Q-Method Extended Kalman Filter

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A new algorithm is proposed that smoothly incorporates the non-linear estimation of the attitude quaternion using Davenport's q-method and the estimation of non-attitude states through an extended Kalman filter. The new algorithm is compared to an existing one and the various similarities and differences are discussed. The validity of the proposed approach is confirmed by numerical simulations.

I. Introduction

The well-known *Wahba Problem* [1] is a non-linear, weighted least-squares performance index that seeks to obtain the optimal attitude matrix from a set of at least two independent vector measurements. The most common technique used to solve the Wahba problem is the so-called q-method, developed by Davenport and documented by Keat [2]. The q-method rearranges the Wahba performance index into a quadratic performance index of the attitude quaternion, which is constrained to have unit norm. The extremals of this performance index are the eigenvalues of the Davenport matrix, and the optimal quaternion is the unit eigenvector corresponding to the largest eigenvalue.

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A variety of numerical approaches exist for calculating the maximum eigenvalue and the corresponding eigenvector of the Davenport matrix. For example, the QUEST algorithm [3] calculates the eigenvalue using a Newton-Raphson method and the eigenvector by factoring the quaternion as a vector of Rodrigues parameters. To avoid the singularity of the Rodrigues parameters, the method of successive rotations is introduced[3]. Alternatively, ESOQ [4] circumvents the singularity by computing the quaternion as a vector cross product in four dimensions. In a follow-on algorithm, ESOQ-2 [5], the Euler axis is computed as the null space of a 3×3 matrix that is derived from the Davenport matrix.

QUEST, ESOQ, and ESOQ-2 are numerical implementations of Davenport's q-method. Other numerical techniques exist that compute the attitude matrix directly rather than the quaternion. One such technique by Markley is based on the Singular Value Decomposition (SVD) [6]. Here, it should be noted that the original Wahba problem objective function is fundamentally just a special case of the *Orthogonal Procrustes Problem*, which has received a considerable amount of study since the 1950s [7].

One of the reasons that the Wahba problem has received significant attention is that it provides a globally optimal solution and it does not make any linearization or small angle approximations. Conversely, the workhorse of aerospace estimation, the extended Kalman filter (EKF) [8] relies on linearization to obtain an estimate. The solution to the Wahba problem provides single point attitude estimates and requires all the measurements to be synchronized. The EKF and its attitude-specific extensions (most notably additive EKF [9] and multiplicative EKF [10]) in contrast are recursive estimators.

With this in mind, a number of algorithms have been developed to reformulate Davenport's solution into a recursive algorithm. Two of the first such methods are Filter QUEST [11] and REQUEST [12], which are filters capable of estimating attitude (but not other states, such as biases). Later, Filter QUEST and REQUEST were shown to be two different formulations of mathematically equivalent filters [13]. Subsequently, the Optimal-REQUEST filter [14] addressed the sub-optimality of these filters, but was still not capable of estimating non-attitude states.

Markley [15] shows how to estimate not only attitude, but also other parameters such sensor biases from vector observations. Extended-QUEST also estimates attitude and non-attitude states [16]. This paper introduces a novel EKF-based estimation algorithm that integrates the q-method to process attitude vector measurements. The existing algorithm that most closely resembles the present work is the Sequential Optimal Attitude Recursion (SOAR) filter by Christian and Lightsey [17]. The key difference is that SOAR uses the information formulation of the Kalman filter for the measurement update, while the proposed method is a covariance formulation. This difference will usually require smaller matrix inversions when the size of the state vector is large. Another difference between the two methods relates to how the initial condition is introduced into the Wahba problem. This paper uses quaternion averaging [18], while SOAR uses the information matrix approach by Shuster [19]. Beyond these differences, the proposed q-method EKF (qEKF) and the SOAR filter are shown to be equivalent to second-order in both the attitude update and the non-attitude state updates. Hence qEKF and SOAR can be considered as the covariance and information approaches to the solution of the same problem.

The herein proposed algorithm smoothly incorporates the q-method into the EKF framework. Similar to the SOAR filter and Extended QUEST, the proposed algorithm processes the vector measurements first and the remaining quantities last. However, unlike Extended QUEST, both the SOAR filter and qEKF do not necessitate numerical iterations (other than solving an eigenvalue problem for a 4×4 matrix; Extended QUEST requires numerically solving a nonlinear equation and to solve the eigenvalue problem for an 8×8 matrix). Shuster [20] suggests that numerical solutions to the q-method such as QUEST could be used as a pre-processor for the EKF. The proposed algorithm takes this concept one step further by integrating the q-method into the EKF. Finally, as in the SOAR filter, the present formulation of the qEKF only considers measurements that are only a function of the attitude portion of the state.

The remainder of this paper is organized as follows. First the Wahba problem and its solution are introduced in Section II followed by a description of the MEKF in Section III. Section IV presents the new algorithm followed by a detailed comparison to SOAR. In Section VI numerical simulations are introduced to validate the proposed approach, finally some conclusions are drawn in section VII.

II. The Wahba Problem

Re-written in terms of the inertial-to-body quaternion $\bar{\mathbf{q}} = [\mathbf{q}_v^{\mathrm{T}} \; q_4]^{\mathrm{T}}$ with vector part $\mathbf{q}_v$ and scalar part q_4, the Wahba problem consists of minimizing the performance index

$$\min_{\hat{\bar{\mathbf{q}}}} \mathcal{J}\left(\hat{\bar{\mathbf{q}}}\right) = \frac{1}{2}\sum_{i=1}^{n} a_i \left\|\tilde{\mathbf{y}}_i - \mathbf{T}(\hat{\bar{\mathbf{q}}})\tilde{\mathbf{n}}_i\right\|^2, \tag{1}$$

where $\tilde{\mathbf{y}}_i$ are vector observations, $\tilde{\mathbf{n}}_i$ are their known representation in the reference frame, a_i are positive scalar weights, and $\mathbf{T}(\hat{\bar{\mathbf{q}}})$ is the direction cosine matrix taking vectors in the reference frame to vectors in the measurement frame.

In the absence of noise, the perfect measurement is simply given by

$$\mathbf{y}_i = \mathbf{T}(\bar{\mathbf{q}})\,\mathbf{n}_i, \tag{2}$$

where $\mathbf{n}_i$ is the perfect (i.e. true) value of $\tilde{\mathbf{n}}_i$. Re-introducing the presence of uncertainty (omitting the dependency on $\bar{\mathbf{q}}$)

$$\tilde{\mathbf{y}}_i = \mathbf{T}\mathbf{n}_i + \delta\mathbf{y}_i \qquad\qquad \tilde{\mathbf{n}}_i = \mathbf{n}_i + \delta\mathbf{n}_i. \tag{3}$$

Since $\|\tilde{\mathbf{y}}_i\| = \|\mathbf{y}_i\| = 1$ and $\|\tilde{\mathbf{n}}_i\| = \|\mathbf{n}_i\| = 1$, the following is also true to first order

$$\tilde{\mathbf{y}}_i^{\mathrm{T}}\delta\mathbf{y}_i \approx \mathbf{y}_i^{\mathrm{T}}\delta\mathbf{y}_i \approx 0 \qquad \tilde{\mathbf{n}}_i^{\mathrm{T}}\delta\mathbf{n}_i \approx \mathbf{n}_i^{\mathrm{T}}\delta\mathbf{n}_i \approx 0. \tag{4}$$

This leads directly to the QUEST measurement model [20] for a unit vector observation

$$\mathbf{R}_{\mathbf{nn},i} = \mathrm{E}\left\{\delta\mathbf{n}_i\delta\mathbf{n}_i^{\mathrm{T}}\right\} = \sigma_{\mathbf{n}_i}^2\left(\mathbf{I}_{3\times3} - \mathbf{n}_i\mathbf{n}_i^{\mathrm{T}}\right) \tag{5a}$$

$$\mathbf{R}_{\mathbf{yy},i} = \mathrm{E}\left\{\delta\mathbf{y}_i\delta\mathbf{y}_i^{\mathrm{T}}\right\} = \sigma_{\mathbf{y}_i}^2\left(\mathbf{I}_{3\times3} - \mathbf{y}_i\mathbf{y}_i^{\mathrm{T}}\right). \tag{5b}$$

Substituting this result into Eq. (1) (and assuming that $\delta\mathbf{y}_i$ and $\delta\mathbf{n}_i$ are uncorrelated) shows that for $\hat{\bar{\mathbf{q}}}$ to be a maximum likelihood estimate of the attitude (to first order and when the errors are distributed as in[19]) the weights a_i should be

$$a_i \approx 1/\left(\sigma_{\mathbf{n}_i}^2 + \sigma_{\mathbf{y}_i}^2\right). \tag{6}$$

Returning to Eq. (1), the goal is next to reformulate the problem in terms of the attitude quaternion. Begin by recalling that the attitude matrix written as a function of the quaternion is given by [21]

$$\mathbf{T} = \mathbf{T}(\bar{\mathbf{q}}) = \mathbf{I}_{3\times3} - 2q_4[\mathbf{q}_v\times] + 2[\mathbf{q}_v\times]^2 \tag{7}$$

$$= \left(q_4^2 - \mathbf{q}_v^{\mathrm{T}}\mathbf{q}_v\right)\mathbf{I}_{3\times3} - 2q_4[\mathbf{q}_v\times] + 2\mathbf{q}_v\mathbf{q}_v^{\mathrm{T}}, \tag{8}$$

where $[\mathbf{v}\times]$ is the 3×3 vector cross product skew symmetric matrix. The minimization of the Wahba performance index in Eq. (1) is thus equivalent to the maximization of [2]

$$\max_{\hat{\bar{\mathbf{q}}}} \mathcal{J}^\star\left(\hat{\bar{\mathbf{q}}}\right) = \mathrm{trace}\left[\mathbf{T}\left(\hat{\bar{\mathbf{q}}}\right)\mathbf{B}^{\mathrm{T}}\right] = \hat{\bar{\mathbf{q}}}^{\mathrm{T}}\mathbf{K}\hat{\bar{\mathbf{q}}}. \tag{9}$$

In this equation the 4×4 Davenport matrix $\mathbf{K}$ is obtained as

$$\mathbf{B} = \sum_{i=1}^{n} a_i\tilde{\mathbf{y}}_i\tilde{\mathbf{n}}_i^{\mathrm{T}} \qquad \mathbf{z} = \sum_{i=1}^{n} a_i\left(\tilde{\mathbf{y}}_i \times \tilde{\mathbf{n}}_i\right)$$

$$\mathbf{S} = \mathbf{B} + \mathbf{B}^{\mathrm{T}} \qquad \sigma^\star = \mathrm{trace}(\mathbf{B})$$

$$\mathbf{K} = \begin{bmatrix} \mathbf{S} - \sigma^\star\mathbf{I}_{3\times3} & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & \sigma \end{bmatrix} \tag{10}$$

the optimal quaternion is the unit eigenvector of $\mathbf{K}$ associated with the maximum eigenvalue.

In this work $\mathbf{K}$ is slightly modified to perform covariance analysis. The performance index is equivalently rewritten as

$$\mathcal{J}^\star\left(\hat{\bar{\mathbf{q}}}\right) = \sigma^\star + \hat{\bar{\mathbf{q}}}^{\mathrm{T}}\begin{bmatrix} \mathbf{M} & \mathbf{z} \\ \mathbf{z}^{\mathrm{T}} & 0 \end{bmatrix}\hat{\bar{\mathbf{q}}} \tag{11}$$

$$\mathbf{M} = \sum_{i=1}^{n} a_i\left([\tilde{\mathbf{y}}_i\times][\tilde{\mathbf{n}}_i\times] + [\tilde{\mathbf{n}}_i\times][\tilde{\mathbf{y}}_i\times]\right) = \mathbf{S} - 2\sigma^\star\mathbf{I}_{3\times3} \tag{12}$$

where the identity $\mathbf{a}\mathbf{b}^{\mathrm{T}} = [\mathbf{b}\times][\mathbf{a}\times] + (\mathbf{a}^{\mathrm{T}}\mathbf{b})\mathbf{I}_{3\times3}$ is used. Hence the optimal quaternion is the unit eigenvector of the matrix in Eq. (11) corresponding to its maximum eigenvalue.

Recall that the perfect measurements $\mathbf{y}_i$ are defined as $\mathbf{y}_i = \mathbf{T}\mathbf{n}_i$ where $\mathbf{T}$ is the true attitude matrix and $\mathbf{n}_i$ are error-free reference vectors. By using $\mathbf{y}_i$ and $\mathbf{n}_i$ in place of $\tilde{\mathbf{y}}_i$ and $\tilde{\mathbf{n}}_i$ in the q-method and solving for the optimal attitude the true quaternion $\bar{\mathbf{q}}$ is obtained. The matrix $\mathbf{B}_{true}$ is computed with the perfect values $\mathbf{y}_i$ and $\mathbf{n}_i$. When the vectors $\mathbf{y}_i$ and $\mathbf{T}(\bar{\mathbf{q}})\mathbf{n}_i$ are used as the inputs in the q-method rather than $\mathbf{y}_i$ and $\mathbf{n}_i$, the identity quaternion is obtained as the optimal solution; with this approach we are estimating the deviation from the true body frame which is denoted as $\delta\bar{\mathbf{q}}^*$, the superscript "$*$" indicates the quaternion conjugate. Using $\mathbf{y}_i$ and $\mathbf{T}(\bar{\mathbf{q}})\mathbf{n}_i$ to calculate matrix $\mathbf{B}$ yields $\mathbf{B}_{true}\mathbf{T}\left(\bar{\mathbf{q}}\right)^{\mathrm{T}}$. Hence the performance index Eq. (9) is rewritten as

$$\mathcal{J}^{\star}(\delta\bar{\mathbf{q}}^*) = \text{trace}\left[\mathbf{T}\left(\delta\bar{\mathbf{q}}^*\right)\mathbf{T}\left(\bar{\mathbf{q}}\right)\mathbf{B}_{true}^{\mathrm{T}}\right]. \tag{13}$$

Notice that the combination of having perfect measurements and replacing $\mathbf{n}_i$ with $\mathbf{T}(\bar{\mathbf{q}})\mathbf{n}_i$ results in $\mathbf{z} = \mathbf{0}$. This makes the performance index

$$\mathcal{J}^{\star}(\delta\bar{\mathbf{q}}^*) = \sigma + \delta\bar{\mathbf{q}}^{*\mathrm{T}}\begin{bmatrix}\mathbf{H}_{true} & \mathbf{0}\\ \mathbf{0}^{\mathbf{T}} & 0\end{bmatrix}\delta\bar{\mathbf{q}}^*, \tag{14}$$

where

$$\mathbf{H}_{true} = \sum_{i=1}^{n} a_i\left([\mathbf{y}_i\times]\,[(\mathbf{T}\mathbf{n}_i)\times] + [(\mathbf{T}\mathbf{n}_i)\times]\,[\mathbf{y}_i\times]\right) = 2\sum_{i=1}^{n} a_i[\mathbf{y}_i\times]^2. \tag{15}$$

Matrix $\mathbf{H}_{true}$ has non-positive eigenvalues, therefore the maximum eigenvalue of the modified Davenport matrix is zero and the optimal solution is the identity quaternion.

Re-introducing the error in the measurements and using $\tilde{\mathbf{y}}_i$ and $\mathbf{T}\left(\bar{\mathbf{q}}\right)\tilde{\mathbf{n}}_i$ in the q-method, the algorithm returns the estimation error since the performance index becomes

$$\mathcal{J}^{\star}\left(\delta\bar{\mathbf{q}}^*\right) = \text{trace}\left[\mathbf{T}\left(\delta\bar{\mathbf{q}}^*\right)\mathbf{T}(\bar{\mathbf{q}})\mathbf{B}^{\mathrm{T}}\right]. \tag{16}$$

Using the same steps as going from Eq. (9) to Eq. (11) (and making use of the definition of the quaternion conjugate $\bar{\mathbf{q}}^* = \begin{bmatrix}-\mathbf{q}_v^{\mathrm{T}} & q_4\end{bmatrix}^{\mathrm{T}}$) one obtains

$$\mathcal{J}^{\star}\left(\delta\bar{\mathbf{q}}\right) = \sigma + \delta\bar{\mathbf{q}}^{\mathrm{T}}\begin{bmatrix}\mathbf{H}_{\boldsymbol{\theta}} & \delta\mathbf{z}\\ \delta\mathbf{z}^{\mathrm{T}} & 0\end{bmatrix}\delta\bar{\mathbf{q}}, \tag{17}$$

where

$$\mathbf{H}_{\boldsymbol{\theta}} = \sum_{i=1}^{n} a_i\left([\tilde{\mathbf{y}}_i\times]\,[(\mathbf{T}\tilde{\mathbf{n}}_i)\times] + [(\mathbf{T}\tilde{\mathbf{n}}_i)\times]\,[\tilde{\mathbf{y}}_i\times]\right) \tag{18}$$

$$\delta\mathbf{z} = -\sum_{i=1}^{n} a_i\left(\tilde{\mathbf{y}}_i \times \mathbf{T}\tilde{\mathbf{n}}_i\right). \tag{19}$$

In the absence of noise, the optimal eigenvalue is equal to zero. With noise, the optimal eigenvalue is a small quantity $\delta\lambda$. The eigenvalue problem requires the following equation to be satisfied

$$\mathbf{H}_{\boldsymbol{\theta}}\,\delta\mathbf{q}_v + \delta q_4\,\delta\mathbf{z} = \delta\lambda\,\delta\mathbf{q}_v. \tag{20}$$

Making a first-order approximation of the quaternion ($\delta q_4 \simeq 1$) and neglecting terms of order higher than one, the estimation error is found to be

$$\delta \mathbf{q}_v = -\mathbf{H}_{\boldsymbol{\theta}}^{-1} \delta \mathbf{z}. \tag{21}$$

Define the attitude estimation error $\delta\boldsymbol{\theta}$ as a rotation vector, as a first order approximation $\delta\boldsymbol{\theta} \simeq 2\delta\mathbf{q}_v$. Therefore, the covariance of the estimation error is given by

$$\mathbf{P}_{\boldsymbol{\theta\theta}} = \mathrm{E}\left\{\delta\boldsymbol{\theta}\delta\boldsymbol{\theta}^{\mathrm{T}}\right\} = 4\mathbf{H}_{\boldsymbol{\theta}}^{-1}\,\mathrm{E}\left\{\delta\mathbf{z}\delta\mathbf{z}^{\mathrm{T}}\right\}\mathbf{H}_{\boldsymbol{\theta}}^{-\mathrm{T}} \tag{22}$$

which is equivalent to the result obtained by Shuster [3] but using a different approach. Since the true attitude is unknown, $\mathbf{H}_{\boldsymbol{\theta}}$ needs to be evaluated at the estimated attitude; the added approximation is a second-order effect. Using Eqs. (2) and (3), Eq. (19) becomes to first-order

$$\delta\mathbf{z} = -\sum_{i=1}^{n} a_i \left\{\mathbf{y}_i \times (\mathbf{T}\delta\mathbf{n}_i) + \delta\mathbf{y}_i \times (\mathbf{T}\mathbf{n}_i)\right\}. \tag{23}$$

Therefore, assuming that each source of error is uncorrelated from the others

$$\mathrm{E}\left\{\delta\mathbf{z}\delta\mathbf{z}^{\mathrm{T}}\right\} = \sum_{i=1}^{n} a_i^2 \left\{[\mathbf{y}_i\times]\,\mathbf{T}\,\mathrm{E}\left\{\delta\mathbf{n}_i\delta\mathbf{n}_i^{\mathrm{T}}\right\}\mathbf{T}^{\mathrm{T}}\,[\mathbf{y}_i\times]^{\mathrm{T}} + [(\mathbf{T}\mathbf{n}_i)\times]\,\mathrm{E}\left\{\delta\mathbf{y}_i\delta\mathbf{y}_i^{\mathrm{T}}\right\}[(\mathbf{T}\mathbf{n}_i)\times]^{\mathrm{T}}\right\}. \tag{24}$$

To calculate $\mathrm{E}\left\{\delta\mathbf{z}\delta\mathbf{z}^{\mathrm{T}}\right\}$ the unknown quantities $\mathbf{y}_i$, $\mathbf{n}_i$, and $\mathbf{T}$ need to be replaced by the known quantities $\tilde{\mathbf{y}}_i$, $\tilde{\mathbf{n}}_i$, and $\hat{\mathbf{T}} = \mathbf{T}\left(\hat{\bar{\mathbf{q}}}\right)$.

Next, for reasons that will become evident in the subsequent section, suppose that one defines $\mathbf{R}_{\mathbf{zz}}$ as

$$\mathbf{R}_{\mathbf{zz}} = 4\,\mathrm{E}\left\{\delta\mathbf{z}\delta\mathbf{z}^{\mathrm{T}}\right\} = 4\sum_{i=1}^{n} a_i^2 \left\{[\tilde{\mathbf{y}}_i\times]\,\hat{\mathbf{T}}\mathbf{R}_{\mathbf{nn}_i}\hat{\mathbf{T}}^{\mathrm{T}}\,[\tilde{\mathbf{y}}_i\times]^{\mathrm{T}} + \left[(\hat{\mathbf{T}}\tilde{\mathbf{n}}_i)\times\right]\mathbf{R}_{\mathbf{yy}_i}\left[(\hat{\mathbf{T}}\tilde{\mathbf{n}}_i)\times\right]^{\mathrm{T}}\right\} \tag{25}$$

III. The Multiplicative Extended Kalman Filter

The Multiplicative Extended Kalman Filter (MEKF) is presently the industry-standard for attitude filtering when both attitude and non-attitude states must be considered. As such, a brief discussion of this approach will be used as a point of departure for our subsequent developments.

The MEKF was developed to account for the fact that attitude errors are multiplicative in nature. This multiplicative relationship is immediately evident when one looks at the relationship between the actual attitude, $\bar{\mathbf{q}}$, and the estimated attitude, $\hat{\bar{\mathbf{q}}}$,

$$\mathbf{T}(\bar{\mathbf{q}}) = \mathbf{T}(\delta\bar{\mathbf{q}})\mathbf{T}(\hat{\bar{\mathbf{q}}}) \tag{26}$$

where $\delta\bar{\mathbf{q}}$ is the attitude estimation error expressed as a quaternion.

In the MEKF, the state update is handled by linearizing the problem about the *a priori* attitude estimate and then using a three dimensional attitude parameterization. In the present work, the rotation vector $\delta\boldsymbol{\theta}$ is used as the attitude state in the filter. It is common to make small angle approximation, such that $\delta\boldsymbol{\theta} \simeq 2\delta\mathbf{q}_v$ and $\delta q_4 \simeq 1$. Furthermore, by construction, the *a priori* state attitude error state is $\delta\hat{\boldsymbol{\theta}}^- = 0$.

With these factors considered, the state vector for an MEKF will look something like

$$\mathbf{x} = \begin{bmatrix} \delta\boldsymbol{\theta} \\ \\ \hat{\mathbf{s}} \end{bmatrix} \tag{27}$$

where $\hat{\mathbf{s}}$ is the estimate of the non-attitude states. The state estimate, $\hat{\mathbf{x}}$, and state covariance, $\mathbf{P}$, may now be computed as in the standard EKF,

$$\hat{\mathbf{x}}^+ = \hat{\mathbf{x}}^- + \mathbf{K}(\mathbf{y} - \mathbf{h}(\hat{\mathbf{x}}^-))\mathbf{P}^+ \qquad = (\mathbf{I} - \mathbf{KH})\mathbf{P}^-(\mathbf{I} - \mathbf{KH})^{\mathrm{T}} + \mathbf{KRK}^{\mathrm{T}} \tag{28}$$

and if $\mathbf{K}$ is chosen to be the optimal Kalman gain, then

$$\mathbf{K} = \mathbf{P}^-\mathbf{H}^{\mathrm{T}}(\mathbf{HP}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1} \tag{29}$$

The quaternion may then be updated according to,

$$\hat{\bar{\mathbf{q}}}^+ = \bar{\mathbf{q}}\big(\delta\hat{\boldsymbol{\theta}}^+\big) \otimes \hat{\bar{\mathbf{q}}}^- \tag{30}$$

where $\bar{\mathbf{q}}\big(\boldsymbol{\theta}\big)$ is the quaternion parameterization of the rotation vector $\boldsymbol{\theta}$ and the quaternion product $\otimes$ is defined such that the quaternions are multiplied in the same order as the attitude matrices.

IV. The Q-Method Extended Kalman Filter

The derivation approach for the qEKF used here was chosen to highlight the connection between this new method and the MEKF. An alternate derivation was presented in the conference version of this paper [22].

For the moment, assume one has a linear measurement model given by

$$\tilde{\mathbf{y}} = \mathbf{Hx} + \mathbf{v} \tag{31}$$

Where $\mathbf{v}$ is zero mean noise with covariance $\mathbf{R}$. From here, one may arrive at the Kalman Filter by minimizing the following objective function,

$$\text{Min} \quad J(\mathbf{x}) = \frac{1}{2}(\mathbf{x} - \hat{\mathbf{x}}^-)^{\mathrm{T}}\mathbf{P}^-(\mathbf{x} - \hat{\mathbf{x}}^-) + \frac{1}{2}(\tilde{\mathbf{y}} - \mathbf{H}\mathbf{x})^{\mathrm{T}}\mathbf{R}^{-1}(\tilde{\mathbf{y}} - \mathbf{H}\mathbf{x}) \tag{32}$$

which, of course, leads to the well known results

$$\hat{\mathbf{x}}^+ = \hat{\mathbf{x}}^- + \mathbf{K}(\mathbf{y} - \mathbf{H}\hat{\mathbf{x}}^-) \tag{33}$$

$$\mathbf{P}^+ = (\mathbf{I} - \mathbf{K}\mathbf{H})\mathbf{P}^-(\mathbf{I} - \mathbf{K}\mathbf{H})^{\mathrm{T}} + \mathbf{K}\mathbf{R}\mathbf{K}^{\mathrm{T}} \tag{34}$$

$$\mathbf{K} = \mathbf{P}^-\mathbf{H}^{\mathrm{T}}(\mathbf{H}\mathbf{P}^-\mathbf{H}^{\mathrm{T}} + \mathbf{R})^{-1} \tag{35}$$

A. Partitioning of the State Vector

Now, as in the MKEF, suppose that the state is partitioned into two parts,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} \tag{36}$$

where $\mathbf{x}_1$ will eventually become the attitude states and $\mathbf{x}_2$ will eventually become the non-attitude states. Likewise, the covariance matrix is partitioned as

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{11} & \mathbf{P}_{12} \\ \mathbf{P}_{21} & \mathbf{P}_{22} \end{bmatrix} \tag{37}$$

As was stated in the introduction, the present work only considered measurements that are only a function of the attitude state. Thus, the measurement sensitivity matrix may be written as

$$\mathbf{H} = \begin{bmatrix} \mathbf{H}_1 & \mathbf{0} \end{bmatrix} \tag{38}$$

Taking advantage of this form of $\mathbf{H}$, it is evident that

$$\mathbf{H}\mathbf{P}\mathbf{H}^{\mathrm{T}} = \mathbf{H}_1\mathbf{P}_{11}\mathbf{H}_1^{\mathrm{T}} \tag{39}$$

meaning that the partitioned Kalman gain is

$$\mathbf{K} = \begin{bmatrix} \mathbf{K}_1 \\ \mathbf{K}_2 \end{bmatrix} = \begin{bmatrix} \mathbf{P}_{11}^-\mathbf{H}_1^{\mathrm{T}}(\mathbf{H}_1\mathbf{P}_{11}\mathbf{H}_1^{\mathrm{T}} + \mathbf{R})^{-1} \\ \mathbf{P}_{21}^-\mathbf{H}_1^{\mathrm{T}}(\mathbf{H}_1\mathbf{P}_{11}\mathbf{H}_1^{\mathrm{T}} + \mathbf{R})^{-1} \end{bmatrix} \tag{40}$$

Now, substituting the partitioned matrix results into Eqs. (33) and (34), the update for $\hat{\mathbf{x}}_1$ and $\mathbf{P}_{11}$ becomes

$$\hat{\mathbf{x}}_1^+ = \hat{\mathbf{x}}_1^- + \mathbf{K}_1(\mathbf{y} - \mathbf{H}_1\hat{\mathbf{x}}_1^-) \tag{41}$$

$$\mathbf{P}_{11}^+ = (\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)\mathbf{P}_{11}^-(\mathbf{I} - \mathbf{K}_1\mathbf{H}_1)^{\mathrm{T}} + \mathbf{K}_1\mathbf{R}\mathbf{K}_1^{\mathrm{T}} \tag{42}$$

What is important about this is that the computation of $\mathbf{K}_1$, $\hat{\mathbf{x}}_1^+$, and $\mathbf{P}_{11}^+$ is completely independent of the terms related to the second part of the partitioned state: $\hat{\mathbf{x}}_2$, $\mathbf{P}_{12}$, $\mathbf{P}_{21}$, or $\mathbf{P}_{22}$. Thus, the update to $\hat{\mathbf{x}}_1$ and $\mathbf{P}_{11}$ (which will soon become the attitude state) may be performed independently of the second part of the state.

To compute the update of $\hat{\mathbf{x}}_2$, expand out the Kalman gain for the update of both $\hat{\mathbf{x}}_1$ and $\hat{\mathbf{x}}_2$.

$$\hat{\mathbf{x}}_1^+ = \hat{\mathbf{x}}_1^- + \mathbf{P}_{11}^-\mathbf{H}_1^{\mathrm{T}}(\mathbf{H}_1\mathbf{P}_{11}\mathbf{H}_1^{\mathrm{T}} + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{H}_1\hat{\mathbf{x}}_1^-) \tag{43}$$

$$\hat{\mathbf{x}}_2^+ = \hat{\mathbf{x}}_2^- + \mathbf{P}_{21}^-\mathbf{H}_1^{\mathrm{T}}(\mathbf{H}_1\mathbf{P}_{11}\mathbf{H}_1^{\mathrm{T}} + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{H}_1\hat{\mathbf{x}}_1^-) \tag{44}$$

Rearranging Eq. (43) produces

$$(\mathbf{P}_{11}^-)^{-1}(\hat{\mathbf{x}}_1^+ - \hat{\mathbf{x}}_1^-) = \mathbf{H}_1^{\mathrm{T}}(\mathbf{H}_1\mathbf{P}_{11}\mathbf{H}_1^{\mathrm{T}} + \mathbf{R})^{-1}(\mathbf{y} - \mathbf{H}_1\hat{\mathbf{x}}_1^-) \tag{45}$$

which, when substituted into Eq. (44) yields the desired update for $\hat{\mathbf{x}}_2$,

$$\hat{\mathbf{x}}_2^+ = \hat{\mathbf{x}}_2^- + \mathbf{P}_{21}^-(\mathbf{P}_{11}^-)^{-1}(\hat{\mathbf{x}}_1^+ - \hat{\mathbf{x}}_1^-) \tag{46}$$

Computing the remaining covariance terms $\mathbf{P}_{12}$, $\mathbf{P}_{21}$, and $\mathbf{P}_{22}$ is a straightforward exercise in applying the definition of the covariance matrix. Define the error in the state estimate as $\mathbf{e}^+ = \mathbf{x} - \hat{\mathbf{x}}^+$ and $\mathbf{e}^- = \mathbf{x} - \hat{\mathbf{x}}^-$. Thus, Eq. (46) may be rewritten as

$$\mathbf{e}_2^+ = \mathbf{e}_2^- + \mathbf{P}_{21}^-(\mathbf{P}_{11}^-)^{-1}(\mathbf{e}_1^+ - \mathbf{e}_1^-) \tag{47}$$

Begin by noting that $\mathbf{P}_{11}^+$ is already known from Eq. (42). Now, the covariance for the cross term $\mathbf{P}_{21}^+$ is defined as

$$\mathbf{P}_{21}^+ = \mathrm{E}\left\{\mathbf{e}_2^+(\mathbf{e}_1^+)^{\mathrm{T}}\right\} \tag{48}$$

where $\mathrm{E}\{\,\}$ is the expected value operator. Substituting the results from Eq. (47) into this and distributing the expected value operator,

$$\mathbf{P}_{21}^{+} = \mathrm{E}\left\{\mathbf{e}_2^{-}(\mathbf{e}_1^{+})^{\mathrm{T}}\right\} + \mathbf{P}_{21}^{-}(\mathbf{P}_{11}^{-})^{-1}\left(\mathrm{E}\left\{\mathbf{e}_1^{+}(\mathbf{e}_1^{+})^{\mathrm{T}}\right\} - \mathrm{E}\left\{\mathbf{e}_1^{-}(\mathbf{e}_1^{+})^{\mathrm{T}}\right\}\right)$$
$$= \mathbf{P}_{21}^{-}(\mathbf{P}_{11}^{-})^{-1}\,\mathrm{E}\left\{\mathbf{e}_1^{+}(\mathbf{e}_1^{+})^{\mathrm{T}}\right\} = \mathbf{P}_{21}^{-}(\mathbf{P}_{11}^{-})^{-1}\mathbf{P}_{11}^{+} \tag{49}$$

The other cross term may be found as

$$\mathbf{P}_{12}^{+} = \mathrm{E}\left\{\mathbf{e}_1^{+}(\mathbf{e}_2^{+})^{\mathrm{T}}\right\} = (\mathbf{P}_{21}^{+})^{\mathrm{T}} \tag{50}$$

Finally, the remaining covariance term $\mathbf{P}_{22}^{+}$ may be computed as,

$$\mathbf{P}_{22}^{+} = \mathrm{E}\left\{\mathbf{e}_2^{+}(\mathbf{e}_2^{+})^{\mathrm{T}}\right\} \tag{51}$$

Substituting Eq. (47), distributing the expected value operator, and combining terms will yield

$$\mathbf{P}_{22}^{+} = \mathbf{P}_{22}^{-} + \mathbf{P}_{21}^{-}\left[(\mathbf{P}_{11}^{-})^{-1}\mathbf{P}_{11}^{+}(\mathbf{P}_{11}^{-})^{-1} - (\mathbf{P}_{11}^{-})^{-1}\right]\mathbf{P}_{12}^{-} \tag{52}$$

B. Application to the Attitude Filtering Problem

This subsection will adapt the above results to the specific problem of attitude estimation. Begin by assuming the same form of the state vector as in the MEKF,

$$\mathbf{x} = \begin{bmatrix} \mathbf{x}_1 \\ \mathbf{x}_2 \end{bmatrix} = \begin{bmatrix} \delta\boldsymbol{\theta} \\ \mathbf{s} \end{bmatrix} \tag{53}$$

and the corresponding covariance matrix,

$$\mathbf{P} = \begin{bmatrix} \mathbf{P}_{\theta\theta} & \mathbf{P}_{\theta s} \\ \mathbf{P}_{s\theta} & \mathbf{P}_{ss} \end{bmatrix} \tag{54}$$

With this selection in mind, recall that it was observed in the previous subsection that the updates to $\hat{\mathbf{x}}_1$ and $\mathbf{P}_{11}$ may be performed independently of the remaining states. Further, note that the solution for $\hat{\mathbf{x}}_1^{+}$ and $\mathbf{P}_{11}^{+}$ from Eqs. (41) and (42) is of the same form as the solution to Eq. (32). Therefore, the optimal solution for $\hat{\mathbf{x}}_1^{+}$ and $\mathbf{P}_{11}^{+}$ for the full problem is equivalent to the solution to

$$\text{Min} \quad J(\mathbf{x}_1) = \frac{1}{2}(\mathbf{x}_1 - \hat{\mathbf{x}}_1^{-})^{\mathrm{T}}(\mathbf{P}_{11}^{-})^{-1}(\mathbf{x}_1 - \hat{\mathbf{x}}_1^{-}) + \frac{1}{2}(\tilde{\mathbf{y}} - \mathbf{H}_1\mathbf{x}_1)^{\mathrm{T}}\mathbf{R}^{-1}(\tilde{\mathbf{y}} - \mathbf{H}_1\mathbf{x}_1) \tag{55}$$

or, equivalently,

$$\text{Min} \quad J(\delta\boldsymbol{\theta}) = \frac{1}{2}\delta\boldsymbol{\theta}^{\mathrm{T}}(\mathbf{P}_{\theta\theta}^{-})^{-1}\delta\boldsymbol{\theta} + \frac{1}{2}(\tilde{\mathbf{y}} - \mathbf{H}_1\delta\boldsymbol{\theta})^{\mathrm{T}}\mathbf{R}^{-1}(\tilde{\mathbf{y}} - \mathbf{H}_1\delta\boldsymbol{\theta}) \tag{56}$$

Note that $\mathbf{H}_1$ is left in this notation to avoid confusion with $\mathbf{H}_\theta$ from Eq. (18).

Two observation about Eq. (56) are now made in regards of the attitude specific problem. First it is noticed that the *a priori* portion of the performance index is nothing more than a first-order approximation to the quaternion averaging performance index [18]

$$\frac{1}{2}\delta\boldsymbol{\theta}^{\mathrm{T}}(\mathbf{P}_{\theta\theta}^{-})^{-1}\delta\boldsymbol{\theta} \simeq \hat{\bar{\mathbf{q}}}^{\mathrm{T}}\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})\mathbf{A}_0\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})^{\mathrm{T}}\hat{\bar{\mathbf{q}}} \tag{57}$$

where $\mathbf{A}_0 = 2(\mathbf{P}_{\boldsymbol{\theta\theta}}^{-})^{-1}$ is a 3×3 symmetric positive definite matrix weight and where

$$\boldsymbol{\Xi}\left(\bar{\mathbf{q}}\right) = \begin{bmatrix} q_4\mathbf{I}_{3\times 3} + [\mathbf{q}_v \times] \\ -\mathbf{q}_v^{\mathrm{T}} \end{bmatrix}.$$

The second observation is that rather than utilizing the linear (or linearized) measurement model, a nonlinear least-squares performance index can be used

$$\mathcal{J}^{\star}(\hat{\bar{\mathbf{q}}}) = -\hat{\bar{\mathbf{q}}}^{\mathrm{T}}\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})\mathbf{A}_0\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})^{\mathrm{T}}\hat{\bar{\mathbf{q}}} - \frac{1}{2}\sum_{i=1}^{n} a_i\left\|\tilde{\mathbf{y}}_i - \mathbf{T}\left(\hat{\bar{\mathbf{q}}}\right)\tilde{\mathbf{n}}_i\right\|^2. \tag{58}$$

The right-hand term in Eq. (58) is the standard Wahba Problem and, therefore, the performance index may be again rewritten as

$$\mathcal{J}^{\star}(\hat{\bar{\mathbf{q}}}) = -\hat{\bar{\mathbf{q}}}^{\mathrm{T}}\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})\mathbf{A}_0\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})^{\mathrm{T}}\hat{\bar{\mathbf{q}}} + \hat{\bar{\mathbf{q}}}^{\mathrm{T}}\mathbf{K}\hat{\bar{\mathbf{q}}} \tag{59}$$

where the $\mathbf{K}$ in this equation is the Davenport matrix and not the Kalman gain. Thus, the optimization problem is equivalent to

$$\mathrm{Max}\,\mathcal{J}^{\star}(\hat{\bar{\mathbf{q}}}) = \hat{\bar{\mathbf{q}}}^{\mathrm{T}}\mathbf{K}^{+}\hat{\bar{\mathbf{q}}} \tag{60}$$

where

$$\mathbf{K}^{+} = \mathbf{K} - \boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})\mathbf{A}_0\boldsymbol{\Xi}(\hat{\bar{\mathbf{q}}}^{-})^{\mathrm{T}} \tag{61}$$

Therefore, from the solution to the Wahba Problem, the optimal attitude estimate $\hat{\bar{\mathbf{q}}}^{+}$ may be found by finding the unit eigenvector associated with the maximum eigenvalue of $\mathbf{K}^{+}$.

The discussion now moves to the computation of the a-posteriori attitude covariance, $\mathbf{P}_{\theta\theta}^{+}$. Define the attitude error quaternion $\delta\bar{\mathbf{q}}^{-}$ according to

$$\bar{\mathbf{q}} = \delta\bar{\mathbf{q}}^{-} \otimes \hat{\bar{\mathbf{q}}}^{-} \tag{62}$$

To perform covariance analysis the same procedure as the previous section is used but instead of estimating the quaternion $\bar{\mathbf{q}}$, the deviation from it is sought. The performance index for the equivalent maximization problem then becomes

$$\mathcal{J}^\star(\delta\bar{\mathbf{q}}) = \delta\bar{\mathbf{q}}^{\mathrm{T}} \begin{bmatrix} \mathbf{H_0} & \delta\mathbf{z_0} \\ \delta\mathbf{z_0}^{\mathrm{T}} & s_0 \end{bmatrix} \delta\bar{\mathbf{q}} + \sigma + \delta\bar{\mathbf{q}}^{\mathrm{T}} \begin{bmatrix} \mathbf{H}_{\boldsymbol{\theta}} & \delta\mathbf{z} \\ \delta\mathbf{z}^{\mathrm{T}} & 0 \end{bmatrix} \delta\bar{\mathbf{q}} \tag{63a}$$

$$\mathbf{H_0} = -\mathbf{A}_0 - \left[\delta\mathbf{q}_v^- \times\right] \mathbf{A}_0 + \mathbf{A}_0 \left[\delta\mathbf{q}_v^- \times\right] + \left[\delta\mathbf{q}_v^- \times\right] \mathbf{A}_0 \left[\delta\mathbf{q}_v^- \times\right] \tag{63b}$$

$$\delta\mathbf{z_0} = \mathbf{A}_0\delta\mathbf{q}_v^- + \left[\delta\mathbf{q}_v^- \times\right] \mathbf{A}_0\delta\mathbf{q}_v^- \tag{63c}$$

$$s_0 = -(\delta\mathbf{q}_v^-)^{\mathrm{T}}\mathbf{A}_0\delta\mathbf{q}_{v0}^- \tag{63d}$$

wherere $\delta\mathbf{q}_v^-$ is the vector part of quaternion $\delta\bar{\mathbf{q}}^-$, while the scalar part is approximated the be equal to one (first order approximation). Making first-order approximation in Eqs. (63b)–(63d), the performance index in Eq. (10) becomes

$$\mathcal{J}^\star(\delta\bar{\mathbf{q}}) = \sigma + \delta\bar{\mathbf{q}}^{\mathrm{T}} \begin{bmatrix} -\mathbf{A}_0 - [\delta\mathbf{q}_v^- \times] \mathbf{A}_0 + \mathbf{A}_0 [\delta\mathbf{q}_v^- \times] + \mathbf{H}_{\boldsymbol{\theta}} & \mathbf{A}_0\delta\mathbf{q}_v^- + \delta\mathbf{z} \\ (\delta\mathbf{q}_v^-)^{\mathrm{T}}\mathbf{A}_0 + \delta\mathbf{z}^{\mathrm{T}} & 0 \end{bmatrix} \delta\bar{\mathbf{q}}. \tag{64}$$

This performance index is maximized when (assuming $\delta q_4 \simeq 1$)

$$\delta\mathbf{q}_v = -(-\mathbf{A}_0 - \left[\delta\mathbf{q}_v^- \times\right] \mathbf{A}_0 + \mathbf{A}_0 \left[\delta\mathbf{q}_v^- \times\right] + \mathbf{H}_{\boldsymbol{\theta}})^{-1}(\mathbf{A}_0\delta\mathbf{q}_v^- + \delta\mathbf{z}) \tag{65a}$$

$$\simeq -(-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}})^{-1}(\mathbf{A}_0\delta\mathbf{q}_v^- + \delta\mathbf{z}), \tag{65b}$$

where the approximation holds to first-order. Using the definition of $\mathbf{R_{zz}}$ in Eq. (25) and assuming $\mathbf{q}_v^-$ and $\delta\mathbf{z}$ are uncorrelated it follows that

$$\mathbf{P}_{\boldsymbol{\theta\theta}} = (-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}})^{-1}(\mathbf{A}_0\mathbf{P}_{\boldsymbol{\theta\theta}}^-\mathbf{A}_0 + \mathbf{R_{zz}})(-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}})^{\mathrm{T}} \tag{66a}$$

$$= \mathbf{K}_{\boldsymbol{\theta}} \left(\mathbf{A}_0\mathbf{P}_{\boldsymbol{\theta\theta}}^-\mathbf{A}_0 + \mathbf{R_{zz}}\right) \mathbf{K}_{\boldsymbol{\theta}}^{\mathrm{T}}, \tag{66b}$$

where

$$\mathbf{K}_{\boldsymbol{\theta}} = \left(-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}}\right)^{-1}. \tag{67}$$

Notice that $\mathbf{A}_0$ is symmetric positive definite by definition while $\mathbf{H}_{\boldsymbol{\theta}}$ is symmetric negative semi-definite when only one measurement is present and negative definate when multiple independent measurements exist. Therefore $\mathbf{K}_{\boldsymbol{\theta}}$ can always be computed. This covariance update equation is next rewritten in the familiar Joseph form [23] by first noting that the following term can be expressed as

$$(-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}})^{-1}(-\mathbf{A}_0) = (-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}})^{-1}(-\mathbf{A}_0 + \mathbf{H}_{\boldsymbol{\theta}} - \mathbf{H}_{\boldsymbol{\theta}}) = \mathbf{I} - \mathbf{K}_{\boldsymbol{\theta}}\mathbf{H}_{\boldsymbol{\theta}}.$$

Applying this result into Eq. (66a), together with Eq. (67), gives

$$\mathbf{P}^{+}_{\boldsymbol{\theta\theta}} = (\mathbf{I} - \mathbf{K}_{\boldsymbol{\theta}}\mathbf{H}_{\boldsymbol{\theta}})\,\mathbf{P}^{-}_{\boldsymbol{\theta\theta}}\,(\mathbf{I} - \mathbf{K}_{\boldsymbol{\theta}}\mathbf{H}_{\boldsymbol{\theta}})^{\mathrm{T}} + \mathbf{K}_{\boldsymbol{\theta}}\mathbf{R}_{\mathbf{zz}}\mathbf{K}^{\mathrm{T}}_{\boldsymbol{\theta}}, \tag{68}$$

which is the Joseph formula. Again, the initial weight is chosen as $\mathbf{A}_0 = 2\mathbf{P}^{-1}_{\boldsymbol{\theta\theta}0}$ because the first term of Eq. (58) does not contain the factor 1/2 and $\delta\hat{\boldsymbol{\theta}} \simeq 2\boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}_0\right)^{\mathrm{T}}\hat{\bar{\mathbf{q}}}$.

With $\delta\hat{\boldsymbol{\theta}}^{+}$ and $\mathbf{P}^{+}$ found using the methods describe here, the update of the non-attitude states may be performed using the relations derived in the previous subsection:

$$\hat{\mathbf{s}}^{+} = \hat{\mathbf{s}}^{-} + \mathbf{P}^{-}_{s\theta}(\mathbf{P}^{-}_{\theta\theta})^{-1}\delta\hat{\boldsymbol{\theta}}^{+} = \hat{\mathbf{s}}^{-} + 2\mathbf{P}^{-}_{s\theta}(\mathbf{P}^{-}_{\theta\theta})^{-1}\boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}_0\right)^{\mathrm{T}}\hat{\bar{\mathbf{q}}}^{+} \tag{69}$$

$$\mathbf{P}^{+}_{s\theta} = \mathbf{P}^{-}_{s\theta}(\mathbf{P}^{-}_{\theta\theta})^{-1}\mathbf{P}^{+}_{\theta\theta} \tag{70}$$

$$\mathbf{P}^{+}_{ss} = \mathbf{P}^{-}_{ss} + \mathbf{P}^{-}_{s\theta}\left[(\mathbf{P}^{-}_{\theta\theta})^{-1}\mathbf{P}^{+}_{\theta\theta}(\mathbf{P}^{-}_{\theta\theta})^{-1} - (\mathbf{P}^{-}_{\theta\theta})^{-1}\right]\mathbf{P}^{-}_{\theta s} \tag{71}$$

In summary, for linear measurements it is equivalent to first update the attitude and subsequently use this updated portion of the state to update the remainder of the state as it is to update the entire state at once. For the attitude estimation case of this work the measurement model is nonlinear. A nonlinear update for the attitude is obtained using the q-method and subsequently used to update the non-attitude states using the optimal gain for the linear measurement case. Therefore, the proposed q-method extended Kalman filter updates the attitude using the q-method and all remaining non-attitude states using the standard extended Kalman filter method. In summary the qEKF filter has a propagation phase exactly the same as in the MEKF and an update phase as follows

1. Calculate the Davenport matrix $\mathbf{K}$ from Eq. (10) associated with all attitude vector measurements

2. Calculate $\mathbf{A}_0 = 2\left(\mathbf{P}_{\theta\theta}^{-}\right)^{-1}$

3. Calculate the updated attitude quaternion as the unit eigenvector associated with the maximum eigenvalue of

$$\mathbf{K}^{+} = \mathbf{K} - \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}^{-}\right)\mathbf{A}_0\boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}^{-}\right)^{\mathrm{T}}$$

4. Calculate the updated attitude covariance partition $\mathbf{P}_{\boldsymbol{\theta\theta}}^{+}$ from Eqs. (18), (25), and (68)

5. Update the non-attitude states using

$$\hat{\mathbf{s}}^{+} \ = \ \hat{\mathbf{s}}^{-} + 2\mathbf{P}_{\mathbf{s}\boldsymbol{\theta}}^{-}\left(\mathbf{P}_{\boldsymbol{\theta\theta}}^{-}\right)^{-1}\boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}^{-}\right)^{\mathrm{T}}\hat{\bar{\mathbf{q}}}^{+}$$

6. Calculate the total covariance update using Eqs. (68), (70), and (71)

V. Comparison with the SOAR Filter

This section establishes the equivalence of the qEKF and the SOAR filter. It begins by making a key observation about the attitude profile matrix, and then proceeds to compare the attitude update and the non-attitude update.

A. Observations on Computation of the Attitude Profile Matrix

Begin by recalling that the Wahba Problem objective function given in Eq. (9) is the negative log-likelihood function when a_i are chosen as shown in Eq. (6). The attitude may be expressed about the estimate using a Taylor Series expansion truncated to second-order

$$J\left(\boldsymbol{\delta\theta}\right) = -\mathrm{trace}\left[\left(\mathbf{I}_{3\times3} + \left[-\boldsymbol{\delta\theta}\times\right] + \frac{1}{2}\left[-\boldsymbol{\delta\theta}\times\right]^2\right)\mathbf{T}\mathbf{B}^{\mathrm{T}}\right]. \tag{72}$$

Under mild conditions, the Fisher information matrix, $\mathcal{F}_{\theta\theta}$ is the expected value of the second-order derivative of the negative log-likelihood function. Recall from the Cramèr-Rao inequality that the attitude covariance, $\mathbf{P}_{\theta\theta}$, is related to the Fisher information matrix by [24]

$$\mathbf{P}_{\theta\theta}^{-1} \leq \mathcal{F}_{\theta\theta} = E\left[\frac{\partial^2 J(\boldsymbol{\delta\theta})}{\partial\boldsymbol{\delta\theta}\,\partial\boldsymbol{\delta\theta}}\right], \tag{73}$$

and that $\mathcal{F}_{\theta\theta}$ approaches $\mathbf{P}_{\theta\theta}^{-1}$ as the number of measurements become large.

Because Eq. (73) requires the second derivative of $J(\boldsymbol{\delta\theta})$ with respect to $\boldsymbol{\delta\theta}$, terms in $J(\boldsymbol{\delta\theta})$ that are independent of $\boldsymbol{\delta\theta}$ or linear in $\boldsymbol{\delta\theta}$ vanish in the computation of $\mathcal{F}_{\theta\theta}$. Therefore,

$$\mathcal{F}_{\theta\theta} = E\left[\frac{\partial^2 J(\boldsymbol{\delta\theta})}{\partial\boldsymbol{\delta\theta}\,\partial\boldsymbol{\delta\theta}}\right] = E\left[\frac{\partial^2}{\partial\boldsymbol{\delta\theta}\,\partial\boldsymbol{\delta\theta}}\left(-\text{trace}\left[\frac{1}{2}\left[-\boldsymbol{\delta\theta}\times\right]^2\mathbf{T}\mathbf{B}^{\mathrm{T}}\right]\right)\right] \tag{74}$$

To compact notation, define the matrix $\mathbf{V} = \mathbf{T}\mathbf{B}^{\mathrm{T}}$,

$$\mathcal{F}_{\theta\theta} = \frac{1}{2}E\left[\frac{\partial^2}{\partial\boldsymbol{\delta\theta}\,\partial\boldsymbol{\delta\theta}}\left(-\text{trace}\left[\left[-\boldsymbol{\delta\theta}\times\right]^2\mathbf{V}\right]\right)\right] \tag{75}$$

Now, making the observation that,

$$\left[-\boldsymbol{\delta\theta}\times\right]^2 = \boldsymbol{\delta\theta}\,\boldsymbol{\delta\theta}^{\mathrm{T}} - (\boldsymbol{\delta\theta}^{\mathrm{T}}\boldsymbol{\delta\theta})\mathbf{I}_{3\times3} \tag{76}$$

Eq. (75) can be rewritten as

$$\mathcal{F}_{\theta\theta} = -\frac{1}{2}E\left[\frac{\partial^2}{\partial\boldsymbol{\delta\theta}\partial\boldsymbol{\delta\theta}}\left(\text{trace}\left[\boldsymbol{\delta\theta}\,\boldsymbol{\delta\theta}^{\mathrm{T}}\mathbf{V}\right] - \boldsymbol{\delta\theta}^{\mathrm{T}}\boldsymbol{\delta\theta}\,\text{trace}\left[\mathbf{V}\right]\right)\right]. \tag{77}$$

Taking advantage of the cyclic properties of the trace operator, this equation can be recast

$$\mathcal{F}_{\theta\theta} = -\frac{1}{2}E\left[\frac{\partial^2}{\partial\boldsymbol{\delta\theta}\partial\boldsymbol{\delta\theta}}\left(\boldsymbol{\delta\theta}^{\mathrm{T}}\mathbf{V}\boldsymbol{\delta\theta} - \text{trace}\left[\mathbf{V}\right]\boldsymbol{\delta\theta}^{\mathrm{T}}\boldsymbol{\delta\theta}\right)\right]. \tag{78}$$

Straightforward differentiation yields,

$$\mathcal{F}_{\theta\theta} = \text{trace}\left[\mathbf{V}\right]\mathbf{I}_{3\times3} - \frac{1}{2}\left(\mathbf{V} + \mathbf{V}^{\mathrm{T}}\right). \tag{79}$$

In the presence of perfect measurements one may note that $\mathbf{V} = \mathbf{V}^{\mathrm{T}}$. It is only under these conditions that one arrives at the result presented by Shuster in Ref. [19].

$$\mathcal{F}_{\theta\theta} = \text{trace}\left[\mathbf{V}\right]\mathbf{I}_{3\times3} - \mathbf{V}. \tag{80}$$

In general, however, this is not the case and using Shuster's formulation will result in non-symmetric (and hence incorrect) information and covariance matrices. However one may correctly compute the Fisher information matrix in the presence of noise as

$$\mathbf{P}_{\theta\theta}^{-1} \approx \mathcal{F}_{\theta\theta} = \text{trace}\left[\mathbf{T}\mathbf{B}^{\mathrm{T}}\right]\mathbf{I}_{3\times3} - \frac{1}{2}\left(\mathbf{T}\mathbf{B}^{\mathrm{T}} + \mathbf{B}\mathbf{T}^{\mathrm{T}}\right). \tag{81}$$

which enforces a symmetric covariance matrix even in the presence of noisy measurements.

The solution provided by Shuster in Ref. [19] to compute $\mathbf{B}$ from $\mathcal{F}_{\theta\theta}$ and $\mathbf{T}$ is still valid. By taking the trace of Eq. (81), note that

$$\text{trace}\left[\mathcal{F}_{\theta\theta}\right] = 3\text{trace}\left[\mathbf{T}\mathbf{B}^{\mathrm{T}}\right] - \text{trace}\left[\mathbf{T}\mathbf{B}^{\mathrm{T}}\right] = 2\text{trace}\left[\mathbf{T}\mathbf{B}^{\mathrm{T}}\right]. \tag{82}$$

Substituting this back into Eq. (81) gives

$$\mathbf{T}\mathbf{B}^{\mathrm{T}} + \mathbf{B}\mathbf{T}^{\mathrm{T}} = \text{trace}\left[\mathcal{F}_{\theta\theta}\right]\mathbf{I}_{3\times 3} - 2\mathcal{F}_{\theta\theta}. \tag{83}$$

Next, it is straightforward to verify that the following solution originally given by Shuster in Ref. [19] is also a solution to this equation

$$\mathbf{B} = \left[\frac{1}{2}\text{trace}\left[\mathcal{F}_{\theta\theta}\right]\mathbf{I}_{3\times 3} - \mathcal{F}_{\theta\theta}\right]\mathbf{T}. \tag{84}$$

B. Equivalence of the Attitude Update

Recall from Ref. [17] that the SOAR filter includes the a-priori attitude information through the following term

$$-\hat{\bar{\mathbf{q}}}^{\mathrm{T}}\mathbf{K}^{-}\hat{\bar{\mathbf{q}}} = -\text{trace}\left[\mathbf{T}\left(\mathbf{B}^{-}\right)^{\mathrm{T}}\right] \tag{85}$$

in the objective function. In Ref. [17] it is also shown that, after a second-order expansion of the matrix exponential of $[-\boldsymbol{\delta\theta}\times]$ about the *a priori* attitude, this objective function may be rewritten as

$$-\hat{\bar{\mathbf{q}}}^{\mathrm{T}}\mathbf{K}^{-}\hat{\bar{\mathbf{q}}} = -\left(\hat{\bar{\mathbf{q}}}^{-}\right)^{\mathrm{T}}\mathbf{K}^{-}\hat{\bar{\mathbf{q}}}^{-} + \frac{1}{2}\boldsymbol{\delta\theta}^{\mathrm{T}}\mathcal{F}_{\theta\theta}\boldsymbol{\delta\theta}. \tag{86}$$

The first term is a constant (not dependent on the *a posteriori* attitude) and disappears when the first differentials are taken to compute the optimal attitude.

It is next straightforward to show that the *a priori* attitude term introduced in Eq. (58) is equivalent to $1/2\,\boldsymbol{\delta\theta}^{\mathrm{T}}\mathcal{F}_{\theta\theta}\boldsymbol{\delta\theta}$ to second-order. Thus, both the qEKF and the SOAR filters can be shown to include the *a priori* attitude information in an equivalent manner to second-order.

To show this, begin by noting that

$$\boldsymbol{\delta}\mathbf{q}_v = \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}^{-}\right)^{\mathrm{T}}\hat{\bar{\mathbf{q}}} = \sin\left(\frac{\boldsymbol{\delta\theta}}{2}\right). \tag{87}$$

Taking the Taylor Series expansion of $\sin(\boldsymbol{\delta\theta}/2)$, one may show that, to second-order,

$$\boldsymbol{\delta}\mathbf{q}_v = \sin\left(\frac{\boldsymbol{\delta\theta}}{2}\right) = \frac{\boldsymbol{\delta\theta}}{2} - \frac{1}{3}\left(\frac{\boldsymbol{\delta\theta}}{2}\right)^3 + \frac{1}{5}\left(\frac{\boldsymbol{\delta\theta}}{2}\right)^5 \ldots \approx \frac{\boldsymbol{\delta\theta}}{2}. \tag{88}$$

Therefore, the first term in Eq. (58) may be rewritten as

$$\hat{\bar{\mathbf{q}}}^{\mathrm{T}} \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}_0\right) \mathbf{A}_0 \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}_0\right)^{\mathrm{T}} \hat{\bar{\mathbf{q}}} \approx \frac{1}{4} \boldsymbol{\delta\theta}^{\mathrm{T}} \mathbf{A}_0 \boldsymbol{\delta\theta}. \tag{89}$$

Noting from before that $\mathbf{A}_0$ was chosen as $\mathbf{A}_0 = 2\left(\mathbf{P}_{\theta\theta}^{-}\right)^{-1} \approx 2\mathcal{F}_{\theta\theta}$, this leads to

$$\hat{\bar{\mathbf{q}}}^{\mathrm{T}} \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}_0\right) \mathbf{A}_0 \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}_0\right)^{\mathrm{T}} \hat{\bar{\mathbf{q}}} \approx \frac{1}{2} \boldsymbol{\delta\theta}^{\mathrm{T}} \mathcal{F}_{\theta\theta} \boldsymbol{\delta\theta}. \tag{90}$$

Therefore, the *a priori* attitude additions to the objective function for both SOAR and the qEKF are equivalent to second-order.

C. Equivalence of the Non-Attitude Update

The non-attitude state update in the SOAR filer is also equivalent to second-order to the qEKF non-attitude update. Partition the Fisher information matrix of the full covariance as

$$\mathbf{P}^{-1} = \begin{bmatrix} \mathbf{P}_{\mathbf{ss}} & \mathbf{P}_{\mathbf{s}\theta} \\ \mathbf{P}_{\theta\mathbf{s}} & \mathbf{P}_{\theta\theta} \end{bmatrix}^{-1} = \mathbf{F}_{\mathbf{xx}} = \begin{bmatrix} \mathbf{F}_{\mathbf{ss}} & \mathbf{F}_{\mathbf{s}\theta} \\ \mathbf{F}_{\theta\mathbf{s}} & \mathbf{F}_{\theta\theta} \end{bmatrix}. \tag{91}$$

Given this definition, recall from Ref. [17] that the optimal update of the non-attitude states in the SOAR filter is given by the equation

$$\mathbf{s}^+ = \mathbf{s}^- - 2\left(\mathbf{F}_{ss}^-\right)^{-1} \mathbf{F}_{s\theta}^- \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}^-\right)^{\mathrm{T}} \hat{\bar{\mathbf{q}}}^+ \tag{92a}$$

$$\mathbf{s}^+ \approx \mathbf{s}^- - \left(\mathbf{F}_{ss}^-\right)^{-1} \mathbf{F}_{s\theta}^- \,\boldsymbol{\delta\theta} \tag{92b}$$

From the definition of the partitioned matrix inverse

$$\mathbf{F}_{\mathbf{s}\boldsymbol{\theta}}^- = -\mathbf{F}_{\mathbf{ss}}^- \mathbf{P}_{\mathbf{s}\boldsymbol{\theta}}^- \left(\mathbf{P}_{\theta\theta}^-\right)^{-1} \tag{93a}$$

$$\left(\mathbf{F}_{\mathbf{ss}}^-\right)^{-1} \mathbf{F}_{\mathbf{s}\boldsymbol{\theta}}^- = -\mathbf{P}_{\mathbf{s}\boldsymbol{\theta}}^- \left(\mathbf{P}_{\theta\theta}^-\right)^{-1} \tag{93b}$$

and, substituting this into Eq. (92) leads to

$$\mathbf{s}^+ = \mathbf{s}^- + 2\mathbf{P}_{\mathbf{s}\boldsymbol{\theta}}^- \left(\mathbf{P}_{\boldsymbol{\theta\theta}}^-\right)^{-1} \boldsymbol{\Xi}\left(\hat{\bar{\mathbf{q}}}^-\right)^{\mathrm{T}} \hat{\bar{\mathbf{q}}}^+ \tag{94a}$$

$$\approx \mathbf{s}^- + \mathbf{P}_{\mathbf{s}\boldsymbol{\theta}}^- \left(\mathbf{P}_{\boldsymbol{\theta\theta}}^-\right)^{-1} \boldsymbol{\delta\theta} \tag{94b}$$

which is equivalent to the qEKF non-attitude state update from Eq. (69) and the approximation is to second-order.

VI. Numerical Example

In this numerical example the spacecraft is placed in a circular orbit with an altitude of 622 km and an inclination of 45 degrees. At the beginning of the simulation the Earth is at vernal equinox 20 March 2012 and the spacecraft is at the ascending node which is located at the inertial X axis. Throughout its orbit the spacecraft is oriented such that its body-fixed X axis is directed in track and the Z axis is Earth-pointing with the Y axis following a right handed coordinate system. As a result the spacecraft has a constant angular velocity equal in magnitude to the orbital mean motion. The sun vector is assumed constant for the duration of the simulation. The magnetic field vector is obtained from the World Magnetic Model in the MATLAB Aerospace toolbox.

A gyro is used to measure the angular velocity of the spacecraft and is defined by the following sensor model [25]

$$\tilde{\boldsymbol{\omega}} = \boldsymbol{\omega} + \boldsymbol{\beta} + \boldsymbol{\eta}_v \tag{95a}$$

$$\dot{\boldsymbol{\beta}} = \boldsymbol{\eta}_u, \tag{95b}$$

where $\boldsymbol{\omega}$ is the true angular velocity, $\tilde{\boldsymbol{\omega}}$ is the measured angular velocity, $\boldsymbol{\beta}$ is the gyro bias vector, and $\boldsymbol{\eta}_v$ and $\boldsymbol{\eta}_u$ are zero-mean Gaussian white-noise processes. Simulated vectors measurements are created by adding noise to the true direction in the spacecraft body frame. The reference vectors remain noise free as the model is assumed perfect for this test case. The scalar weights a_i of the Wahba problem follow the QUEST measurement model and are given by $1/\sigma_{sun}^2$ and $1/\sigma_{mag}^2$ for the sun sensor and magnetometer measurements respectively.

The state vector consists of the three component gyro bias vector and the three component attitude angle representation $\mathbf{x}^{\mathrm{T}} = \begin{bmatrix}\boldsymbol{\beta}^{\mathrm{T}} & \boldsymbol{\theta}^{\mathrm{T}}\end{bmatrix}$. The initial gyro bias covariance is 0.2^2 $(\text{deg/hr})^2$ in each axis and the initial attitude covariance is 0.1^2 deg^2 in each axis. The initial estimated quaternion is obtained by perturbing the true quaternion according to the initial attitude covariance while the initial estimated gyro bias is always zero. The simulation spans 6000 seconds which is slightly more than one full orbit and uses a step size between observations of 1 second.

Error Source	Symbol	Value
Sun-sensor noise ($\boldsymbol{\eta}_{sun}$)	σ_{sun}	0.1 deg
Magnetometer noise ($\boldsymbol{\eta}_{mag}$)	σ_{mag}	0.5 deg
Angular Random Walk ($\boldsymbol{\eta}_{v}$)	σ_{v}	$\sqrt{10} \times 10^{-7}$ rad/sec$^{1/2}$
Gyro Bias Random Walk ($\boldsymbol{\eta}_{u}$)	σ_{u}	$\sqrt{10} \times 10^{-10}$ rad/sec$^{3/2}$

Table 1 Sensors Errors

Figures 1 and 2 show the performance of 100 Monte Carlo runs. The figures show the 100 instances of the estimation error, the thicker line is the 3-sigma value of the sample standard deviation. Underneath the thick lines there are the 100 instances of the 3-sigma filter's prediction of its own uncertainty. Since the predicted uncertainty matches the actual uncertainty it follows that the filter is consistent. Figure 3 shows the performance of SOAR under the same circumstances. It can be seen that there is no visible difference between the two algorithms.

Another case of interest is when only one set of measurements is available. While the standard q-method requires at least two independent vector measurements in order to determine the attitude, the method proposed in the qEKF includes an initial condition which replaces the requirement for the second measurement. Furthermore, an advantage of the nonlinear attitude update used in the qEKF over the standard MEKF is the ability to converge to an accurate estimate for highly nonlinear systems and poor initial estimates. Figures 4 and 5 show the attitude performance of 100 Monte Carlo runs for the proposed qEKF and the standard MEKF respectively. Figures 6 and 7 show the corresponding gyro bias performance. The same inputs except that only magnetometer measurements are included and the errors have been increased. The noise values from Table 1 have been increased by one order of magnitude, the initial gyro bias covariance is now 20^2 $(\text{deg/hr})^2$ in each axis and the initial attitude covariance is 200^2 deg^2 in each axis. In this case the qEKF quickly converges to an accurate estimate of both the attitude and the gyro bias. In the presence of such large initial errors the MEKF does converge, however, many of the runs remain outside the 3-sigma value of the filter's predicted covariance. As a result, the MEKF predicts a more accurate estimate than is actually achieved in the Monte Carlo simulation. Therefore, the nonlinear attitude update of the qEKF is advantageous over the linearized method of MEKF for very poor initial estimates.

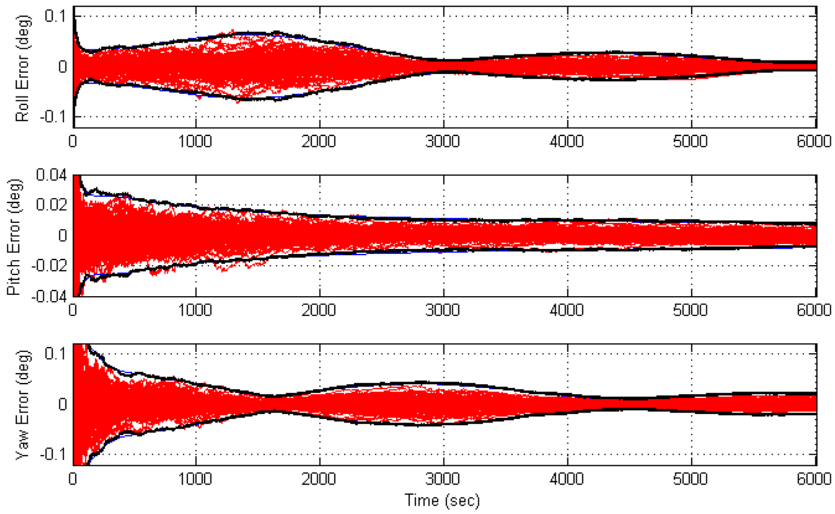

Fig. 1 Attitude estimation error of qEKF expressed in body frame

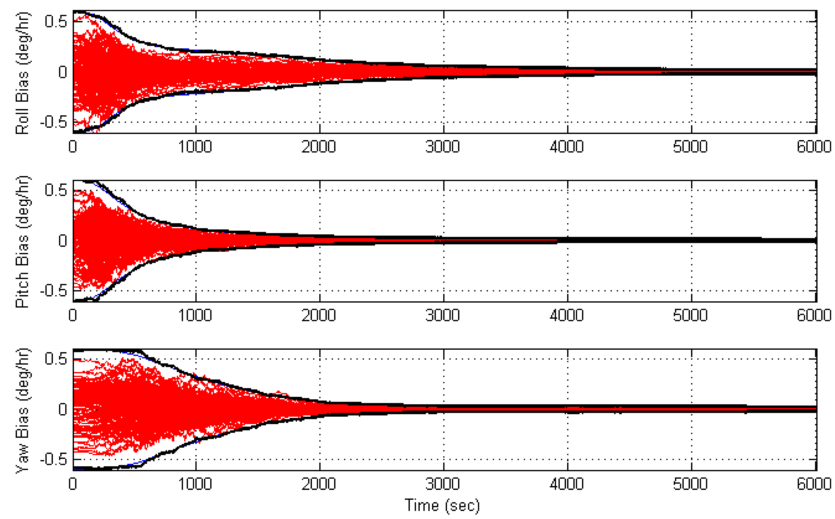

Fig. 2 Gyro bias estimation error of qEKF

VII. Conclusions

The q-method for quaternion estimation has been integrated into an extended Kalman filter (EKF) to produce the novel qEKF filter for attitude estimation which is capable of treating both attitude and non-attitude states without additional numerical iterations. Within the filter, attitude vector measurements are first processed using the q-method which solves the non-linear Wahba problem directly without any linearizing assumptions. Remaining measurements are processed to

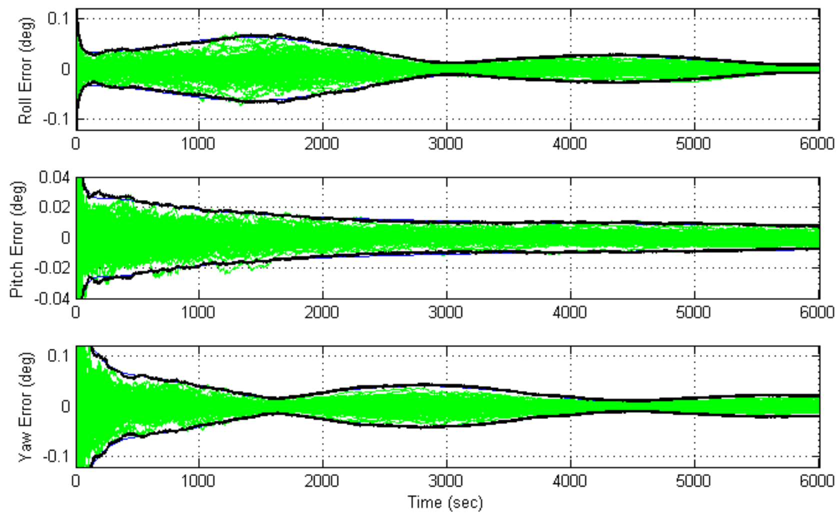

Fig. 3 Attitude estimation error of SOAR expressed in body frame

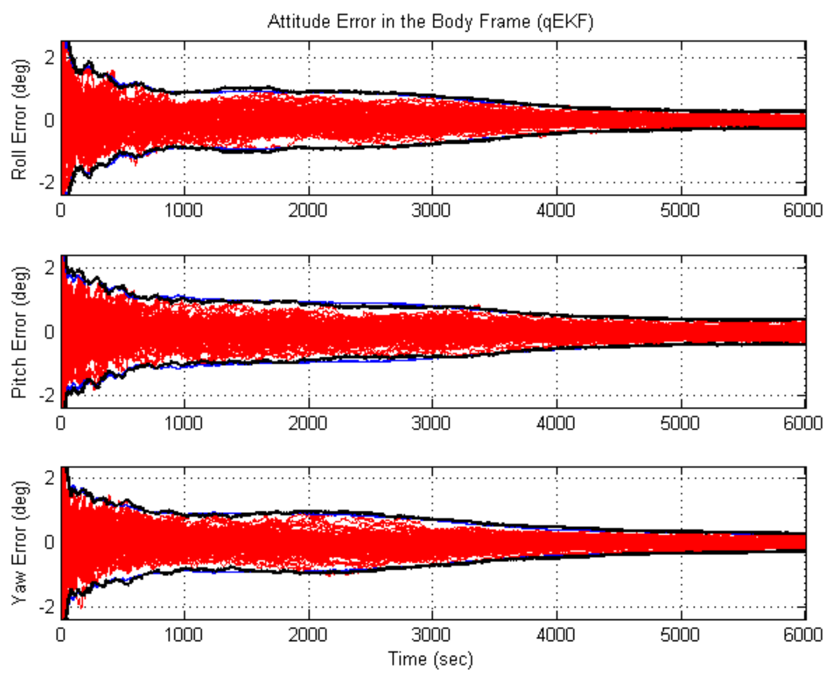

Fig. 4 Attitude estimation error of qEKF expressed in body frame with only magnetometer measurements and large initial errors

update the non-attitude states using the standard multiplicative extended Kalman filter algorithm. The proposed algorithm is shown to be equivalent to the Sequential Optimal Attitude Recursion (SOAR) filter to second-order in both the attitude and non-attitude updates where each method

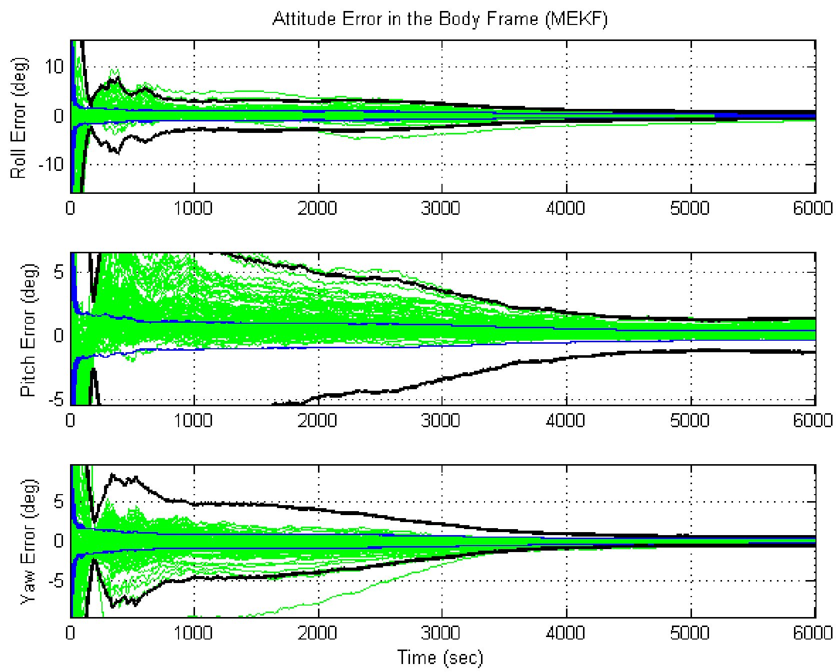

Fig. 5 Attitude estimation error of MEKF expressed in body frame with only magnetometer measurements and large initial errors

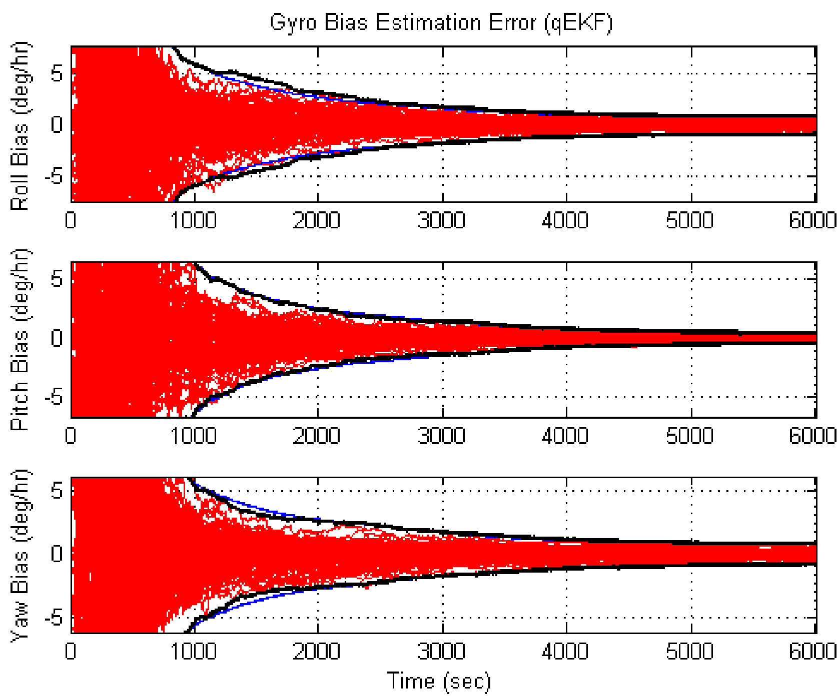

Fig. 6 Gyro bias estimation error of qEKF with only magnetometer measurements and large initial errors

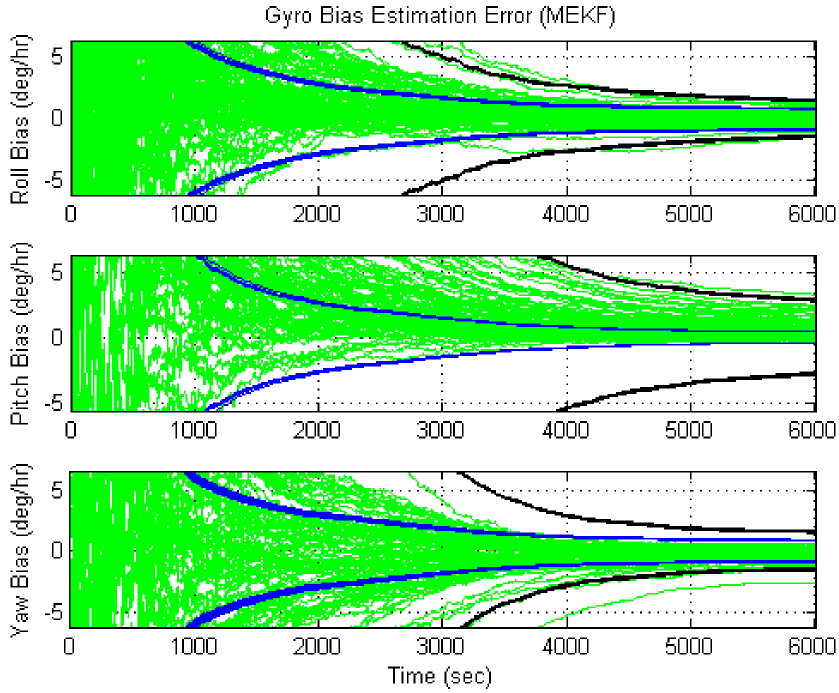

Fig. 7 Gyro bias estimation error of qEKF with only magnetometer measurements and large initial errors

represents the covariance and information matrix formulation respectively. In qEKF the initial condition is introduced into the Wahba problem through quaternion averaging where the SOAR filter relies on the information matrix approach. The equivalence of qEKF and SOAR was also validated by simulation results in which the filter estimated the attitude and gyro bias.

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