends):

$$T = T' = 0, \quad M = 0 \quad \text{at} \quad s = 0\,\&\,L. \tag{34}$$

Replacing (ii) and (iii) of (17), the above boundary conditions conform with the notion of "simple supports", i.e., they cannot provide bending moment or twisting torque to the beam.

Cantilever. For a curved "cantilever", we can apply a point force at the free end:

(*i*) $\mathbf{c}-\mathbf{c}_0=\mathbf{c}'-\mathbf{c}_0'=\sin\varphi-\tilde{\mathbf{e}}_2\cdot\mathbf{e}_3=0$ at $s=0$,
(*ii*) A point "dead" force $\mathbf{p}$ acting on the free end $s = L$. (35)

The free energy of the system shall include the potential energy associated with the point force and is given by

$$F[\mathbf{c},\varphi;\mathbf{c}_0] = U_e[\mathbf{c},\varphi;\mathbf{c}_0] - \int_0^L \mathbf{q}\cdot\mathbf{c}ds - \mathbf{p}\cdot\mathbf{c}(L). \tag{36}$$

By the principle of minimum free energy, the equilibrium state of the beam shall be determined by minimizing the total free energy subject to the kinematic constraints in (35)(i). By (25), (26) and (28), we integrate by parts and obtain

$$\frac{dF[\mathbf{c}+\varepsilon\mathbf{v},\varphi+\varepsilon\varphi_1;\mathbf{c}_0]}{d\varepsilon}\bigg|_{\varepsilon=0} = \int_0^L \boxed{\text{EL1}}\cdot\mathbf{v}+\boxed{\text{EL2}}\cdot\varphi_1 ds + \left[T\frac{\mathbf{e}_3\cdot\mathbf{v}''}{\kappa}+M\mathbf{e}_2\cdot\mathbf{v}'-\frac{T'}{\kappa}\mathbf{e}_3\cdot\mathbf{v}'+T\varphi_1\right]\bigg|_{s=L} + \left\{\left[\frac{T'}{\kappa}\mathbf{e}_3-M\mathbf{e}_2\right]'+T(\kappa\mathbf{e}_3+\tau\mathbf{e}_1)-\mathbf{p}\right\}\cdot\mathbf{v}\bigg|_{s=L}. \tag{37}$$

Therefore, we identify that (λ again arises from the constraint (22) as a Lagrange multiplier)

$$\mathbf{M} := M\mathbf{e}_2 - \frac{T'}{\kappa}\mathbf{e}_3 + T\mathbf{e}_1,$$
$$\mathbf{P} := (\lambda+T\tau)\mathbf{e}_1 - \left(M'+\frac{T'\tau}{\kappa}\right)\mathbf{e}_2 + \left[\left(\frac{T'}{\kappa}\right)'+T\kappa-M\tau\right]\mathbf{e}_3 \tag{38}$$

as the internal moment and force, respectively. Moreover, at the absence of applied boundary moment and dead force $\mathbf{p}$ we shall have $\mathbf{M} = 0$ and $\mathbf{P} = \mathbf{p}$ at $s = L$, i.e.,

$$T = T' = 0, \quad M = 0, \quad \lambda\mathbf{e}_1 - M'\mathbf{e}_2 + \left(\frac{T'}{\kappa}\right)'\mathbf{e}_3 - \mathbf{p} = 0, \tag{39}$$

which shall be supplemented to the kinematic boundary conditions $(35)_1$ at $s = 0$ for solving the Euler–Lagrange Eq. (31).

In summary, the boundary conditions for clamped support, simple support and free end are given by (17), (32) & (34), and (39), respectively. Other types of boundary conditions may be similarly discussed in our variational framework.

2.3. Kirchhoff's helical solutions

The full geometrically nonlinear system (29) for curved beams are not amenable to explicit solutions. Exceptions include the case that the beam is initially straight, symmetric with equal bending stiffness in the two lateral directions and at the absence of distributed load $\mathbf{q}$. In this case, $\kappa_0 = \tau_0 = 0$, $I_3 = I_2 =: I$, $M = EI\kappa$, $T = GJ(\varphi' + \tau)$, and by (31) we obtain

$$\begin{cases} EI\kappa^2 + GJ(\varphi'+\tau)\tau + \lambda + V_1 = 0, \\ (EI\kappa)' + \dfrac{1}{\kappa}GJ(\varphi'+\tau)'\tau - V_2 = 0, \\ EI\kappa\tau - GJ(\varphi'+\tau)\kappa - \left[\dfrac{GJ(\varphi'+\tau)'}{\kappa}\right]' - V_3 = 0, \\ GJ(\varphi'+\tau) = T_0, \end{cases} \tag{40}$$

where $\mathbf{V} \in \mathbb{R}^3$ and T_0 are integration constants. For a trial solution of helix with constant κ, constant torsion τ, and parametrization as

$$\mathbf{c}(\xi^1) = (a\cos\alpha\xi^1, a\sin\alpha\xi^1, b\xi^1),$$
$$\left(\alpha = (\kappa^2+\tau^2)^{1/2}, a = \frac{\kappa}{\alpha^2}, b = \frac{\tau}{\alpha}\right),$$

the Frenet frame is given by

$$\mathbf{e}_1 = (-a\alpha\sin\alpha\xi^1, a\alpha\cos\alpha\xi^1, b),$$
$$\mathbf{e}_2 = (-\cos\alpha\xi^1, -\sin\alpha\xi^1, 0), \quad \mathbf{e}_3 = \mathbf{e}_1\times\mathbf{e}_2.$$

Therefore, if $\mathbf{V} = (0, 0, V_z)$ is a constant vector along z-axis, then

$$V_1 = \mathbf{e}_1\cdot\mathbf{V} = \frac{\tau V_z}{\alpha}, \quad V_2 = \mathbf{e}_2\cdot\mathbf{V} = 0, \quad V_3 = \mathbf{e}_3\cdot\mathbf{V} = \frac{\kappa V_z}{\alpha}$$

happen to be constant as well. By inspection, we see that (40) are satisfied if

$$-(EI\kappa^2 + T_0\tau) = V_1 + \lambda, \quad EI\kappa\tau - T_0\kappa = V_3. \tag{41}$$

For given V_z and T_0, we can solve the above equations for κ and τ; the loading conditions at the ends and Lagrangian multiplier λ can then be determined by (39). We remark that the helical solution (41) is slightly more general than the classic Kirchhoff's solution (Love, 1944, pg. 414) since it allows a twisting torque T_0 at the ends.

2.4. Linearized theory

The nonlinear boundary value problem formed by the differential system (29) and one of the boundary conditions: (i) clamped supports (17), (ii) simple supports (32) and (34), or (iii) cantilever $(35)_1$ and (39), for the centroid line $\mathbf{c}$ and relative twist angle φ, though can be used to determine the large-deformation, large-twist and small strain equilibria, are not amenable to theoretical analysis. For many applications of practical importance, it may suffice to consider small strain and small relative angle of twist in the sense that

$$|\mathbf{u}'(s)| \sim \varepsilon \ll 1, \quad |\varphi(s)| \sim \eta \ll 1, \tag{42}$$

where $\mathbf{u}(s) = \mathbf{c}(s) - \mathbf{c}_0(s)$ is the displacement. Moreover, we may assume the scaling relationship

$$\varepsilon = \eta. \tag{43}$$

We remark that the scaling assumptions (42) and (43) are not the only possible asymptotic behavior of the minimizer (or minimizing sequence); the validity of these assumptions shall be *a posteriori* checked for self-consistency. In Section 3.1 we will present theories of different asymptotic behaviors than (42) and (43).

By (42), (43) and direct calculation we find that (cf. (24))

$$\kappa = \kappa_0 + \triangle\kappa + o(\varepsilon), \quad \tau = \tau_0 + \triangle\tau + o(\varepsilon),$$
$$\triangle\kappa := \tilde{\mathbf{e}}_2\cdot\mathbf{u}'', \quad \triangle\tau := (\kappa_0\tilde{\mathbf{e}}_3 + \tau_0\tilde{\mathbf{e}}_1)\cdot\mathbf{u}' + \left(\frac{\tilde{\mathbf{e}}_3\cdot\mathbf{u}''}{\kappa_0}\right)'. \tag{44}$$

Let $\mathbf{u} = u_1\tilde{\mathbf{e}}_1 + u_2\tilde{\mathbf{e}}_2 + u_3\tilde{\mathbf{e}}_3$. By the Frenet Eq. (1) we have

$$\begin{aligned}\mathbf{u}' &= (u_1' - \kappa_0 u_2)\tilde{\mathbf{e}}_1 + (u_2' + \kappa_0 u_1 - \tau_0 u_3)\tilde{\mathbf{e}}_2 + (u_3' + u_2\tau_0)\tilde{\mathbf{e}}_3,\\ \mathbf{u}'' &= [-\kappa_0(u_2' + \kappa_0 u_1 - \tau_0 u_3)]\tilde{\mathbf{e}}_1\\ &\quad + [u_2'' + (\kappa_0 u_1 - \tau_0 u_3)' - \tau_0(u_3' + u_2\tau_0)]\tilde{\mathbf{e}}_2\\ &\quad + [u_3'' + (u_2\tau_0)' + \tau_0(u_2' + \kappa_0 u_1 - \tau_0 u_3)]\tilde{\mathbf{e}}_3,\end{aligned} \tag{45}$$

where we have noticed that

$$u_1' - \kappa_0 u_2 = 0 \quad \text{whence} \quad [\mathbf{c}_0(s) + \mathbf{u}(s)]' \cdot [\mathbf{c}_0(s) + \mathbf{u}(s)]' = 1. \tag{46}$$

Then by (44) the change of curvature and torsion are given by

$$\begin{aligned}\triangle\kappa &= u_2'' + (\kappa_0 u_1 - \tau_0 u_3)' - \tau_0(u_3' + u_2\tau_0),\\ \triangle\tau &= \kappa_0(u_3' + u_2\tau_0) + \left\{\frac{1}{\kappa_0}[u_3'' + (u_2\tau_0)' + \tau_0(u_2' + \kappa_0 u_1 - \tau_0 u_3)]\right\}'.\end{aligned} \tag{47}$$

Inserting (44) into (16), (29) and keeping only the leading-order terms, we rewrite the strain energy (16) as:

$$U_e[\mathbf{c}, \varphi; \mathbf{c}_0] = \int_0^L \left\{\frac{EI_3}{2}(\triangle\kappa)^2 + \frac{EI_2}{2}(\kappa_0\varphi)^2 + \frac{GJ}{2}(\varphi' + \triangle\tau)^2\right\} + o(\varepsilon^2). \tag{48}$$

As in (26) and with an abuse of notation, we denote the bending moment and torque by

$$M = EI_3\triangle\kappa, \quad T = GJ(\varphi' + \triangle\tau). \tag{49}$$

Then upon repeating the standard variational calculation for the strain energy (48), we obtain the linearized differential equations for the displacement $\mathbf{u}(s)$, twist angle $\varphi(s)$ and Lagrangian multiplier $\lambda(s)$ (cf. (7) and (31)):

$$\begin{cases} M\kappa_0 + T\tau_0 + \tilde{Q}_1 + \lambda + \tilde{V}_1 = 0,\\ M' + \dfrac{T'\tau_0}{\kappa_0} - \tilde{Q}_2 - \tilde{V}_2 = 0,\\ M\tau_0 - T\kappa_0 - \left(\dfrac{T'}{\kappa_0}\right)' - \tilde{Q}_3 - \tilde{V}_3 = 0,\\ -T' + EI_2\kappa_0^2\varphi = 0 \end{cases} \quad \forall s \in (0, L), \tag{50}$$

where $\tilde{V}_i = \mathbf{V} \cdot \tilde{\mathbf{e}}_i$ and $\mathbf{V} \in \mathbb{R}^3$ are integration constants.

In particular, for planar curved beams with $\tau_0 = 0$ and at the absence of distributed load $\mathbf{q}$ (i.e., $\tilde{Q}_i = 0$), the strain energy (48) can be rewritten as

$$U_e[\mathbf{u}, \varphi; \mathbf{c}_0] = \int_0^L \left\{\frac{EI_3}{2}[u_2'' + (\kappa_0 u_1)']^2 + \frac{EI_2}{2}(\kappa_0\varphi)^2 + \frac{GJ}{2}\left[\varphi' + \kappa_0 u_3' + \left(\frac{u_3''}{\kappa_0}\right)'\right]^2\right\} ds, \tag{51}$$

and the differential system (50) can be rewritten as (recall that $M = EI_3(u_2'' + (\kappa_0 u_1)')$ and $T = GJ(\varphi' + \kappa_0 u_3' + \frac{u_3''}{\kappa_0})$)

$$\begin{cases} M\kappa_0 + \lambda + \tilde{V}_1 = 0,\\ M' - \tilde{V}_2 = 0,\\ -T\kappa_0 - \left(\dfrac{T'}{\kappa_0}\right)' - \tilde{V}_3 = 0, \quad \forall s \in (0, L).\\ -T' + EI_2\kappa_0^2\varphi = 0 \end{cases} \tag{52}$$

2.5. Explicit solutions for linearized circular beams

For explicit solutions, we consider a planar semi-circle beam $\mathbf{c}_0(s) = \frac{1}{\kappa_0}(\cos\kappa_0\ s, \sin\kappa_0\ s)$ $(0 \le s \le L = \pi/\kappa_0)$ of constant curvature κ_0 and uniform rectangular cross section area (see Fig. 2(a)):

$$\mathcal{A}_0(\xi^1) = \{\xi^2\tilde{\mathbf{e}}_2 + \xi^3\tilde{\mathbf{e}}_3 : \xi^2 \in (-\frac{w}{2}, \frac{w}{2}),\ \xi^3 \in (-\frac{h}{2}, \frac{h}{2}\}.$$

It is clear that

$$\tilde{\mathbf{e}}_1 = (-\sin\kappa_0\ s, \cos\kappa_0\ s, 0),$$
$$\tilde{\mathbf{e}}_2 = -(\cos\kappa_0\ s, \sin\kappa_0\ s, 0), \quad \tilde{\mathbf{e}}_3 = (0, 0, 1),$$

and the moment of inertia tensor is given by

$$\mathbf{I} = \ \mathrm{diag}[I_3, I_2], \quad I_3 = \frac{hw^3}{12}, \quad I_2 = \frac{h^3 w}{12}. \tag{53}$$

Noticing that $\tilde{V}_2' = \mathbf{V} \cdot \tilde{\mathbf{e}}_2' = -\kappa_0\mathbf{V} \cdot \tilde{\mathbf{e}}_1 = -\kappa_0\tilde{V}_1$, we see that (52) implies

$$\begin{cases} M'' + \kappa_0^2 M = \lambda, \quad M = EI_3(u_2'' + \kappa_0^2 u_2),\\ \left(\dfrac{T'}{\kappa_0}\right)'' + (\kappa_0 T)' = 0,\\ -T' + EI_2\kappa_0^2\varphi = 0, \quad T = GJ\tilde{\varphi}', \quad \tilde{\varphi} = \varphi + \kappa_0 u_3 + \dfrac{u_3''}{\kappa_0}. \end{cases} \tag{54}$$

We can solve the linear differential system (54) or (52) for the unknown displacement $\mathbf{u}$ and relative twist angle φ upon specifying the boundary conditions. Below we separately consider three different boundary conditions as discussed in Section 2.2.

1. **Clamped supports.** If (17) is enforced with $\mathbf{u}_0 = \delta\tilde{\mathbf{e}}_2/2 = -\delta\tilde{\mathbf{e}}_x/2$, $\mathbf{u}_L = \delta\tilde{\mathbf{e}}_2/2 = \delta\tilde{\mathbf{e}}_x/2$, we have

$$\begin{cases} u_1(0) = u_1(L) = 0, \quad u_2(0) = u_2(L) = \delta/2,\\ \quad u_3(0) = u_3(L) = 0,\\ u_2'(0) = u_3'(0) = u_2'(L) = u_3'(L) = 0,\\ \kappa_0\varphi(0) + u_3''(0) = 0, \quad \kappa_0\varphi(L) + u_3''(L) = 0, \end{cases} \tag{55}$$

We remark that the mixed boundary conditions for φ and u_3 in $(55)_3$ follow from part (iii) of (17), i.e., $\varphi = \tilde{\mathbf{e}}_2 \cdot \mathbf{e}_3 + o(\varepsilon)$, and (in account of (42)) $\mathbf{e}_3 = \tilde{\mathbf{e}}_3 + \{\frac{1}{\kappa_0}\tilde{\mathbf{e}}_1 \times [\mathbf{u}'' - (\tilde{\mathbf{e}}_2 \cdot \mathbf{u}'')\tilde{\mathbf{e}}_2] + \mathbf{u}' \times \tilde{\mathbf{e}}_2\} + o(\varepsilon)$:

$$\varphi = \frac{1}{\kappa_0}\tilde{\mathbf{e}}_2 \cdot \{\tilde{\mathbf{e}}_1 \times [\mathbf{u}'' - (\tilde{\mathbf{e}}_2 \cdot \mathbf{u}'')\tilde{\mathbf{e}}_2]\} = -u_3''/\kappa_0.$$

It is not hard to verify that the boundary value problem formed by (54) and (55) *uniquely* admits the following planar solution:

$$\begin{aligned} u_2(s) &= \frac{\delta_0}{2} + \frac{\delta - \delta_0}{2}\cos\kappa_0 s + \frac{\delta - \delta_0}{\pi}\sin\kappa_0 s\\ &\quad - \frac{(\delta - \delta_0)\kappa_0 s}{\pi}\cos\kappa_0\ s, \quad \varphi = u_3 = 0,\\ u_1(s) &= \int_0^s \kappa_0 u_2(t)dt, \quad u_1(L) = 0 \ \Rightarrow \delta_0 = \frac{8}{8 - \pi^2}\delta. \end{aligned} \tag{56}$$

The associated critical free energy is given by

$$\begin{aligned} &F[\mathbf{u}, \varphi; \mathbf{c}_0]\\ &= \int_0^L \left\{\frac{EI_3}{2}(u_2'' + \kappa_0^2 u_2)^2 + \frac{EI_2}{2}(\kappa_0\varphi)^2 + \frac{GJ}{2}\left(\varphi' + \kappa_0 u_3' + \frac{u_3'''}{\kappa_0}\right)^2\right\}\\ &= \frac{\pi}{\pi^2 - 8}EI_3\kappa_0(\delta\kappa_0)^2. \end{aligned} \tag{57}$$

2. **Simple supports**. If (32) is enforced with $\mathbf{u}_0 = \delta\tilde{\mathbf{e}}_2/2 = -\delta\tilde{\mathbf{e}}_x/2$, $\mathbf{u}_L = \delta\tilde{\mathbf{e}}_2/2 = \delta\tilde{\mathbf{e}}_x/2$ (see Fig. 3(b)), by (34) we have boundary conditions $(55)_1$, $(55)_3$, and

$$\tilde{\varphi}'' = \left(\varphi + \kappa_0 u_3 + \frac{u_3''}{\kappa_0}\right)'' = 0, \quad \tilde{\kappa} = u_2'' + \kappa_0^2 u_2 = 0,$$
$$\text{at } s = 0 \,\&\, L. \tag{58}$$

It is not hard to verify that the boundary value problem formed by (54) and (58) *uniquely* admits the following planar solution:

$$u_2(s) = \frac{\delta}{2\pi}[(\pi - 2\kappa_0 s)\cos\kappa_0 s - 2\sin\kappa_0 s],$$

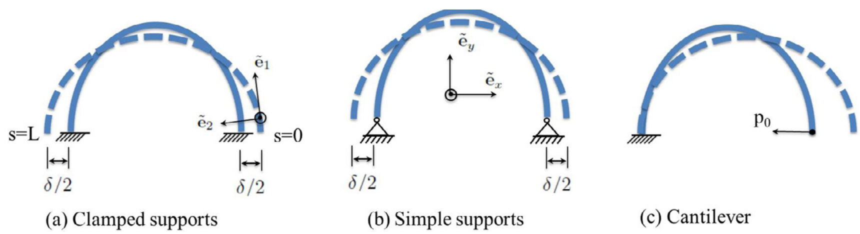

Fig. 3. Typical boundary conditions for curved space beams: (a) clamped supports with prescribed boundary displacements; (b) simple supports with prescribed boundary displacements; (c) cantilever with a point force on the free end. Dashed line: reference undeformed curved beam; solid line: deformed beam.

$$u_1(s) = \int_0^s \kappa_0 u_2(t)dt, \quad \varphi = u_3 = 0. \tag{59}$$

and the associated critical free energy is given by

$$F[\mathbf{u}, \varphi; \mathbf{c}_0] = U_e[\mathbf{u}, \varphi; \mathbf{c}_0] = \int_0^L \frac{EI_3}{2}(u_2'' + \kappa_0^2 u_2)^2 = \frac{1}{\pi} EI_3 \kappa_0 (\delta\kappa_0)^2.$$

3. **Cantilever.** If (35) is enforced with $\mathbf{p} = p_0\tilde{\mathbf{e}}_2 = -p_0\tilde{\mathbf{e}}_x$ (see Fig. 3(c)), by (39) we have the following boundary conditions:

$$\begin{cases} u_1(0) = u_2(0) = u_3(0) = 0, & u_2'(0) = u_3'(0) = 0, \quad \kappa_0\varphi(0) + u_3''(0) = 0, \\ \tilde{\varphi}' = \tilde{\varphi}'' = \tilde{\varphi}''' + \kappa_0^2\tilde{\varphi}' = 0, & \tilde{\kappa} = u_2'' + \kappa_0^2 u_2 = 0, \quad \tilde{\kappa}' + \frac{p_0}{EI_3} = 0 \quad \text{at } s = L. \end{cases} \tag{60}$$

It is not hard to verify that the boundary value problem formed by (54) and (58) *uniquely* admits the following planar solution:

$$u_2(s) = -\frac{p_0}{2EI_3\kappa_0^2}(s\cos\kappa_0 s - \frac{1}{\kappa_0}\sin\kappa_0\ s),$$
$$u_1(s) = \int_0^s \kappa_0 u_2(t)dt, \quad \varphi = u_3 = 0. \tag{61}$$

and the associated critical free energy is given by

$$F[\mathbf{u}, \varphi; \mathbf{c}_0] = U_e[\mathbf{u}, \varphi; \mathbf{c}_0] - p_0 u_2(L) = -\frac{\pi p_0^2}{4EI_3\kappa_0^3}.$$

3. Instabilities of curved beams

3.1. Nonlinear asymptotic theories

A fully linearized boundary value problems for planar curved beam admits a unique planar solution. This can be seen from the linearized strain energy functional (51) where the out-of-plane displacement and twist is not coupled with in-plane displacement and always cost positive energy. This is however inconsistent with experimental observations: thin curved beams in fact twist and bend out-of-plane even if all loadings are in-plane as shown in Fig. 1. Moreover, upon releasing external loadings, the beam recovers the original undeformed geometry, which implies that the strain (12) in the beam remains to be small so that the linear stress-strain relation (13) is applicable.

To capture the possible out-of-plane displacement and twist of a planar beam, we shall come back to our original nonlinear strain energy functional (16). Simplified theories can be obtained for two cases that will be discussed separately as follows.

3.1.1. Ribbon-like beams

If the beam is very thin, from (53) we see that $I_3/I_2 = (w/h)^2 = 1/\hat{r} \gg 1$ and hence the beam would prefer out-of-plane bending than in-plane bending. In regard of the nonlinear strain energy functional (16), in the limit of $\hat{r} \to 0$ for fixed EI_2, the energies of bending about $\mathbf{e}_3$-axis and twisting around $\mathbf{e}_1$-axis being finite implies that the curve $\mathbf{c}$ and relative twist angle φ shall satisfy the constraints:

$$-\kappa\cos\varphi + \kappa_0 = 0, \quad \varphi' + \tau - \tau_0 = 0 \quad \text{on } (0, L), \tag{62}$$

and minimize the (Γ-limit) free energy functional with strain energy given by

$$U_e^\infty[\mathbf{c}, \varphi; \mathbf{c}_0] = \int_0^L \frac{EI_2}{2}(\kappa\sin\varphi)^2 ds. \tag{63}$$

By the method of Lagrange's multipliers and at the absence of external loading (i.e., $\mathbf{q} = 0$), the Euler–Lagrange's equations associated with (63), (62) and (22) are given by (cf., (29))

$$\begin{cases} [(\kappa\sin^2\varphi - \Lambda_1\cos\varphi)\mathbf{e}_2]'' - [\Lambda_2(\kappa\mathbf{e}_3 + \tau\mathbf{e}_1) + \lambda\mathbf{e}_1]' \\ \quad - \left[\frac{\Lambda_2'}{\kappa}\mathbf{e}_3\right]'' = 0, \\ -\Lambda_2' + \Lambda_1\kappa\sin\varphi + \kappa^2\sin\varphi\cos\varphi = 0, \end{cases} \tag{64}$$

where, $EI_2\Lambda_1$ and $EI_2\Lambda_2$ are the Lagrange's multipliers associated with the two equations in (62), respectively. The above asymptotic limit of (sequence of) solutions at $\hat{r} \to 0$ may be rigorously justified by the Γ-convergence method (Maso, 1993) which will not be addressed here.

3.1.2. Beams with moderately small out-of-plane displacement and twist

It is not hard to see that the planar solutions for an originally planar beam, e.g., (56), is the global minimizer of the fully linearized free energy (i.e., the strain energy is given by (48)). Therefore, the out-of-plane and twist solutions, if exist, must scale differently from (42) and (43). Also, in applications it is desirable to precisely relate the critical loading with the geometry of the beam, e.g., the ratio $\hat{r}$. In this regard the asymptotic theory (62) and (63) will not be useful. Nevertheless, for planar beams and keeping only the leading order we can rewrite (62) as

$$u_2'' + (\kappa_0 u_1)' - \frac{1}{2}\kappa_0\varphi^2 \approx 0, \quad \varphi' + \kappa_0 u_3' + \left(\frac{u_3''}{\kappa_0}\right)' \approx 0, \tag{65}$$

which hints the following scaling relations for small strain and twist:

$$u_1' \sim u_2' \sim \varepsilon \ll 1, \quad \varphi(s) \sim u_3' \sim \eta \sim \varepsilon^{1/2}. \tag{66}$$

Again we emphasize that the above scaling relation shall be *a posteriori* verified upon solutions to the simplified problem.

We now compute the strain energy to the leading order according to the scaling relation (66). First of all, to keep $s \mapsto \mathbf{c}_0(s) + \mathbf{u}(s)$ as an arc-length parametrization, we shall require

$$[\mathbf{c}_0(s) + \mathbf{u}(s)]' \cdot [\mathbf{c}_0(s) + \mathbf{u}(s)]' = 1 + [2(u_1' - \kappa_0 u_2) + (u_3')^2] + o(\varepsilon) = 1, \tag{67}$$

and hence

$$u_1' - \kappa_0 u_2 + \frac{1}{2}(u_3')^2 = 0. \tag{68}$$

Repeating calculations as for (44), by (66) we find that (recall that $\mathbf{u} = u_1\tilde{\mathbf{e}}_1 + u_2\tilde{\mathbf{e}}_2 + u_3\tilde{\mathbf{e}}_3$)

$$\begin{aligned}
\mathbf{e}_1 &= \tilde{\mathbf{e}}_1 - \frac{1}{2}(u_3')^2\tilde{\mathbf{e}}_1 + (u_2' + \kappa_0 u_1)\tilde{\mathbf{e}}_2 + u_3'\tilde{\mathbf{e}}_3 + o(\varepsilon),\\
\mathbf{u}' &= -\frac{1}{2}(u_3')^2\tilde{\mathbf{e}}_1 + (u_2' + \kappa_0 u_1)\tilde{\mathbf{e}}_2 + u_3'\tilde{\mathbf{e}}_3 + o(\varepsilon),\\
\mathbf{u}'' &= [-\kappa_0(u_2' + \kappa_0 u_1) - u_3' u_3'']\tilde{\mathbf{e}}_1\\
&\quad + [u_2'' + (\kappa_0 u_1)' - \frac{1}{2}\kappa_0 (u_3')^2]\tilde{\mathbf{e}}_2 + u_3''\tilde{\mathbf{e}}_3 + o(\varepsilon),\\
\mathbf{e}_2 &= \frac{\kappa_0\tilde{\mathbf{e}}_2 + \mathbf{u}''}{|\kappa_0\tilde{\mathbf{e}}_2 + \mathbf{u}''|} = \tilde{\mathbf{e}}_2 + \frac{1}{\kappa_0}[\mathbf{u}'' - (\tilde{\mathbf{e}}_2\cdot\mathbf{u}'')\tilde{\mathbf{e}}_2] + o(\varepsilon),\\
\mathbf{e}_3 &= \tilde{\mathbf{e}}_3 + \left\{\frac{1}{\kappa_0}\tilde{\mathbf{e}}_1 \times [\mathbf{u}'' - (\tilde{\mathbf{e}}_2\cdot\mathbf{u}'')\tilde{\mathbf{e}}_2] + \mathbf{u}' \times \tilde{\mathbf{e}}_2\right\} + o(\varepsilon),\\
\kappa &= |\kappa_0\tilde{\mathbf{e}}_2 + \mathbf{u}''| = \kappa_0 + [u_2'' + (\kappa_0 u_1)' - \frac{1}{2}\kappa_0(u_3')^2] + o(\varepsilon),\\
\tau &= \kappa_0 u_3' + \left(\frac{u_3''}{\kappa_0}\right)' + o(\varepsilon).
\end{aligned} \tag{69}$$

Therefore, up to the order of $O(\varepsilon^2)$ the strain energy (16) is given by

$$\begin{aligned}
U_e[\mathbf{u}, \varphi; \mathbf{c}_0] &= \int_0^L \left\{\frac{EI_3}{2}\tilde{\kappa}^2 + \frac{EI_2}{2}(\kappa_0\varphi)^2 + \frac{GJ}{2}(\tilde{\varphi}')^2\right\},\\
\tilde{\kappa} &= u_2'' + (\kappa_0 u_1)' - \frac{1}{2}\kappa_0(u_3')^2 - \frac{1}{2}\kappa_0\varphi^2, \quad \tilde{\varphi} = \varphi + \kappa_0 u_3 + \frac{u_3''}{\kappa_0}.
\end{aligned} \tag{70}$$

Taking into account of the constraint (68) by a Lagrangian multiplier, we can find the Euler–Lagrange equations associated with the principle of minimum free energy as well as the Neumann-type boundary conditions at the ends.

In particular, for the semi-circle planar beam discussed in Section 2.5 with κ_0 being constant, by (68) we have

$$\begin{aligned}
\tilde{\kappa} &= u_2'' + (\kappa_0 u_1)' - \frac{1}{2}\kappa_0(u_3')^2 - \frac{1}{2}\kappa_0\varphi^2\\
&= u_2'' + \kappa_0^2 u_2 - \kappa_0(u_3')^2 - \frac{1}{2}\kappa_0\varphi^2,
\end{aligned} \tag{71}$$

and the Euler–Lagrange equations are given by

$$\begin{cases}
EI_3\tilde{\kappa}'' + EI_3\kappa_0^2\tilde{\kappa} - \kappa_0\lambda = 0,\\
-GJ\tilde{\varphi}'' + EI_2\kappa_0^2\varphi - EI_3\tilde{\kappa}\kappa_0\varphi = 0, \quad \tilde{\varphi} = \varphi + \kappa_0 u_3 + \dfrac{u_3''}{\kappa_0},\\
GJ\tilde{\varphi}'''' + \kappa_0^2 GJ\tilde{\varphi}'' - \kappa_0^2[2EI_3\tilde{\kappa}u_3']' + \kappa_0(\lambda u_3')' = 0,
\end{cases} \tag{72}$$

where λ is the Lagrangian multiplier associated with the constraint (68); the first variation with respect to u_1 implies that $\lambda' = 0$, i.e., λ is a constant.

The nonlinear differential system (72) characterizes the asymptotic behaviors of the curved beam in the scaling regime (66). Upon specifying the boundary conditions, we can in principle solve (72) and determine the critical loadings when the system bifurcates between two or more equilibria. The precise procedure is, however, not so easy to explicitly carry out because of the nonlinearity. Below we present approximate solutions and associated criteria for instabilities based on some simple trial solutions.

3.2. Trial solutions and critical loadings

Since explicit solutions to the nonlinear differential system (72) are generally impossible, we employ the method of trial solutions to obtain insights on the behaviors of the system, particularly, the critical loadings such that the planar solutions presented in Section 2.5 are no longer globally stable.

The method of trial solutions is in the same vein as the Saint-Venant's semi-inverse method (Love, 1944). Instead of directly solving the nonlinear differential system (72), we conjecture trial solutions, typically a special ansatz on the kinematic variables, and then insert this ansatz into the free energy functional. Upon evaluating the free energy functional (or integral) we obtain the free energy in *algebraic* terms of adjustable coefficients of the ansatz. We finally minimize this algebraic form of free energy with respect to these adjustable coefficients and obtain the "optimal" solutions to our problem within the functional form as prescribed by the ansatz. In this method, it is clear that the quality of the final solution is dictated by the quality of trial solutions, i.e., the ansatz. The best trial solution is clearly the actual solution to the original nonlinear system, e.g., (72). Predictions based on this method could be trivial or even erroneous if the quality of the trial solutions deteriorates on one hand, and on the other hand explicit predictions might be impossible if the ansatz is too general to have a simple and explicit parametrization. Therefore, the implementation of this approach for a particular problem can be delicate.

Based on the numerical results and experimental observations, we will consider the following simple trial solutions such that

$$\begin{aligned}
\varphi(s) &= A_0 + A_1\sin\kappa_0 s + A_2\cos\kappa_0\, s,\\
\tilde{\varphi}(s) &= \frac{u_3''}{\kappa_0} + \kappa_0 u_3 + \varphi = B_1\sin\kappa_0\, s,\\
\tilde{\kappa}(s) &= u_2'' + \kappa_0^2 u_2 - \kappa_0(u_3')^2 - \frac{1}{2}\kappa_0\varphi^2\\
&= \kappa_0[C_0 + C_1\sin\kappa_0 s + C_2\cos\kappa_0 s],
\end{aligned} \tag{73}$$

where $A_i,\ B_i,\ C_i\ (i = 0, 1, 2)$ are adjustable dimensionless parameters. Inserting (73) into (70), we obtain the strain energy in terms of these adjustable parameters as ($\hat{r} := I_3/I_2$ and $\tilde{r} := GJ/EI_2$)

$$\begin{aligned}
U_e(A_i, B_i, C_i) = \frac{EI_2\kappa_0}{2}\Big[&\hat{r}\left(\pi C_0^2 + 4C_0C_1 + \frac{\pi}{2}C_1^2 + \frac{\pi}{2}C_2^2\right)\\
&+ \pi A_0^2 + 4A_0A_1 + \frac{\pi}{2}A_1^2 + \frac{\pi}{2}A_2^2 + \tilde{r}\frac{\pi}{2}B_1^2\Big].
\end{aligned} \tag{74}$$

In addition, to be qualified as trial solutions, $(\mathbf{u}, \varphi)$ satisfying (73) shall further conform with the boundary conditions which place a number of restrictions on the adjustable parameters $A_i,\ B_i,\ C_i\ (i = 0, 1, 2)$. Finally, we minimize the free energy against $A_i,\ B_i,\ C_i\ (i = 0, 1, 2)$ and within these restrictions. If the obtained minimum free energy is lower than that of the planar solution, we conclude that the trial out-of-plane and twist solutions are more favorable; the planar solutions are unstable or metastable.

We now present the instability criteria for the semi-circular arches in Fig. 2(a) based on the trial solution satisfying (73). Again, we have three separate cases.[4]

1. **Clamped supports.** If boundary conditions (17) is enforced with $\mathbf{u}_0 = \delta\tilde{\mathbf{e}}_2/2 = -\delta\tilde{\mathbf{e}}_x/2$, $\mathbf{u}_L = \delta\tilde{\mathbf{e}}_2/2 = \delta\tilde{\mathbf{e}}_x/2$ and at the absence of distributed force $\mathbf{q} = 0$, the variational principle for an equilibrium can be written as

$$\min\{F[\mathbf{u}, \varphi; \mathbf{c}_0] = U_e = (70) : \mathbf{u}, \varphi \text{ satisfies (55) and (68)}\}. \tag{75}$$

First, by (71) we notice that for circular arc with constant κ_0 the energy functional (70) does not explicitly depend on u_1. Therefore, we can solve (68) for u_1 and the boundary conditions $u_1(0) = u_1(L) = 0$ is equivalent to

$$\int_0^L [2\kappa_0 u_2 - (u_3')^2]ds = 0. \tag{76}$$

Solving $(73)_2$ for u_3 and applying the boundary conditions $u_3(0) = u_3(L) = u_3'(0) = u_3'(L) = 0$, we find that

$$A_1 - B_1 = -\frac{4A_0}{\pi}, \quad A_2 = 0, \tag{77}$$

[4] Mathematica Notebooks for subsequent calculations are available at the author's (L.L.) homepage http://math.rutgers.edu/ll502/Curvedbeams/.

and

$$u_3(s) = \frac{A_0}{\kappa_0 \pi}[-\pi + (\pi - 2\kappa_0\, s)\cos\kappa_0 s + 2\sin\kappa_0 s]. \tag{78}$$

Solving $(73)_3$ for u_2 and applying the constraint (76), the boundary conditions $u_2(0) = u_2(L) = \frac{\delta}{2}$ and $u_2'(0) = u_2'(L) = 0$, we solve for C_0, C_1, C_2 in terms of A_0, B_1. Functions $\mathbf{u}$ and φ obtained in this way clearly satisfy all of the boundary conditions (55) and constraint (68), and are henceforth qualified as trial solutions.

To proceed, by (74) we evaluate the free energy of this trial solution in terms of adjustable parameters A_0, B_1 and obtain

$$\begin{aligned} F[\mathbf{u}, \varphi; \mathbf{c}_0] = U_e[\mathbf{u}, \varphi; \mathbf{c}_0] &= \frac{EI_2}{2}\int_0^L [\hat{r}\tilde{\kappa}^2 + (\kappa_0\varphi)^2 + \tilde{r}(\tilde{\varphi}')^2] \\ &= \frac{\pi}{\pi^2 - 8} EI_3\kappa_0(\delta\kappa_0)^2 + Q_2(A_0, B_1) + Q_4(A_0, B_1), \end{aligned} \tag{79}$$

where Q_4 is a quartic monomial,

$$\begin{aligned} Q_2 &= EI_2\kappa_0[(a_{11}A_0^2 + a_{12}A_0B_1 + a_{22}B_1^2)\hat{r}(\kappa_0\delta) \\ &\quad + b_{11}A_0^2 + b_{22}(1+\tilde{r})B_1^2], \\ a_{11} &\approx 0.14, \quad a_{12} \approx 0.43, \quad a_{22} \approx -0.56, \quad b_{11} \approx 0.30, \\ b_{22} &\approx 0.79. \end{aligned} \tag{80}$$

We remark that the above constants a_{ij}, b_{ij}, independent of cross-sectional properties of the beam, admit closed-form expressions which are too long to be presented here. From the above expression of free energy, we see that if

$$\begin{cases} \kappa_0\delta > \dfrac{0.44\tilde{r} - 0.23}{\hat{r}} + (1.94 + 1.69\tilde{r} + 0.20\tilde{r}^2)^{1/2} \quad \text{or} \\ \kappa_0\delta < \dfrac{0.44\tilde{r} - 0.23}{\hat{r}} - (1.94 + 1.69\tilde{r} + 0.20\tilde{r}^2)^{1/2}, \end{cases} \tag{81}$$

then $Q_2 + Q_4 < 0$ for some infinitesimal A_0, B_1. Therefore, the free energy (79) is less than (57) of the planar solution, meaning that the planar solution (56) is no longer the global minimizer (or loses its global stability) if the applied displacement exceeds the critical displacements defined by (81).

2. **Simple supports**. If the boundary conditions (32) is enforced with $\mathbf{u}_0 = \delta\tilde{\mathbf{e}}_2/2 = \delta\tilde{\mathbf{e}}_x/2$, $\mathbf{u}_L = \delta\tilde{\mathbf{e}}_2/2 = -\delta\tilde{\mathbf{e}}_x/2$, and at the absence of distributed force $\mathbf{q} = 0$, the variational principle for the equilibrium state can be written as

$$\min\{F[\mathbf{u}, \varphi; \mathbf{c}_0] = U_e = (70) : \mathbf{u}, \varphi \text{ satisfies (55) and (68)}\}. \tag{82}$$

For simplicity, we further restrict ourselves to trial solutions (73) with $A_2 = B_1 = 0, A_1 = -4A_0/\pi$, i.e., u_3 is given by (78). It is clear that the trial functions (73) satisfy $u_3(0) = u_3(L) = 0$. Moreover, solving $(73)_3$ for u_2 and applying the boundary conditions $u_2(0) = u_2(L) = \frac{\delta}{2}$ and constraint (76) yield

$$C_0 = C_2 = 0, \quad C_1 = \frac{(64 - 18\pi^2)A_0^2}{27\pi} + \frac{2\kappa_0\delta}{\pi},$$

and henceforth, the free energy of this trial solution is given by

$$\begin{aligned} F[\mathbf{u}, \varphi; \mathbf{c}_0] = U_e[\mathbf{u}, \varphi; \mathbf{c}_0] &= \int_0^L \left[\frac{EI_3}{2}\tilde{\kappa}^2 + \frac{EI_2}{2}(\kappa_0\varphi)^2\right] \\ &= \frac{1}{\pi}EI_3\kappa_0(\delta\kappa_0)^2 + [-\beta EI_3\kappa_0(\kappa_0\delta) + \frac{\pi^2 - 8}{2\pi}EI_2\kappa_0]A_0^2 \\ &\quad + \alpha EI_3\kappa_0 A_0^4. \end{aligned} \tag{83}$$

where $\alpha \approx 0.014$ and $\beta \approx 0.136$. From the above equation we see that if

$$-\beta EI_3\kappa_0(\kappa_0\delta) + \frac{\pi^2 - 8}{2\pi}EI_2\kappa_0 < 0, \quad \text{i.e.,} \quad \kappa_0\delta > \frac{(\pi^2 - 8)I_2}{2\beta\pi I_3}, \tag{84}$$

the planar solution is no longer the global minimizer of the free energy. Moreover, the constant A_0 can be determined by minizing the right hand side of (79) with respect to A_0 and we obtain

$$(A_0)_{\min} = \left[\frac{1}{2\alpha I_3}[\beta\kappa_0\delta - \frac{(\pi^2 - 8)I_2}{2\pi I_3}\right]^{1/2}.$$

3. **Cantilever.** If (35) is enforced with $\mathbf{p} = p_0\tilde{\mathbf{e}}_2 = p_0\tilde{\mathbf{e}}_x$ at $x = L$ and zero distributed force $\mathbf{q} = 0$, by (36) and (70) we write the free energy functional as

$$\begin{aligned} &F[\mathbf{c}, \varphi; \mathbf{c}_0] \\ &= \int_0^L \left\{\frac{EI_3}{2}[u_2'' + \kappa_0^2 u_2 - \kappa_0(u_3')^2 - \frac{1}{2}\kappa_0\varphi^2]^2 + \frac{EI_2}{2}(\kappa_0\varphi)^2 \right. \\ &\quad \left. + \frac{GJ}{2}(\varphi' + \kappa_0 u_3' + \frac{u_3'''}{\kappa_0})^2\right\} ds - p_0 u_2(L). \end{aligned} \tag{85}$$

and the variational principle for an equilibrium can be written as

$$\min\{F[\mathbf{u}, \varphi; \mathbf{c}_0] : \mathbf{u}, \varphi \text{ satisfies } (60)_1\}. \tag{86}$$

For simplicity, we again restrict ourselves to trial solutions (73) with $A_2 = B_1 = 0$. We first solve $(73)_2$ for u_3, i.e.,

$$\begin{cases} u_3'' + \kappa_0^2 u_3 = -\kappa_0\varphi(s) \quad \text{on } (0, L), \\ u_3(0) = u_3'(0) = 0, \end{cases} \tag{87}$$

and then solve $(73)_3$ for u_2. In account of the boundary condition $u_2(0) = u_2'(0) = 0$, $\tilde{\kappa}'(L) = -\frac{p_0}{EI_3}$ and constraint (76), we find that

$$C_0 = C_2 = 0, \quad C_1 = \frac{p_0}{EI_3\kappa_0^2}.$$

Evaluation the free energy (85) in terms of our trial solution, we obtain

$$\begin{aligned} &F(A_0, A_1; p_0) \\ &= -\frac{\pi p_0^2}{4EI_3\kappa_0^3} + \frac{p_0(-126A_0^2 - 63\pi A_0A_1 + 4A_1^2 - 9\pi^2 A_1^2)}{54\kappa_0} \\ &\quad + \frac{EI_2\kappa_0}{4}(2\pi A_0^2 + 8A_0A_1 + \pi A_1^2). \end{aligned}$$

Therefore, if

$$\begin{aligned} p_0 &> \min\left\{\frac{EI_2 54\kappa_0^2(2\pi A_0^2 + 8A_0A_1 + \pi A_1^2)}{4(126A_0^2 + 63\pi A_0A_1 - 4A_1^2 + 9\pi^2 A_1^2)} : A_0, A_1 \in \mathbb{R}\right\} \\ &= 0.479EI_2\kappa_0^2 \\ &\quad \times \text{(the minimum is achieved if } A_1 \approx -3.71A_0 \neq 0), \end{aligned} \tag{88}$$

the planar solution is no longer the global minimizer of the free energy; any perturbation of form (87) with $A_1 \approx -3.71A_0 \neq 0$ has lower free energy.

We remark that the criteria for instability obtained by considering trial solutions (cf., (73)) shall be regarded as "upper bounds" of what would be obtained if (72) is exactly solved. Moreover, restricting to trial solutions (73) may miss many modes of instability and even give spurious predictions. For example, for simple supports, the selected trial solutions cannot capture the instability that occurs under tension. A full understanding of instability of curved beams inevitably requires exact solutions to the nonlinear system (72) with suitable boundary conditions.

4. Conclusion and discussion

We present a variational framework for curved beams subjected to a variety of boundary conditions. We show that the variational formulation is consistent with the classic field equations which, to some extent, is not obvious at all. Mathematical boundary conditions are systematically derived for clamped supports, simple supports and free ends. Further, explicit solutions for linearized theory and explicit solutions to instabilities and critical loadings are obtained for semi-circular arches. We anticipate that the variational theories, stability analysis and explicit solutions will give us important insight on designing stretchable electronic structures and predictive modeling of biological macromolecules.

A final remark is in order here regarding the application of the present formulation to curved beams of less regularity. First of all, if the centroid curve $\mathbf{c}_0 = \mathbf{c}_0(s)$ is continuously differentiable and piecewisely of C^3-class with finitely many singular points, the Euler–Lagrange Eq. (29) (or (31)) shall hold on each C^3 (open) interval. Meanwhile, the displacement and rigid rotation defined by (5) and (7) shall remain continuous at each singular point, i.e., $[\![\mathbf{c}]\!] = 0$ and $[\![\mathbf{Q}]\!] = 0$, where $[[(\cdot)]]$ denotes the jump of $(\cdot)$. By (5)–(7) and in terms of kinematic variables $(\mathbf{c}, \varphi; \mathbf{c}_0)$, we have the following interfacial conditions at each singular point:

$$[\![\mathbf{c}]\!] = 0, \quad [\![\tilde{\mathbf{e}}_i \cdot \mathbf{f}_j]\!] = 0 \ \ (i, j = 1, 2, 3).$$

In addition, by similar calculations as in (37) we have ((38))

$$[\![\mathbf{P}]\!] = 0, \quad [\![\mathbf{M}]\!] = 0,$$

which may be interpreted as the force and moment balance of an infinitesimal segment containing the singular point. For curved beams that are not continuously differentiable with cusp singularities, extension at the singularities would be important and shall be addressed using the strain energy (14).

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Appendix. Field-equation approach to a geometrically nonlinear beam theory.

In the framework of Cosserat theory, we describe the kinematics of the rod by the deformation of the centroid line $\mathbf{c} : (0, L) \to \mathbb{R}^3$ and the orthonormal basis vectors $\mathbf{f}_i : (0, L) \to \mathbb{R}^3$ $(i = 1, 2, 3)$ attached to a material cross section with $\mathbf{f}_1$ being the unit normal. In general, the basis vectors $\mathbf{f}_i$ may be independent of the centroid line $\mathbf{c}$. Let $\mathbf{e}_1 = \mathbf{c}'/|\mathbf{c}'|$ be the unit tangential vector along the centroid line and define constants Ω_{ij} such that

$$\mathbf{f}_i' = \Omega_{ij}\mathbf{f}_j, \qquad \text{i.e.,} \qquad \Omega_{ij} = \mathbf{f}_i' \cdot \mathbf{f}_j.$$

Since $\mathbf{f}_i \cdot \mathbf{f}_j = \delta_{ij}$ implies that $0 = (\mathbf{f}_i \cdot \mathbf{f}_j)' = \mathbf{f}_i' \cdot \mathbf{f}_j + \mathbf{f}_i \cdot \mathbf{f}_j' = \Omega_{ij} + \Omega_{ji}$, i.e., Ω_{ij} is skew-symmetric, we set $(\omega_1, \omega_2, \omega_3) = (\Omega_{23}, -\Omega_{13}, \Omega_{12})$, i.e., $\omega_i = \frac{1}{2}\mathcal{E}_{ijk}\mathbf{f}_j' \cdot \mathbf{f}_k$, and find that

$$\mathbf{f}_i' = \boldsymbol{\omega} \times \mathbf{f}_i \quad (i = 1, 2, 3). \tag{89}$$

The extension of the centroid line can be described by

$$s = \gamma(\xi^1) = \int_0^{\xi^1} |\mathbf{c}'(t)| dt.$$

In a field-equation approach to an extensible elastic rod, the *internal* Piola–Kirchhoff cross-sectional moment vector $\mathbf{M} : (0, L) \to \mathbb{R}^3$ and force $\mathbf{P} : (0, L) \to \mathbb{R}^3$ are primitive concepts. Under the application of an external (dead) distributed load $\mathbf{q} : (0, L) \to \mathbb{R}^3$, by the balances of linear and angular momenta we obtain (Reissner, 1973; Su et al., 2012)

$$\mathbf{P}' + \mathbf{q} = 0, \quad \mathbf{M}' + \mathbf{c}' \times \mathbf{P} = 0 \quad \text{on } (0, L). \tag{90}$$

Constitutive relations between kinematic variables and internal moment and force (**M, P**) are necessary to close the system. A common choice is to postulate the strain (or internal) energy of the rod is given by, e.g.,

$$U_e[\mathbf{c}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3] = \int_0^L W\left(\frac{1}{2}|\mathbf{c}' - \mathbf{f}_1|^2, \boldsymbol{\omega}\right) dS, \tag{91}$$

where $\mathbb{R} \times \mathbb{R}^3 \ni (a, \mathbf{b}) \mapsto W(a, \mathbf{b})$ is the strain energy density function (Simo, 1985). We remark that the general form of strain energy proposed in Simo (1985) has to be of the above form in order to be consistent with the requirement of frame indifference, i.e., is invariant with respect to an overall rigid motion of the body. It has been proposed that the constitutive relations can be given by Simo (1985)

$$\mathbf{P} = \frac{\partial W}{\partial \mathbf{c}'} = (\mathbf{c}' - \mathbf{f}_1)W_a, \quad \mathbf{M} = \frac{\partial W}{\partial \boldsymbol{\omega}} = W_{\mathbf{b}}. \tag{92}$$

where W_a and $W_{\mathbf{b}}$ represent partial derivatives of the strain energy density function $W = W(a, \mathbf{b})$. Eqs. (90) and (92) compose the governing field equations for curved beams.

We now show that Eqs. (90) and (92) are indeed consistent with our variational approach. In particular, both (90) and (92) follow as the Euler–Lagrange equations to the principle of minimum free energy if one admits that the strain energy is given by (91). As an example, for clamped boundary conditions at the two ends and under the application of the (dead) distributed load $\mathbf{q} : (0, L) \to \mathbb{R}^3$, we see the free energy of the system is given by

$$F[\mathbf{c}, \mathbf{f}_1, \mathbf{f}_2, \mathbf{f}_3] = \int_0^L \left[W\left(\frac{1}{2}|\mathbf{c}' - \mathbf{f}_1|^2, \boldsymbol{\omega}\right) - \mathbf{q} \cdot \mathbf{c}\right] dS.$$

By the principle of minimum free energy, an equilibrium state necessarily satisfies that

$$\frac{d}{d\varepsilon} F[\mathbf{c} + \varepsilon\mathbf{u}, \mathbf{f}_1 + \varepsilon\mathbf{v}_1, \mathbf{f}_2 + \varepsilon\mathbf{v}_3, \mathbf{f}_3 + \varepsilon\mathbf{v}_3]\Big|_{\varepsilon=0} = 0 \tag{93}$$

for all smooth perturbations $\mathbf{u}, \mathbf{v}_i : (0, L) \to \mathbb{R}^3$ satisfying that $\mathbf{u} = \mathbf{v}_i = \mathbf{u}' = \mathbf{v}_i' = \mathbf{u}'' = \mathbf{v}_i'' = 0$ at $\xi^1 = 0 \,\&\, L$ and that

$$(\mathbf{f}_i + \varepsilon\mathbf{v}_i) \cdot (\mathbf{f}_j + \varepsilon\mathbf{v}_j) = \delta_{ij} + o(\varepsilon) \quad \Longrightarrow \quad \mathbf{v}_i = \tilde{\boldsymbol{\omega}} \times \mathbf{f}_i \ \text{ for some } \tilde{\boldsymbol{\omega}} : (0, L) \to \mathbb{R}^3.$$

By (93), integrating by parts we find that

$$\int_0^L \mathbf{u} \cdot \{-[(\mathbf{c}' - \mathbf{f}_1)W_a]' - \mathbf{q}\} + \tilde{\boldsymbol{\omega}} \cdot [-(W_{\mathbf{b}})' - \mathbf{f}_1 \times (\mathbf{c}' - \mathbf{f}_1)W_a] d\xi^1 = 0, \tag{94}$$

where we have used the identity (Steigmann and Faulkner, 1993)

$$\begin{aligned}
&(W_{\mathbf{b}})_i \frac{1}{2}\mathcal{E}_{ijk}[(\tilde{\boldsymbol{\omega}} \times \mathbf{f}_j)' \cdot \mathbf{f}_k + \mathbf{f}_j' \cdot (\tilde{\boldsymbol{\omega}} \times \mathbf{f}_k)] \\
&= \frac{1}{2}(W_{\mathbf{b}})_i \mathcal{E}_{ijk}(\tilde{\boldsymbol{\omega}}' \times \mathbf{f}_j) \cdot \mathbf{f}_k = \frac{1}{2}(W_{\mathbf{b}})_i \mathcal{E}_{ijk}\tilde{\boldsymbol{\omega}}' \cdot (\mathbf{f}_j \times \mathbf{f}_k) \\
&= \frac{1}{2}(W_{\mathbf{b}})_i \mathcal{E}_{ijk}\tilde{\boldsymbol{\omega}}' \cdot (\mathcal{E}_{mjk}\mathbf{f}_m) = \tilde{\boldsymbol{\omega}}' \cdot W_{\mathbf{b}}.
\end{aligned}$$

Since $\mathbf{u}, \tilde{\boldsymbol{\omega}} : (0, L) \to \mathbb{R}^3$ are arbitrary and independent of each other, by (94) we obtain the following Euler–Lagrangian equations for an equilibrium state:

$$[(\mathbf{c}'-\mathbf{f}_1)W_a]'+\mathbf{q}=0,\quad (W_{\mathbf{b}})'+\mathbf{f}_1\times(\mathbf{c}'-\mathbf{f}_1)W_a=0 \quad \text{on } (0,L). \tag{95}$$

Comparing (95) with (90), we justify the constitutive relation (92) by noticing

$$\mathbf{f}_1\times\mathbf{P}=\mathbf{f}_1\times(\mathbf{c}'-\mathbf{f}_1)W_a=\mathbf{f}_1\times\mathbf{c}'W_a=\mathbf{c}'\times\mathbf{P}.$$

If we impose the Bernoulli–Euler kinematics, we have the constraint $\mathbf{c}'=\gamma'\mathbf{f}_1$. Then by the method of Lagrange's multiplier we find the following necessary conditions for an equilibrium state:

$$[\mathbf{f}_1W_a(\gamma'-1)+\boldsymbol{\lambda}]'+\mathbf{q}=0,$$
$$(W_{\mathbf{b}})'+\mathbf{f}_1\times[\mathbf{f}_1W_a(\gamma'-1)+\gamma'\boldsymbol{\lambda}]=0 \quad \text{on } (0,L).$$

Comparing the above equation with (90), we justify the second of (92) and identify $\mathbf{P}=\mathbf{f}_1W_a(\gamma'-1)+\boldsymbol{\lambda}$ by noticing

$$\mathbf{f}_1\times[\mathbf{f}_1W_a(\gamma'-1)+\gamma'\boldsymbol{\lambda}]=\gamma'\mathbf{f}_1\times\boldsymbol{\lambda}=\mathbf{c}'\times\mathbf{P}.$$

If the rod is further assumed to be inextensible, i.e., $\mathbf{c}'=\mathbf{f}_1$ and $\gamma'\equiv 1$, the variational principle (93) implies

$$\boldsymbol{\lambda}'+\mathbf{q}=0,\quad (W_{\mathbf{b}})'+\mathbf{c}'\times\boldsymbol{\lambda}=0 \quad \text{on } (0,L), \tag{96}$$

which implies $\mathbf{P}=\boldsymbol{\lambda}$ and the second of (92).

Applying the above framework to the scenario discussed in Section 2, we identify the orthonormal frame $\{\mathbf{f}_1,\mathbf{f}_2,\mathbf{f}_3\}=\mathbf{Q}\{\tilde{\mathbf{e}}_1,\tilde{\mathbf{e}}_2,\tilde{\mathbf{e}}_3\}$, and by (5), (7) and Frenet equation, find that

$$\begin{cases}\mathbf{f}_1'=\kappa(\cos\varphi\mathbf{f}_2-\sin\varphi\mathbf{f}_3),\\ \mathbf{f}_2'=-\kappa\cos\varphi\mathbf{f}_1+(\varphi'+\tau)\mathbf{f}_3,\\ \mathbf{f}_3'=\kappa\sin\varphi\mathbf{f}_1-(\varphi'+\tau)\mathbf{f}_2.\end{cases} \tag{97}$$

By (89), the above equation implies the curvature vector $\boldsymbol{\omega}:(0,L)\to\mathbb{R}^3$ is given by

$$\boldsymbol{\omega}=\omega_i\mathbf{f}_i,\quad (\omega_1,\omega_2,\omega_3):=(\varphi'+\tau,\kappa\sin\varphi,\kappa\cos\varphi).$$

Therefore, comparing (16) and (91) we identify the strain energy density function as

$$W(\boldsymbol{\omega})=\frac{GJ}{2}(\omega_1-\omega_1^0)^2+\frac{EI_2}{2}(\omega_2-\omega_2^0)^2+\frac{EI_3}{2}(\omega_3-\omega_3^0)^2,$$

where $(\omega_1^0,\omega_2^0,\omega_3^0)=(\tau_0,0,\kappa_0)$. By (92) and (96), we can write the equilibrium equation in terms of curvature vector $\boldsymbol{\omega}$ as

$$\begin{cases}\mathbf{P}'+\mathbf{q}=0 & \text{on } (0,L),\\ \mathbf{M}'+\mathbf{f}_1\times\mathbf{P}=0 & \text{on } (0,L),\\ \mathbf{M}=\dfrac{\partial W}{\partial\boldsymbol{\omega}}=GJ(\omega_1-\omega_1^0)\mathbf{f}_1+EI_2(\omega_2-\omega_2^0)\mathbf{f}_2+EI_3(\omega_3-\omega_3^0)\mathbf{f}_3.\end{cases} \tag{98}$$

From $(98)_2$, we have

$$\mathbf{f}_1\times[\mathbf{M}'+\mathbf{f}_1\times\mathbf{P}]=\mathbf{f}_1\times\mathbf{M}'+\mathbf{f}_1(\mathbf{P}\cdot\mathbf{f}_1)-\mathbf{P}=0,\quad \mathbf{f}_1\cdot\mathbf{M}'=0.$$

In account of $(98)_1$, we have

$$\begin{cases}[\mathbf{f}_1\times\mathbf{M}'+\mathbf{f}_1(\mathbf{P}\cdot\mathbf{f}_1)]'+\mathbf{q}=0 & \text{on } (0,L),\\ \mathbf{f}_1\cdot\mathbf{M}'=0 & \text{on } (0,L).\end{cases}$$

Inserting the last of (98) into the above equation, upon tedious algebraic calculations we can show the above differential system is equivalent to the Euler–Lagrangian Eq. (29). The advantage of the variational formulation lies in the explicit parametrization of the centroid line and relative twist and the systematic derivation of boundary conditions.

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